Solutions for Euler and Navier-Stokes Equations in Powers of Time

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Abstract – We present a solution for the Euler and Navier-Stokes equations for incompressible case given any smooth (C^∞) initial velocity, pressure and external force in $N = 3$ spatial dimensions, based on expansion in Taylor's series of time. Without major difficulties, it can be adapted to any spatial dimension, $N \geq 1$.

Keywords – Lagrange, Mécanique Analitique, exact differential, Euler's equations, Navier-Stokes equations, Taylor's series, Cauchy, Mémoire sur la Théorie des Ondes, Lagrange's theorem, Bernoulli's law, non-uniqueness solutions.

§ 1

Let p, q, r be the three components of velocity of an element of fluid in the 3-D orthogonal Euclidean system of spatial coordinates (x, y, z) and t the time in this system.

Lagrange in his *Mécanique Analitique*, firstly published in 1788, proved that if the quantity $(p\,dx + q\,dy + r\,dz)$ is an exact differential when $t = 0$ it will also be an exact differential when t has any other value. If the quantity $(p\,dx + q\,dy + r\,dz)$ is an exact differential at an arbitrary instant, it should be such for all other instants. Consequently, if there is one instant during the motion for which it is not an exact differential, it cannot be exact for the entire period of motion. If it were exact at another arbitrary instant, it should also be exact at the first instant.[1]

To prove it Lagrange used

$$(1.1)\qquad \begin{cases} p = p^I + p^{II}t + p^{III}t^2 + p^{IV}t^3 + \cdots \\ q = q^I + q^{II}t + q^{III}t^2 + q^{IV}t^3 + \cdots \\ r = r^I + r^{II}t + r^{III}t^2 + r^{IV}t^3 + \cdots \end{cases}$$

in which the quantities p^I, p^{II}, p^{III}, etc., q^I, q^{II}, q^{III}, etc., r^I, r^{II}, r^{III}, etc., are functions of x, y, z but without t.

Here we will finally solve the equations of Euler and Navier-Stokes using this representation of the velocity components in infinite series, as pointed by Lagrange. We assume satisfied the condition of incompressibility, for brevity. Without it the resulting equations are more complicated, as we know, but the method of solution is essentially the same in both cases. We focus our attention in the general case of the Navier-Stokes equations, and for the Euler equations simply set the viscosity coefficient as $\nu = 0$.

To facilitate and abbreviate our writing, we represent the fluid velocity by its three components in indicial notation, i.e., $u = (u_1, u_2, u_3)$, as well as the external force will be $f = (f_1, f_2, f_3)$ and the spatial coordinates $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$. The pressure, a scalar function, will be represented as p. As frequently used in mathematics approach, the density mass will be $\rho = 1$.

The representation (1.1) is as the expansion of the velocity in a Taylor´s series in relation to time around $t = 0$, considering x, y, z as constant, i.e., for $1 \le i \le 3$,

$$(1.2) \qquad u_i = u_i|_{t=0} + \frac{\partial u_i}{\partial t}|_{t=0}\, t + \frac{\partial^2 u_i}{\partial t^2}|_{t=0}\, \frac{t^2}{2} + \frac{\partial^3 u_i}{\partial t^3}|_{t=0}\, \frac{t^3}{6} + \cdots + \frac{\partial^k u_i}{\partial t^k}|_{t=0}\, \frac{t^k}{k!} + \cdots$$

or

$$(1.3) \qquad u_i = u_i^0 + \sum_{k=1}^{\infty} \frac{\partial^k u_i}{\partial t^k}|_{t=0}\, \frac{t^k}{k!}.$$

For the calculation of $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \frac{\partial^3 u_i}{\partial t^3}, \ldots$ we use the values that are obtained directly from the Navier-Stokes equations and its derivatives in relation to time, i.e.,

$$(1.4) \qquad \frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu\, \nabla^2 u_i + f_i,$$

and therefore

$$(1.5) \qquad \frac{\partial^2 u_i}{\partial t^2} = -\frac{\partial^2 p}{\partial t\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial u_j}{\partial t} \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} \right) + \nu\, \nabla^2 \frac{\partial u_i}{\partial t} + \frac{\partial f_i}{\partial t},$$

$$(1.6) \qquad \frac{\partial^3 u_i}{\partial t^3} = -\frac{\partial^3 p}{\partial t^2\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_i}{\partial x_j} + 2 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2} \right) + \nu\, \nabla^2 \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial^2 f_i}{\partial t^2},$$

$$(1.7) \qquad \frac{\partial^4 u_i}{\partial t^4} = -\frac{\partial^4 p}{\partial t^3\, \partial x_i} - \sum_{j=1}^{3} N_j^3 + \nu\, \nabla^2 \frac{\partial^3 u_i}{\partial t^3} + \frac{\partial^3 f_i}{\partial t^3},$$

$$N_j^3 = \frac{\partial}{\partial t} N_j^2, \;\; N_j^2 = \frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_i}{\partial x_j} + 2 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2},$$

$$N_j^3 = \frac{\partial^3 u_j}{\partial t^3} \frac{\partial u_i}{\partial x_j} + 3 \frac{\partial^2 u_j}{\partial t^2} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + 3 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2} + u_j \frac{\partial}{\partial x_j} \frac{\partial^3 u_i}{\partial t^3},$$

$$(1.8) \qquad \frac{\partial^5 u_i}{\partial t^5} = -\frac{\partial^5 p}{\partial t^4\, \partial x_i} - \sum_{j=1}^{3} N_j^4 + \nu\, \nabla^2 \frac{\partial^4 u_i}{\partial t^4} + \frac{\partial^4 f_i}{\partial t^4},$$

$$N_j^4 = \frac{\partial}{\partial t} N_j^3 = \frac{\partial^4 u_j}{\partial t^4}\frac{\partial u_i}{\partial x_j} + 4\frac{\partial^3 u_j}{\partial t^3}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} + 6\frac{\partial^2 u_j}{\partial t^2}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2} +$$
$$+4\frac{\partial u_j}{\partial t}\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3} + u_j\frac{\partial}{\partial x_j}\frac{\partial^4 u_i}{\partial t^4},$$

and using induction we come to

(1.9)
$$\frac{\partial^k u_i}{\partial t^k} = -\frac{\partial^k p}{\partial t^{k-1}\,\partial x_i} - \sum_{j=1}^3 N_j^{k-1} + \nu\,\nabla^2\frac{\partial^{k-1} u_i}{\partial t^{k-1}} + \frac{\partial^{k-1} f_i}{\partial t^{k-1}},$$
$$N_j^{k-1} = \frac{\partial}{\partial t} N_j^{k-2} = \sum_{l=0}^{k-1}\binom{k-1}{l}\partial_t^{k-1-l} u_j\,\frac{\partial}{\partial x_j}\partial_t^l u_i,$$
$$\partial_t^0 u_n = u_n,\ \partial_t^m u_n = \frac{\partial^m u_n}{\partial t^m},\ \binom{k-1}{l} = \frac{(k-1)!}{(k-1-l)!\,l!}.$$

In (1.2) and (1.3) it is necessary to know the values of the derivatives $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \dots, \frac{\partial^k u_i}{\partial t^k}$ in $t = 0$ then we must to calculate, from (1.4) to (1.9),

(1.10)
$$\frac{\partial u_i}{\partial t}\Big|_{t=0} = -\frac{\partial p^0}{\partial x_i} - \sum_{j=1}^3 u_j^0\frac{\partial u_i^0}{\partial x_j} + \nu\,\nabla^2 u_i^0 + f_i^0,$$

the superior index 0 meaning the value of the respective function at $t = 0$, and

(1.11)
$$\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} = -\frac{\partial^2 p}{\partial t\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^3 N_j^1\Big|_{t=0} +$$
$$+\nu\,\nabla^2\frac{\partial u_i}{\partial t}\Big|_{t=0} + \frac{\partial f_i}{\partial t}\Big|_{t=0},$$
$$N_j^1\Big|_{t=0} = \sum_{j=1}^3\left(\frac{\partial u_j}{\partial t}\Big|_{t=0}\frac{\partial u_i^0}{\partial x_j} + u_j^0\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0}\right),$$

(1.12)
$$\frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0} = -\frac{\partial^3 p}{\partial t^2\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^3 N_j^2\Big|_{t=0} +$$
$$+\nu\,\nabla^2\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} + \frac{\partial^2 f_i}{\partial t^2}\Big|_{t=0},$$
$$N_j^2\Big|_{t=0} = \frac{\partial^2 u_j}{\partial t^2}\Big|_{t=0}\frac{\partial u_i^0}{\partial x_j} + 2\frac{\partial u_j}{\partial t}\Big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0} +$$
$$+u_j^0\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0},$$

(1.13)
$$\frac{\partial^4 u_i}{\partial t^4}\Big|_{t=0} = -\frac{\partial^4 p}{\partial t^3\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^3 N_j^3\Big|_{t=0} +$$
$$+\nu\,\nabla^2\frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0} + \frac{\partial^3 f_i}{\partial t^3}\Big|_{t=0},$$
$$N_j^3\Big|_{t=0} = \frac{\partial^3 u_j}{\partial t^3}\Big|_{t=0}\frac{\partial u_i^0}{\partial x_j} + 3\frac{\partial^2 u_j}{\partial t^2}\Big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0} +$$
$$+3\frac{\partial u_j}{\partial t}\Big|_{t=0}\frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} + u_j^0\frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0},$$

(1.14) $$\frac{\partial^5 u_i}{\partial t^5}|_{t=0} = -\frac{\partial^5 p}{\partial t^4\,\partial x_i}|_{t=0} - \sum_{j=1}^{3} N_j^4\,|_{t=0} + \\ + \nu\,\nabla^2 \frac{\partial^4 u_i}{\partial t^4}|_{t=0} + \frac{\partial^4 f_i}{\partial t^4}|_{t=0},$$

$$N_j^4|_{t=0} = \frac{\partial^4 u_j}{\partial t^4}|_{t=0} \frac{\partial u_i^0}{\partial x_j} + 4\frac{\partial^3 u_j}{\partial t^3}|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} + \\ + 6\frac{\partial^2 u_j}{\partial t^2}|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}|_{t=0} + 4\frac{\partial u_j}{\partial t}|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3}|_{t=0} + \\ + u_j^0 \frac{\partial}{\partial x_j}\frac{\partial^4 u_i}{\partial t^4}|_{t=0},$$

and of generic form,

(1.15) $$\frac{\partial^k u_i}{\partial t^k}|_{t=0} = -\frac{\partial^k p}{\partial t^{k-1}\,\partial x_i}|_{t=0} - \sum_{j=1}^{3} N_j^{k-1}\,|_{t=0} + \\ + \nu\,\nabla^2 \frac{\partial^{k-1} u_i}{\partial t^{k-1}}|_{t=0} + \frac{\partial^{k-1} f_i}{\partial t^{k-1}}|_{t=0},$$

$$N_j^{k-1}|_{t=0} = \sum_{l=0}^{k-1} \binom{k-1}{l}\,\partial_t^{k-1-l} u_j|_{t=0}\,\frac{\partial}{\partial x_j}\partial_t^l u_i|_{t=0},$$

$$\partial_t^0 u_n|_{t=0} = u_n^0,\ \ \partial_t^m u_n|_{t=0} = \frac{\partial^m u_n}{\partial t^m}|_{t=0}.$$

If the external force is conservative there is a scalar potential U such as $f = \nabla U$ and the pressure can be calculated from this potential U, i.e.,

(1.16) $$\frac{\partial p}{\partial x_i} = f_i = \frac{\partial U}{\partial x_i},$$

and then

(1.17) $$p = U + \theta(t),$$

$\theta(t)$ a generic function of time of class C^∞, so it is not necessary the use of the pressure p and external force f, and respective derivatives, in (1.4) to (1.15) if the external force is conservative. In this case, the velocity can be independent of the both pressure and external force, otherwise it will be necessary to use both the pressure and external force derivatives to calculate the velocity in powers of time.

The result that we obtain here in this development in Taylor's series seems to me a great advance in the search of the solutions of the Euler's and Navier-Stokes equations. It is possible now to know on the possibility of non-uniqueness solutions as well as breakdown solution respect to unbounded energy of another manner.

We now can choose previously an infinity of different pressures such that the calculation of $\frac{\partial u}{\partial t}$ and derivatives can be done, for a given initial velocity and external force, although such calculation can be very hard.

It is convenient say that Cauchy[2] in his memorable and admirable *Mémoire sur la Théorie des Ondes*, winner of the Mathematical Analysis award, year 1815, firstly does a study on the equations to be obeyed by three-dimensional molecules in a homogeneous fluid in the initial instant $t = 0$, coming to the conclusion which the initial velocity must be irrotational, i.e., a potential flow. Of this manner, after, he comes to conclusion that the velocity is always irrotational, potential flow, if the external force is conservative, which is essentially the Lagrange's theorem described in the begin of this article, but it is shown without the use of series expansion (a possible exception to the theorem occurs if one or two components of velocity are identically zero, when the reasonings on 3-D molecular volume are not valid). The solution obtained by Cauchy for Euler's equations is the Bernoulli's law, as almost always happens. Now a more generic solution is obtained, in special when it is possible a solution be expanded in polynomial series of time. Though not always a function can be expanded in Taylor's series, there is certainly an infinity of possible cases of solution where this is possible.

If the mentioned series is divergent in some point or region may be an indicative of that the correspondent velocity and its square diverge, again going to the case of breakdown solution due to unbounded energy. With the three functions initial velocity, pressure and external force belonging to Schwartz Space is expected that the solution for velocity also belongs to Schwartz Space, obtaining physically reasonable and well-behaved solution throughout the space.

The method presented here in this first section can also be applied in other equations, of course, for example in the heat equation, Schrödinger equation, wave equation and many others. Always will be necessary that the remainder in the Taylor's series goes to zero when the order k of the derivative tends to infinity (Courant[3], chap. VI). Applying this concept in (1.3) and (1.9), substituting t by τ, the remainder $R_{i,k}$ of order k for velocity component i is

$$(1.18) \qquad R_{i,k} = \frac{1}{k!}\int_0^t (t-\tau)^k \frac{\partial^{k+1} u_i}{\partial t^{k+1}}\, d\tau,$$

which can be estimated by Lagrange's remainder,

$$(1.19) \qquad R_{i,k} = \frac{t^{k+1}}{(k+1)!}\frac{\partial^{k+1} u_i}{\partial t^{k+1}}(\xi),$$

or by Cauchy's remainder,

$$(1.20) \qquad R_{i,k} = \frac{t^{k+1}}{k!}(1-\theta)^k \frac{\partial^{k+1} u_i}{\partial t^{k+1}}(\xi),$$

with $0 \le \xi \le t$ and $0 \le \theta \le 1$.

§ 2

In this section we will build a series of powers of time solving the Navier-Stokes equations, differently than that used in the previous section. From theorem of uniqueness of series of powers (Courant[3], chap. VIII), both solutions need be the same.

Defining

(2.1) $$u_i = u_i^0 + X_{i,1}t + X_{i,2}t^2 + \cdots + X_{i,n}t^n + \cdots = \sum_{n=0}^{\infty} X_{i,n}t^n,$$
$$X_{i,0} = u_i^0 = u_i(x_1, x_2, x_3, 0),$$

where each $X_{i,n}$ is a function of position (x_1, x_2, x_3), without t, and

(2.2) $$\frac{\partial p}{\partial x_i} = q_i^0 + q_{i,1}t + q_{i,2}t^2 + \cdots + q_{i,n}t^n + \cdots = \sum_{n=0}^{\infty} q_{i,n}t^n,$$
$$q_{i,0} = q_i^0 = \frac{\partial p^0}{\partial x_i},\ p^0 = p(x_1, x_2, x_3, 0),$$

(2.3) $$f_i = f_i^0 + f_{i,1}t + f_{i,2}t^2 + \cdots + f_{i,n}t^n + \cdots = \sum_{n=0}^{\infty} f_{i,n}t^n,$$
$$f_{i,0} = f_i^0 = f_i(x_1, x_2, x_3, 0),$$

we can put these series in the Navier-Stokes equation,

(2.4) $$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu\, \nabla^2 u_i + f_i.$$

The velocity derivative in relation to time is

(2.5) $$\frac{\partial u_i}{\partial t} = X_{i,1} + 2X_{i,2}t + 3X_{i,3}t^2 + \cdots + nX_{i,n}t^{n-1} + \cdots =$$
$$= \sum_{n=0}^{\infty}(n+1)X_{i,n+1}t^n,$$

the nonlinear terms are, of order zero (constant in time)

(2.6) $$\sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j},$$

of order 1,

(2.7) $$\sum_{j=1}^{3} \left(u_j^0 \frac{\partial X_{i,1}}{\partial x_j} + X_{j,1} \frac{\partial u_i^0}{\partial x_j} \right) t,$$

of order 2,

(2.8) $$\sum_{j=1}^{3} \left(u_j^0 \frac{\partial X_{i,2}}{\partial x_j} + X_{j,1} \frac{\partial X_{i,1}}{\partial x_j} + X_{j,2} \frac{\partial u_i^0}{\partial x_j} \right) t^2,$$

of order 3,

(2.9) $$\sum_{j=1}^{3}\left(u_j^0\frac{\partial X_{i,3}}{\partial x_j}+X_{j,1}\frac{\partial X_{i,2}}{\partial x_j}+X_{j,2}\frac{\partial X_{i,1}}{\partial x_j}+X_{j,3}\frac{\partial u_i^0}{\partial x_j}\right)t^3,$$

and of order n, of generic form, equal to

(2.10) $$\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}t^n,$$

with $X_{j,0}=u_j^0$, $\frac{\partial X_{i,0}}{\partial x_j}=\frac{\partial u_i^0}{\partial x_j}$.

Applying these sums in (2.4) we have

(2.11) $$\sum_{n=0}^{\infty}(n+1)X_{i,n+1}t^n=-\sum_{n=0}^{\infty}q_{i,n}t^n-$$
$$-\sum_{n=0}^{\infty}\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}t^n+\nu\sum_{n=0}^{\infty}\nabla^2X_{i,n}\,t^n+$$
$$+\sum_{n=0}^{\infty}f_{i,n}t^n,$$

and then

(2.12) $$(n+1)X_{i,n+1}=-q_{i,n}-\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}+$$
$$+\nu\nabla^2X_{i,n}+f_{i,n},$$

which allows us to obtain, by recurrence, $X_{i,1}$, $X_{i,2}$, $X_{i,3}$, etc., that is, for $1\le i\le 3$ and $n\ge 0$,

(2.13) $$X_{i,n+1}=\frac{1}{n+1}S_n,$$
$$S_n=-q_{i,n}-\sum_{j=1}^{3}\sum_{k=0}^{n}X_{j,k}\frac{\partial X_{i,n-k}}{\partial x_j}+\nu\nabla^2X_{i,n}+f_{i,n}.$$

You can see how much will become increasingly difficult calculate the terms $X_{i,n}$ with increasing the values of n, for example, will appear terms in $\nu^n,\nabla^2\nabla^2\ldots\nabla^2u_i^0$, etc. If $\nu>1$ certainly there is a specific problem to be studied with relation to convergence of the series, which of course also occurs in the representation given in section § 1. The same can be said for $t\to\infty$. In fact, I do not understand why a particle fluid initially in motion, without any collision with another particle and submitted to an permanent impulsive force need be with finite velocity as $t\to\infty$.

September-11,27-2016

What good would living on a planet without destruction, greed and envy, where the nations were dedicated to building a beautiful world

and to the salvation of those in need.
That there were no enemies and everyone could be happy where they live,
in their own way.

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