Oppermann's Conjecture and the Growth of Primes Between Pronic Numbers

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Abstract:

In this paper, we are going to prove Oppermann's conjecture which states there are at least one prime presents between first and second halves of two consecutive pronic numbers greater than one. So, there must be at least: $n(n-1) < p_1 < n^2$ and $n^2 < p_2 < n(n+1)$ for *n > 1*. Subsequently, we are going to prove the logarithmic sum of primes between two pronic numbers increase highest magnitude of *log(4)*.

Definition

According to Moivre-Stirling Approximation of factorial [1]:

$$\int_1^n \log(x)dx < \log(n!) < \int_1^{n+1} \log(x)dx \quad , \quad \int \log(x)dx = xlog(x)-x+C \quad , \ldots.. (1)$$

If we assume $\Delta_2=\frac{1}{2}\log(\frac{2}{1})$, $\Delta_3=\frac{1}{2}\log(\frac{3}{2})$,..., $\Delta_{n-1}=\frac{1}{2}\log(\frac{n-1}{n-2})$, $\Delta_n=\frac{1}{2}\log(\frac{n}{n-1})$,

we can get better Moivre-Stirling Approximation using simple geometric arguments from figure below [1]:

$$\int_{n-1}^n \log(n)dn-\log(n-1) = nlog(\frac{n}{n-1})-1 = 2n\Delta_n-1 \leq \Delta_{n-1} \quad \text{for } n > 2$$

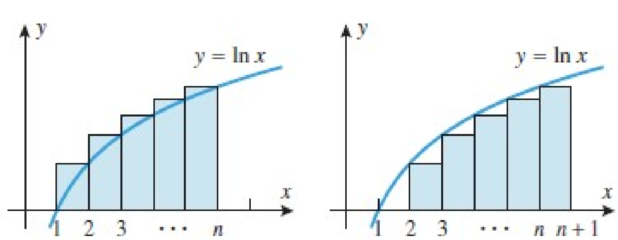
$$\text{because } \frac{d}{dn}(2n\Delta_n-\Delta_{n-1}) > 0 \text{ and } \lim_{n\to\infty}(2n\Delta_n-\Delta_{n-1}) = 1$$

so,
$$\int_2^{n+1} \log(x)dx - (\Delta_2+\Delta_3+...+\Delta_{n-1}+\Delta_n) < \log(n!)$$
$$< \int_1^n \log(x)dx + (\Delta_2+\Delta_3+...+\Delta_{n-1}+\Delta_n)$$

hence,
$$\int_2^{n+1} \log(x)dx - \frac{1}{2}\log(n) < \log(n!) < \int_1^n \log(x)dx + \frac{1}{2}\log(n)$$

or,
$$(n+1)\log(n+1)-n-\frac{1}{2}\log(n)-b+1 < \log(n!) < nlog(n)-n+\frac{1}{2}\log(n)+1 \quad , \ldots.. (2)$$

where *b = log(4)*



▲ Figure Ex-30

We define function $\upsilon(x)$ and $\psi(x)$ conventionally as [2]:

$$\upsilon(x) = \sum_{p\le x} \log(p) = \log\prod_{p\le x} p \quad , \quad \psi(x) = \sum_{p^m\le x} \log(p)$$

Since $p^2 \le x, p^3 \le x, \ldots$ are equivalent to $p \le x^{1/2}, p \le x^{1/3}, \ldots$, we have [2], [3] :

$$\psi(x) = \upsilon(x)+\upsilon(x^{1/2})+\upsilon(x^{1/3})+\ldots = \sum_{m\ge 1} \upsilon(x^{1/m}) \quad , \ldots.. (3)$$

and so, $\psi(2n) = \upsilon(2n)+\upsilon((2n)^{1/2})+\upsilon((2n)^{1/3})+\ldots = \sum_{m\ge 1} \upsilon((2n)^{1/m})$

Proof

We know [3]: $\log((2n)!) = \psi(2n)+\psi(n)+\psi(\frac{2n}{3})+\ldots$, ….. (4)

Let, $N_{2n} = \frac{(2n)!}{n!n!}$, then from (4) [3]: $\log(N_{2n}) = \psi(2n)-\psi(n)+\psi(\frac{2n}{3})-\ldots$, ….. (5)

As $\psi(x)$ is a steadily increasing function,

$$\log(\frac{N_{m_1}}{N_{m_2}}) = (\psi(m_1)-\psi(m_2))-(\psi(\frac{m_1}{2})-\psi(\frac{m_2}{2}))+(\psi(\frac{m_1}{3})-\psi(\frac{m_2}{3}))-\ldots$$

$$< (\psi(m_1)-\psi(m_2)) \quad \text{where} \quad m_1 \ge m_2 \quad , \ldots.. (6)$$

First Half:

From (2) and (5), we get:

$$\log(\frac{N_{n^2}}{N_{n(n-1)}}) >$$

$$(n^2+1)\log(n^2+1)-n^2-\log(n)-b+1-n(n-1)\log(n(n-1))+n(n-1)-\frac{1}{2}\log(n(n-1))-1$$

$$-n^2\log(\frac{n^2}{2})+n^2-\log(\frac{n^2}{2})-2+(n(n-1)+2)\log(\frac{n(n-1)}{2}+1)-n(n-1)-\log(\frac{n(n-1)}{2})-2b+2$$

$$= n^2\log(\frac{2n^2+2}{n^2})-n(n-1)\log(\frac{2n(n-1)}{n(n-1)+2})+2\log(n(n-1)+2)$$

$$+\log(n^2+1)-3\log(n)-\frac{3}{2}\log(n(n-1))-3b \quad , \ldots.. (7a)$$

Again, form relation of $\upsilon(x)$ and $\psi(x)$, we get from (3):

$$\psi(n^2)-\psi(n(n-1)) =$$
$$(\upsilon(n^2)-\upsilon(n(n-1)))+(\upsilon(n)-\upsilon(n(n-1))^{\frac{1}{2}})+(\upsilon(n^{\frac{2}{3}})-\upsilon((n(n-1))^{\frac{1}{3}}))+\ldots$$

We assume $2^m=n$, then $(\upsilon(n)-\upsilon((n(n-1))^{\frac{1}{2}}))+(\upsilon(n^{\frac{2}{3}})-\upsilon((n(n-1))^{\frac{1}{3}}))+\ldots <$
$$m\log(2)+(\frac{2m}{3})\log(2)+\ldots+\log(2) = 2m\log(2)(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2m}) < 2m\log(2)\int_1^{(2m)}\frac{1}{x}dx =$$
$$2m\log(2)\log(2m) = 2\log(n)\log(2\,lg(n)) \qquad \text{where } lg(x)=\frac{\log(x)}{\log(2)}$$

because $1 > n-(n(n-1))^{\frac{1}{2}} > n^{\frac{2}{3}}-(n(n-1))^{\frac{1}{3}} > n^{\frac{1}{2}}-(n(n-1))^{\frac{1}{4}} > \ldots$

hence, $(\psi(n^2)-\psi(n(n-1))) < (\upsilon(n^2)-\upsilon(n(n-1)))+2\log(n)\log(2\,lg(n))$, ….. (8a)

Finally, we get from (6), (7a) and (8a):

$$(\upsilon(n^2)-\upsilon(n(n-1))) > n^2\log(\frac{2n^2+2}{n^2})-n(n-1)\log(\frac{2n(n-1)}{n(n-1)+2})+2\log(n(n-1)+2)$$
$$+\log(n^2+1)-3\log(n)-\frac{3}{2}\log(n(n-1))-3b-2\log(n)\log(2lg(n)) = F_1(n) \quad , \ldots.. (9a)$$

Second Half:

Again, From (2) and (5), we get:

$$\log(\frac{N_{n(n+1)}}{N_{n^2}}) >$$
$$(n(n+1)+1)\log(n(n+1)+1)-n(n+1)-\frac{1}{2}\log(n(n+1))-b+1-n^2\log(n^2)+n^2-\log(n)-1$$
$$-n(n+1)\log(\frac{n(n+1)}{2})+n(n+1)-\log(\frac{n(n+1)}{2})-2+(n^2+2)\log(\frac{n^2}{2}+1)-n^2-\log(\frac{n^2}{2})-2b+2$$
$$= n(n+1)\log(\frac{2n(n+1)+2}{n(n+1)})-n^2\log(\frac{2n^2}{n^2+2})+2\log(n^2+2)$$
$$+\log(n(n+1)+1)-3\log(n)-\frac{3}{2}\log(n(n+1))-3b \quad , \ldots.. (7b)$$

Again, form relation of $\upsilon(x)$ and $\psi(x)$, we get from (3):

$$\psi(n(n+1))-\psi(n^2) =$$
$$(\upsilon(n(n+1))-\upsilon(n^2))+(\upsilon((n(n+1))^{\frac{1}{2}})-\upsilon(n))+(\upsilon((n(n+1))^{\frac{1}{3}})-\upsilon(n^{\frac{2}{3}}))+\ldots$$

We assume $2^m=(n(n+1))^{\frac{1}{2}}$, then $(\upsilon((n(n+1))^{\frac{1}{2}})-\upsilon(n))+(\upsilon((n(n+1))^{\frac{1}{3}})-\upsilon(n^{\frac{2}{3}}))+\ldots <$

$$mlog(2)+(\frac{2m}{3})\log(2)+\ldots+\log(2) = 2mlog(2)(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2m}) < 2mlog(2)\int_1^{(2m)}\frac{1}{x}dx =$$

$$2mlog(2)\log(2m) = \log(n(n+1))\log(lg(n(n+1)))$$ where $lg(x)=\frac{\log(x)}{\log(2)}$

because $1 > (n(n+1))^{\frac{1}{2}}-n > (n(n+1))^{\frac{1}{3}}-n^{\frac{2}{3}} > (n(n+1))^{\frac{1}{4}}-n^{\frac{1}{2}} > \ldots$

hence, $(\psi(n(n+1))-\psi(n^2)) < (\upsilon(n(n+1))-\upsilon(n^2))+\log(n(n+1))\log(lg(n(n+1)))$

, ….. (8b)

Finally, we get from (6), (7b) and (8b):

$$(\upsilon(n(n+1))-\upsilon(n^2)) > n(n+1)\log(\frac{2n(n+1)+2}{n(n+1)})-n^2\log(\frac{2n^2}{n^2+2})+2\log(n^2+2)$$

$$+\log(n(n+1)+1)-3\log(n)-\frac{3}{2}\log(n(n+1))-3b-\log(n(n+1))\log(lg(n(n+1))) = F_2(n)$$

, ….. (9b)

Now, $F'_1(n) = 2nlog(\frac{(n(n-1)+2)(n^2+1)}{(n-1)n^3})+\log(\frac{2n(n-1)}{n(n-1)+2})-\frac{3}{n}-\frac{3}{2}\frac{2n-1}{n(n-1)}$

$$-\frac{2}{n}\log(2lg(n))-\frac{2}{n}$$

and $F'_2(n) = 2nlog(\frac{(n(n+1)+1)(n^2+2)}{(n+1)n^3})+\log(\frac{2n(n+1)+2}{n(n+1)})-\frac{3}{n}-\frac{3}{2}\frac{2n+1}{n(n+1)}$

$$-\frac{2n+1}{n(n+1)}\log(lg(n(n+1)))-\frac{2n+1}{n(n+1)}$$

Now, $F_1(21) > 0$ and $F_2(21) > 0$

As $F'_1(n)$ and $F'_2(n)$ are increasing functions of *n* and $F'_1(9) > 0$ and $F'_2(8) > 0$,

hence, $F_1(n)$, $F_2(n) > 0$ for $n > 20$.

It is easy to verify Oppermann's conjecture for $1 < n \le 20$, we have actually proved the conjecture.

Again, $\lim_{n \to \infty} F'_1(n) = \lim_{n \to \infty} F'_2(n) = \frac{b}{2}$,

hence, the growth of prime numbers between two pronic numbers < $\frac{b}{2} + \frac{b}{2} = b = \log(4)$.

Reference

[1] H. Anton, I. Bivens and S. Davis, "Calculus," John Wiley & Sons, Inc, NY, p. 528, p. 664, 2002.

[2] G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers," Oxford University Press, NY, pp. 340-341, 1979.

[3] Edited: G. H. Hardy, P. V. S. Aiyar and B. M. Wilson, "Collected Papers of Srinivasa Ramanujan," Cambridge University Press, pp. 208-209, 1927.