Schrödinger Equation for a System with Exponential-Type Restoring Force Function


*. Department of Physics, University of Abomey-Calavi, Abomey-Calavi, 01.B.P. 526, Cotonou, BENIN.

**. Department of Industrial and Technical Sciences, ENSET-Lokossa, University of Lokossa, Lokossa, BENIN.

Abstract

In this paper the Schrödinger equation is derived for a dynamical system with exponential-type restoring force function.

1. Let us consider the exponential-type restoring force function arisen in a class of exactly integrable quadratic Liénard-type nonlinear dissipative oscillator equations which admit also a position-dependent mass dynamics as well as in a class of generalized Duffing-type equations [1-4]

\[ F(x) = -\omega^2 xe^{2\gamma \varphi(x)} \]  \hspace{1cm} (1)

where \( \gamma \) and \( \omega \) are arbitrary parameters, and \( \varphi(x) \) an arbitrary function of \( x \). The associated potential reads

\[ V(x) = \omega^2 \int xe^{2\gamma \varphi(x)} \, dx \]  \hspace{1cm} (2)

2. Derivation of Schrödinger equation for \( \varphi(x) = \frac{1}{2} x^2 \)

The potential (2) becomes

\[ V(x) = \frac{\omega^2}{2\gamma} e^{\gamma x^2} \]  \hspace{1cm} (3)

For \( \gamma < 0 \), this potential tends to zero as \( x \to \pm \infty \), and becomes \( V(x) = \frac{\omega^2}{2\gamma} e^{\gamma x^2} \), for \( x = 0 \), so that the time-independent Schrödinger equation

\[ \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi = E\psi \]  \hspace{1cm} (4)

where \( \psi = \psi(x) \), takes the form

\[ \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{\omega^2}{2\gamma} e^{\gamma x^2} \right] \psi = E\psi \]  \hspace{1cm} (5)

1 Corresponding author.
E-mail: jeanakande7@gmail.com
After a few algebraic treatment, (5) may be written in the form

$$\frac{d^2\psi}{dx^2} - \frac{m\omega^2}{\gamma h^2} \left(e^{\gamma x^2} - \frac{2\gamma E}{\omega^2}\right)\psi = 0$$

(6)

With \( \lambda^2 = \frac{m\omega^2}{\gamma h^2} \) and \( \alpha = \frac{2\gamma E}{\omega^2} \)

the equation (6) may be rewritten as

$$\frac{d^2\psi}{dx^2} + \lambda^2 \left(e^{\gamma x^2} - \alpha\right)\psi = 0$$

(7)

The potential, obtained for the restoring force function \( F(x) = -\omega^2 xe^{2\gamma x} \)

$$V(x) = \frac{\omega^2}{4\gamma^2} (2\gamma x - 1)e^{2\gamma x}$$

(8)

gives as Schrödinger equation

$$\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{\omega^2}{4\gamma^2} (2\gamma x - 1)e^{2\gamma x} \right]\psi = E\psi$$

(9)

which becomes

$$\frac{d^2\psi}{dx^2} - \frac{m\omega^2}{2h^2 \gamma^2} \left[(2\gamma x - 1)e^{2\gamma x} - \frac{4\gamma^2 E}{\omega^2}\right]\psi = 0$$

(10)

Considering here \( \lambda^2 = \frac{m\omega^2}{2\gamma^2 h^2} \) and \( \alpha = \frac{4\gamma^2 E}{\omega^2} \), yields

$$\frac{d^2\psi}{dx^2} - \lambda^2 \left[(2\gamma x - 1)e^{2\gamma x} - \alpha\right]\psi = 0$$

(11)

The obtained Schrödinger equations (7) and (11) are under investigation. However, it is interesting to note that other potentials generated from the exponential-type restoring force functions introduced in previous studies [1-4] may also be considered to investigate the Schrödinger equation. In this perspective as example, the potential, that is the Schrödinger equation, associated to the following restoring force function

$$F(x) = -\omega^2 h(x)e^{\mu \phi(x)}$$

(12)

where \( \mu \) and \( \omega \) are arbitrary parameters, and \( \phi(x) \) and \( h(x) \) are arbitrary functions of \( x \),

will be studied in future works.

References

