Quantum Hall Effect for Dyons

Pawan Kumar Joshi and O.P.S.Negi

September 9, 2016

Department of Physics
Kumaun University
S. S. J. Campus
Almora – 263601 (Uttarakhand)

Email: pj4234@gmail.com
ops_negi@yahoo.co.in

Considering the generalized charge and generalized four potential associated of dyons as complex quantities with their real and imaginary parts as electric and magnetic constituents, in this present discussion we have constructed a gauge covariant and rotational symmetric angular momentum operator for dyons in order to analyze the integer and fractional quantum Hall effect. It has been shown that the commutation relations of angular momentum operator possesses a higher symmetry to reproduce the eigen values and eigen function Lowest Landau Level (LLL) for quantum Hall system. The LLL has also been constructed in terms of I^{st} Hopf map ($S^3 \to S^2$) and it is concluded that dyons are more suitable object to investigate the existence of quantum Hall effect (both integer and fractional)

1 Introduction

The two discoveries of [1, 2]quantum hall effect(Integer and Fractional) open a new area in theoretical and experimental physics. The quantum Hall effect was discovered by von Klitzing, Dorda and Pepper[2]. This effect has been considered as one of the most remarkable phenomena in condensed-matter physics and theoretical physics. Klitzing et al [2] discovered that two dimensional electron gas,[1, 3] at very low temperatures($\sim 4K$) and strong magnetic fields($\sim 10T$), displays a remarkable quantization of the Hall conductance. Robert Laughlin [4] put forward an argument for the quantization of the Hall conductance $\left(\sigma_{xy} = \nu \frac{e^2}{h}\right)$ where ν is an integer. This argument plays a seminal role in the development of the theory of the Integer quantum

Hall effect [1]. The next revolution occurred with the discovery of the "Fractional Quantum Hall effect (FQHE)" by Tsui-Stormer-Gossard[5] (TSG). It has shown that the Hall conductance is quantized at rational fractional (f). The striking feature of this fraction quantum Hall effect was that all fractional filling factor appear with odd denominator. In other words, the Hall conductivity can take on rational fractional values in the units of $\frac{e^2}{h}$, and denominator of the fractional ($i.e. f = \frac{p}{q}$) was necessarily odd. The leading members given rise to be $\frac{1}{3}$, $\frac{2}{3}$, $\frac{2}{5}$, etc. The first fractional $\left(\frac{1}{3}\right)$ was observed in 1982 in GaAs hetro-structures. This effect was called the anomalous quantum Hall effect. The integer quantum Hall effect can be explained by free electrons. In contrast, the fractional quantum Hall effect needs us to take interactions between electrons. Fractional quantum Hall effect theory can not be explained simply by landau level structure. In FQHE the conductivity at very high magnetic [5, 6] field gives the fractionally field lowest landau level. The major breakthrough in this theory was made by Laughlin, who proposed a justrow-type[4] trial wave function for filling factor $\nu = \frac{1}{m}$ with m an odd integer. Laughlin's wave function described a quantum liquid and provided an explicit example of a correlated many body state. However Laughlin wave functions [4] are not translation-ally invariant [7], but described a circular droplet of fluid, which must be confined in an external potential. Laughlin circumvented this problem by comparing the properties of the fluid to those of the classical $2D$ one component plasma. Haldane demonstrated construct the hierarchy of fractional quantum Hall effect[7] by spherical geometry. which is very convenient geometry for studies of FQHE systems. In this geometry a $2D$ electron gas of N particles problem was discussed on a spherical surface R, in a radial (monopole) magnetic field. So, the first attempt to understand Hall effect from magnetic monopoles was described Haldane. This technique allows the construction of homogeneous states with finite N; in the limit R, N and total magnetic flux$\rightarrow \infty$, the Euclidean group of the plane has been is recovered [8, 9] from the rotation group $SO(3)$ of the sphere. In the spherical geometry [7], the two dimensional sheet containing charge particles are wrapped around the sphere, and a perpendicular magnetic field is generated by placing a Dirac monopole [10] at the center of the sphere. The popularity of this geometry may be discussed due to two reasons: First, it does not have edges (which makes it suitable for an investigation of the bulk properties) and second, Landau levels have finite degeneracy. This theory has been extensively studied later on by Fano [8]. The question of existence of monopole [10] and dyons [11, 12] has become a challenging new frontier and more interest in high energy physics. Dirac showed [10, 11, 13] that the quantum mechanics of an electrically charged particle of charge e and a magnetically charged particle of charge g is consistent only if $eg = 2\pi I$, I being an integer. Schwinger-Zwanziger [11, 12] generalized this condition to allow for the possibility of particles (dyons) these carry both electric and magnetic charges. As seen there exists a quantum mechanical theory which can have two particles of electric and magnetic charges (e_1, g_1) and (e_2, g_2) only if $e_1 g_2 - e_2 g_1 = 2\pi I$ [11]. The angular momentum in the field of the two particle system can be calculated readily with the magnitude $\frac{e_1 g_2 - e_2 g_1}{4\pi c}$. This has an integer or half-integer value, as expected in quantum mechanics, only if $e_1 g_2 - e_2 g_1 = 2\pi I$. Many physicists [14]-[16] has been claiming now a days the possibility of indirect evidences of magnetic monopoles in the condensed matter physics. So, it is being speculated that magnetic materials may provide a new context for observing magnetic monopoles which plays an important role in condensed matter physics.

2 Why dyons

Particles carrying simultaneous existence of electric and magnetic charge are called dyons. In addition to magnetic monopole solutions, the Yang - Mills-Higgs theory with an adjoint Higgs field has dyon solutions. By definition, a dyon is a particle or soliton with both magnetic and electric charge. The name was coined by Schwinger [17]. Dyons are not strictly static, although they are stationary in certain gauges, and they have non - zero kinetic energy. In physics, a dyon is a hypothetical particle in 4-dimensional theories with both electric and magnetic charges. A dyon with a zero electric charge is usually referred to as a magnetic monopole. Many grand unified theories predict the existence of both magnetic monopoles and dyons. Dyons were first proposed [11] by Julian Schwinger in 1969 as a phenomenological alternative to quarks. He extended the Dirac quantization condition to the dyon and used the model to predict the existence of a particle with the properties of the J/ψ meson prior to its discovery in 1974. The allowed charges of dyons are restricted by the Dirac quantization condition. This states, in particular that the magnetic charge on dyons must be integral, while the electric charges corresponding to dyons must all be equal modulo 1. The Witten effect, demonstrated [18] the dyons of Charge $e\theta/2\pi$ states that the electric charges of dyons must all be equal, modulo one, to the product of their magnetic charge and the theta angle of the theory. In particular, if a theory preserves $\mathcal{CP}$ symmetry then the electric charges of all dyons are integers. If monopoles carry electric charge in addition to their magnetic charge, they are called dyons. Dyons are not strictly static, although they are stationary in certain gauges and they have non - zero kinetic energy. In spite of the enormous potential importance of Dirac's monopoles and the fact that they have been extensively studied recently, there has been presented no reliable theory which is as conceptually transparent and practically tractable as the usual electrodynamics. However, the problem raised be Dirac's veto were eventually solved when Wu and Yang [19] introduced the fibre bundle formulation into gauge theories. Still there are following paradoxes faced by the theory of pure monopoles.

2.1 Wrong connection between spin and statistics

If the monopole exists, then classical physics tells that a system of pole g and electric charge e has an angular momentum of magnitude eg directed from charge to pole i.e.

$$\overrightarrow{J} = \overrightarrow{L} - eg\hat{r}. \tag{1}$$

This is the sum of orbital angular momentum of the particle and the angular momentum of the electromagnetic fields. In quantum mechanics that spin adds to orbital and intrinsic angular momentum, so that for $eg = \left(n + \frac{1}{2}\right)$, an otherwise integral spin system will have net half integral total angular momentum. This holds equally good in the $SU\left(2\right)$ gauge field formulation of charge pole interactions. In fact this spin may be used to derive the gauge field shown by Jackiw and Rebbi [20]and demonstrating that in an $SU\left(2\right)$ quantum gauge field theory, with iso - spin symmetry broken spontaneously by triplet of scalar mesons, iso-spinors degrees of freedom and converted into spin degrees of freedom in the field of magnetic monopole consequently in solution - monopole sector of quantum theory total angular momentum $\left(\overrightarrow{J}\right)$ is the sum of conventional orbital plus spin $\left(\overrightarrow{M}\right)$ and iso - spin $\left(\overrightarrow{I}\right)$;

$$\overrightarrow{J} = \overrightarrow{L} + \overrightarrow{M} + \overrightarrow{I}. \tag{2}$$

As such, perhaps an object whose half integral spin comes from charge pole contribution obeys Fermi - Dirac statistics, so that a fermion may be made out of bosons.
It led Goldhaber [21] to prove a theorem which states that if electric charge e can combine with magnetic monopole g to from the cluster with half integral values for the product eg then there must entities with wrong connection between spin and statistics, considering electric and magnetic charge on the same particle (a dyons) could solve this problem. They showed that dyon (provides carrying electric and magnetic charges) wave function (which is diagonal in both angular momentum and parity) lead to correct spin statistics relationship. Such particle (carrying electric and magnetic charges) were named as dyons by Schwinger [17]suggesting that quark and dyons.

3 Field Associated with Dyons

The generalized duality invariant Dirac Maxwell's equation in presence of electric and magnetic[22, 23] charges are expressed as

$$\overrightarrow{\nabla}\cdot\overrightarrow{E}=\rho_{e}\quad;\qquad\overrightarrow{\nabla}\times\overrightarrow{H}=\overrightarrow{j}+\frac{\partial\overrightarrow{E}}{\partial t},$$
$$\overrightarrow{\nabla}\cdot\overrightarrow{H}=\rho_{m}\quad;\qquad\nabla\times\overrightarrow{E}=-\overrightarrow{k}-\frac{\partial\overrightarrow{H}}{\partial t},\tag{3}$$

where ρ_e is the charge source density due to electric charge , ρ_m is the charge source density due to magnetic charge (monopole), $\overrightarrow{j}$ is the current source density due to electric charge (e) and $\overrightarrow{k}$ is the current source density due to magnetic charge(g). This hypothesis of existence of magnetic charge(monopole) provides an explanation for the quantization of electric charge, Dirac [12, 13], gave an interesting result was that the product of electric charge (e) with magnetic monopole charge (g) must be quantized.

$$eg=I.\tag{4}$$

where I is an integer which could assume the values $1,2,3$........

In spite of many good point, Dirac's monopole theory encounters the difficulty of string. The vector potential can not be defined uniquely and definitely along this string. This condition was referred as Dirac's veto [11]. It is unnatural and undesirable condition because string are unphysical object. The name dyon was coined by Schwinger [17] for the particles carrying simultaneously the existence of electric and magnetic charges. Dyons are not strictly static, although they are stationary in certain gauges, and they have non - zero kinetic energy. A dyon with a zero electric charge is usually referred to as a magnetic monopole. Schwinger extended the Dirac quantization condition (4) to the dyon. So an alternative approach which is free from Dirac string involve a second potential in addition to the electric four potential. The electric and magnetic fields of dyons satisfying the generalized Dirac Maxwell's equations are now expressed in terms of components of two four potentials in a symmetrical manner i.e.

$$\overrightarrow{E}=-\overrightarrow{\nabla}\phi_{e}-\frac{\partial\overrightarrow{A}}{\partial t}-\overrightarrow{\nabla}\times\overrightarrow{B},$$
$$\overrightarrow{H}=-\overrightarrow{\nabla}\phi_{g}-\frac{\partial\overrightarrow{B}}{\partial t}-\overrightarrow{\nabla}\times\overrightarrow{A},\tag{5}$$

Where $\{A^{\mu}\}=\left\{\phi_{e},\,\vec{A}\right\}$ and $\{B^{\mu}\}=\left\{\phi_{g},\,\vec{B}\right\}$ are the component of two four potential associated respectively

The complex vector electrodynamic field $\overrightarrow{\psi}=\overrightarrow{E}-i\overrightarrow{H}$ reduces the four GDM [22] equations to two differential equations as

$$\overrightarrow{\nabla}.\overrightarrow{\psi} = \rho;$$

$$\overrightarrow{\nabla}\times\overrightarrow{\psi} = -i\overrightarrow{j} - i\frac{\partial\overrightarrow{\psi}}{\partial t}. \tag{6}$$

Consequently, the Lorentz force equation of motion may be written in following form as

$$m\frac{d^2x}{dt^2} = \left(eF_{\mu\nu} + g\widetilde{F}_{\mu\nu}\right)u^{\nu}; \tag{7}$$

which may further be reduced to

$$m\frac{dv}{dt} = e\left(\overrightarrow{E} + \overrightarrow{u}\times\overrightarrow{H}\right) + g\left(\overrightarrow{H} - \overrightarrow{u}\times\overrightarrow{E}\right); \tag{8}$$

where m is the mass of the particle, e is the electric charge, $\{u^{\nu}\}$ is four-velocity of particle, space-time four vector is defined as $\{x^{\mu}\}= \{t, \overrightarrow{x}\}$ and g is magnetic charge. Electric and magnetic four-current are related as $j^{\mu} = eu^{\mu}$ and $k^{\mu} = gu^{\mu}$. As such the duality invariance is an intrinsic property of Maxwell's Lorentz theory of electrodynamics in presence of monopole(ie. for dyons).

let us introduce the generalized charge for dyon as $q = e - ig$, so that the Generalized four potential $V^{\mu} = \left(\phi, \vec{V}\right)$ associated with dyons is defined as

$$V^{\mu} = A^{\mu} - iB^{\mu}; \tag{9}$$

So the duality transformations for $\{A^{\mu}\}$ and $\{B^{\mu}\}$ are described as

$$\begin{aligned} A_{\mu} = &\ A_{\mu}cos\theta + B_{\mu}sin\theta; \\ B_{\mu} = &\ A_{\mu}sin\theta - B_{\mu}cos\theta; \end{aligned} \tag{10}$$

Hence, the covariant tensorial form of generalized Dirac-Maxwell's equations of dyons may be written as,

$$\begin{aligned} \partial_{\nu}F^{\mu\nu} = &\ j^{\mu}; \\ \partial_{\nu}\widetilde{F_{\mu\nu}} = &\ k^{\mu}; \end{aligned} \tag{11}$$

These equation are invariant under the duality transformations

$$\begin{aligned} \left(F, \widetilde{F}\right) = &\ \left(Fcos\theta + \widetilde{F}sin\theta;\ Fsin\theta - \widetilde{F}cos\theta\right); \\ (j_{\mu}, k_{\mu}) = &\ (j_{\mu}cos\theta + k_{\mu}sin\theta;\ j_{\mu}sin\theta - k_{\mu}cos\theta). \end{aligned} \tag{12}$$

where

$$\frac{g}{e} = \frac{B_\mu}{A_\mu} = \frac{k_\mu}{j_\mu} = \frac{\widetilde{F}}{F} = -tan\theta. \tag{13}$$

is described as constancy condition. The generalized charge may also be written as

$$q = \mid q \mid e^{-i\theta}. \tag{14}$$

In addition to the dual symmetry,the equation of motion (7) and the GDM field equation (11) leads to the following symmetries;

(a) invariance under a pure rotation in charge space or its combination with a transformation containing simultaneously space and time reflection(strong symmetry);

(b) a weak symmetry under charge reflection combined with space reflection or time reflection (not both);

(c) a weak symmetry under PT (combined operation of parity and time reversal) and strong symmetry under CPT (combined operation of charge conjugation, parity and time reversal).

using equation (13), the Interaction of i^{th}dyon in the field of j^{th} dyon may be written as follows

$$I_{ij} = \frac{A^j_\mu}{e_j} q^*_j q_i u^i_\mu,$$

where A^j_μ is the electric four potential describing the field of j^{th} dyon e_j is its electric charge and u^i_μ is the four-velocity of i^{th} dyon in the field of j^{th} this equation shows that

(a) interaction between two dyons is zero, when their generalized charges are orthogonal in their combined charge space.

(b) interaction depends on electric coupling parameter

$$\alpha_{ij} = e_i e_j + g_i g_j. \tag{15}$$

under the constancy condition $\frac{e_i}{g_i} = \frac{e_j}{g_j}$=constant.

(c) interaction depends on the magnetic coupling parameter(i.e. chirality)

$$\mu_{ij} = e_i g_j - g_i e_j; \tag{16}$$

under the condition$\frac{e_i}{g_j} = -\frac{e_j}{g_j}$ The coupling between two generalized charges q_i and $q_j$is described as

$$q_i^* \cdot q_j = (e_i e_j + g_i g_j) - i\,(e_i g_j - g_i e_j) = \alpha_{ij} - i\,\mu_{ij}; \tag{17}$$

where the real part α_{ij} is called the electric coupling parameter (the Coulomb like term) responsible for the existence of either electric charge or magnetic monopole while the imaginary part μ_{ij} is the magnetic coupling parameter and plays an important role for the existence of magnetic charge. Both of these parameters are invariant under the duality transformations. The parameter μ_{ij} has also been named as Chirality quantization parameter for dyons and leads the following charge quantization condition i.e.

$$\mu_{ij} = \pm I \ (I \in \mathbb{Z}); \tag{18}$$

where the half integral quantization is forbidden by chiral invariance and locality in commutator of the electric and magnetic vector potentials.

If we consider two dyons with $q_i = (e, 0)$ and $q_j = (0, g)$ the quantization condition (18) reduces to well known Dirac quantization condition $eg = \pm I$. While if we do not consider Dirac particle as dyon the Dirac quantization condition loses its dual invariance. Thus dyon plays an important role in electromagnetic duality with the association of Chirality quantization parameter and it is important to consider the consistent quantum field theory for the simultaneous existence of electric and magnetic charges (dyons).

4 Quantum Hall Effect For Dyons

Let us construct a more general situation, where the interaction between i^{th} generalized charge $q_i\,(e_i,\,g_i)$ and the j^{th} generalized charge $q_j\,(e_j,\,g_j)$take place. This interaction (given by equation (17)), depends on electric coupling parameter $\alpha_{ij}\ (= e_i e_j + g_i g_j)$ and magnetic coupling parameter $\mu_{ij}\,(= e_i g_j - e_j g_i)$. The gauge invariant angular-momentum operator[24] associated with dyons is defined as

$$\overrightarrow{J} = \overrightarrow{r} \times (\overrightarrow{p} - \mu_{ij} V)\,; \tag{19}$$

where $\overrightarrow{r}$ is position vector, $\mu_{ij}(\mu_{ij} = e_i g_j - e_j g_i)$ is magnetic coupling parameter (or chirality parameter) and V is the transverse generalized vector potential for dyons.This angular momentum vector(also called dynamical angular momentum) in above equation (19), is not rotationally symmetric because it satisfies the following commutation relation;

$$\left[\overrightarrow{j_{\alpha}},\overrightarrow{j_{\beta}}\right]=i\varepsilon_{\alpha\beta\gamma}\left(\overrightarrow{j_{\gamma}}-\mu_{ij}\hat{r}\right); \tag{20}$$

where $\varepsilon_{\alpha\beta\gamma}$ is usual Levi-Civita symbol, Since the angular momentum vector is not rotationally symmetric it does not satisfy the angular momentum algebra, so the eigen values of $\mid J\mid^{2}$are not equal to $j\left(j+1\right)\hbar^{2}$ with j an integer.

The rotationally symmetric and gauge invariant angular momentum operator for dyon may then be written [24]as

$$\begin{aligned}\vec{\Lambda}=&\overrightarrow{r}\times(\overrightarrow{p}-\mu_{ij}V)+\mu_{ij}\frac{\vec{r}}{r}\\=&r\times(\Pi)+\mu_{ij}\frac{\vec{r}}{r};\end{aligned} \tag{21}$$

where $\Pi\left(=\overrightarrow{p}-\mu_{ij}V\right)$ is the linear momentum of this system. So the compact form of angular momentum operator in case of dyonic charge is written as

$$\vec{\Lambda}=\vec{J}+\mu_{ij}\frac{\vec{r}}{r}; \tag{22}$$

Consequently the Hamiltonian of this combined system responsible for our quantum hall effect problem may now be written as

$$\mathbb{H}=\frac{\Pi^{2}}{2m}-\frac{\alpha_{ij}}{r}+\frac{\mu_{ij}^{2}}{2mr^{2}}; \tag{23}$$

It is quite obvious that the Hamiltonian, given by equation (23) possesses the higher symmetry than the pure coulomb Hamiltonian. This higher symmetry is provided by the addition of $(\frac{\mu_{ij}^{2}}{2mr^{2}})$ term. This Hamiltonian equation (23) may now be reduced as

$$\mathbb{H}=\frac{\mid\Lambda\mid^{2}}{2Mr^{2}}; \tag{24}$$

It should be noted that the Hamiltonian (24) is defined on a two sphere, the Hamiltonian (24) is reduced to the $SO(3)$ Landau model [7]. So we may readily obtain the following commutation relations described as

$$
\begin{aligned}
[\Lambda_j \Lambda_k] =& i\varepsilon_{jkl}\Lambda_l \quad (\forall j,\, k,\, l\, = 1,\, 2,\, 3)\,; \\
[\Lambda^2 \mathbb{H}] =& [\Lambda, \mathbb{H}] = 0.
\end{aligned} \tag{25}
$$

where ε_{jkl} is usual Levi-Civita symbol. From equation (19), the scalar $\frac{\vec{r}}{r}\cdot\overrightarrow{\Lambda} = \mu_{ij} = \overrightarrow{\Lambda}\cdot\frac{\vec{r}}{r}$, commutes with all the observables, so that we may easily solve the eigen values of the operators Λ^2 and Λ_z respectively as

$$
(\mid \mu_{ij} \mid +n)\,(\mid \mu_{ij} \mid +n+1) \Longrightarrow l\,(l+1)\,; \tag{26}
$$

and

$$
-\,(\mid \mu_{ij} \mid +n)\,,\,(\mid \mu_{ij} \mid +n+1)\,,\,...............\,(\mid \mu_{ij} \mid +n) \Longrightarrow (m=-l,\;\, -l+1,........,l\,)\,; \tag{27}
$$

It is customary to say that the angular momentum commutation relations imply that l must have values $\left(l = 0,\, 1,\, \frac{1}{2},\, 1,\, 2,\, \frac{3}{2}, ...\right)$where $n\,(=0,1,2,3.....)$ is a non negative integer therefore

$$
\mid J \mid^2 = \mid \Lambda \mid^2 - \mid \mu_{ij} \mid^2; \tag{28}
$$

The above equation demands that $l\,(l+1) \geq \mid \mu_{ij} \mid^2$, so that the eigenvalues of J^2 is written as$[(\mid \mu_{ij} \mid +n)\,(\mid \mu_{ij} \mid +n+1) - \mid \mu_{ij} \mid^2]$. We may now write the energy eigenvalues obtained as

$$
E_n = \frac{((\mid \mu_{ij} \mid +n)\,(\mid \mu_{ij} \mid +n+1) - \mid \mu_{ij} \mid^2)}{2Mr^2} = \frac{(n\,(n+1) + \mu_{ij}\,(2n+1))}{2Mr^2}. \tag{29}
$$

Where n plays the role of Landau level index, for $(n = 0)$ corresponds to the lowest Landau level. Then the lowest landau level and its energy (29) may now be defined as;

$$
E = \frac{\mu_{ij}\,(\mu_{ij}+1) - \mid \mu_{ij} \mid^2}{2Mr^2}; \tag{30}
$$

Substituting the value of μ_{ij} from equation (18), in to equation (29) and solving further we get the energy eigen values as

$$
E_n = \frac{(n\,(n+1) + I\,(2n+1))}{2Mr^2}. \tag{31}
$$

It shows that the degeneracy of the landau level is written as $d_n = 2\,(\mu_{ij}+n)+1$. Hence for $n=0$ we get the lowest landau level degeneracy $(d_0 = 2\mu_{ij}+1)$. Therefore the ground state are described as $2\mu_{ij}+1$(or $2I+1$) fold degeneracy. The cyclotron frequency of the system may be defined as (ω_d)

$$\omega_d = \frac{q\psi^*}{M}; \tag{32}$$

which is the generalized cyclotron frequency. Equation (28) may now be decomposed as

$$\omega_d = \frac{(eE + gH) + i\,(eH - gE)}{M}; \tag{33}$$

It is clear that real term of equation (33) does not have any physical significance related to the cyclotron frequency, since the electric and magnetic charges are not interacting their originating field linearly. Rather the imaginary part has been obtained due to interaction of electric charge to the magnetic field and magnetic charge to the electric field that gives rise to the significance of cyclotron frequency [25]. So the modified form of cyclotron frequency (called dyonic cyclotron frequency) may be written as

$$\omega_d = \frac{(eH - gE)}{M}; \tag{34}$$

Now let us define the following forms of the electric and magnetic field strengths at the point with $\overrightarrow{r}$ of magnitude r i.e.

$$\begin{aligned}\overrightarrow{E} &= e_2 \frac{\overrightarrow{r}}{r^3}, \\ \overrightarrow{H} &= g_2 \frac{\overrightarrow{r}}{r^3};\end{aligned} \tag{35}$$

in which the electric e_2 and magnetic g_2 charges for a stationary body are located at origin. Substituting $\overrightarrow{E}$ and $\overrightarrow{H}$ from equation (35) into the equation (34), we get the following expressions respectively for cyclotron frequency(ω_d), and the magnetic length (l_d) of this system

$$\omega_d = \frac{(\mu_{ij})}{M}\frac{\vec{r}}{r^3}; \tag{36}$$

$$l_d = \sqrt{\frac{1}{\omega_d\, M}}; \tag{37}$$

Substituting ω_d from equation (36) into the equation (24) , we get the following expression for Hamiltonian

$$\mathbb{H} = \frac{\omega_d}{2\mu_{ij}} \mid \Lambda \mid^2 . \tag{38}$$

which has given same expression as discussed earlier by some author [7, 9, 14]

5 Hopf maps and Eigenstates for Lowest Landau levels

The Hopf fibration, named after Heinz Hopf [9, 26, 27]who was the studied it,is an important tool which deals both mathematics and physics.It relates topology and the theory of Lie groups. These Hopf fibration find to be suitable for discussing the magnetic monopoles The n- sphere $S^n$is the set of all points $(x_0, x_1..........x_n)$ in real space (R^{n+1}) that must satisfied this condition

$$x_0^2 + x_1^2 + ...x_n^2 = 1$$

Thus by this equation the 1-sphere $S^1$is the unit circle in the plane and 2-sphere $S^2$is the surface of three dimensional sphere. By the Hurwitz theorem there exists four normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathcal{O}$ respectively named as the algebras of real numbers, complex numbers, quaternions and octonions. Quaternion discovered by Hamilton in 1843 to generalize the basics elements $(1, i)$of a complex number field to the basis (1, i, j, k)of quaternion number field. Interestingly, the division algebras are closely related to topological maps from sphere to sphere in different dimensions; i.e. the Hopf maps. In existence there are only four (Hopf) fibrations of spheres[9] over spheres: $S^1 \overset{Z_2}{\rightarrow} S^1$ for $\mathbb{R}$, $S^3 \overset{S^1}{\rightarrow} S^2$ for $\mathbb{C}$, $S^7 \overset{S^3}{\rightarrow} S^4$ for $\mathbb{H}$ and $S^{15} \overset{S^7}{\rightarrow} S^8$ for $\mathcal{O}$.This gives us maps from the spheres S^0, S^1, $S^3$and S^7.The algebraic structure of the four dimensional Quantum Hall Effect in presence of dyon by the 2^{nd} Hopf map is represented as $S^7 \overset{S^3}{\rightarrow} S^4$. According to the Hopf map, the gauge field is obtained from the so-called Hopf spinor. The Hopf spinor plays an important role in the quantum Hall system.There all three Hopf maps are closely related bundle structures of $U(1)$, $SU(2)$, $SU(8)$ magnetic monopole. The first Hopf map is defined as a map from S^3 to S^2. The first Hopf map may be constructed as by first introducing a normalized complex two spinor.

In the quantum Hall effect problem eigenstate are known as Landau levels.The eigenstates of this system is formulated by spinor coordinate [7, 9, 14, 27]. The first Hopf map now be constructed as introducing a normalized two component complex spinor i.e.

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}; \tag{39}$$

which satisfy

$$\varphi^\dagger \varphi = \mid \varphi_1 \mid^2 + \mid \varphi_2 \mid^2 = 1; \tag{40}$$

Here φ is the 1^{st} Hopf spinor on the space S^3 and may be realized as

$$\varphi \rightarrow x_i = \varphi^\dagger \sigma_i \varphi; \tag{41}$$

Where $\sigma_i\,(i=1,2,3)$ are well known Pauli spin matrices i.e.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{42}$$

The x_i satisfy the condition for S^2

$$x_i x_i = \left(\varphi^\dagger \varphi\right)^2 = 1; \tag{43}$$

using equation (39 -42), we get

$$\begin{aligned} x_i =& \varphi^\dagger \sigma_i \varphi; \\ x_1 =& \left(\varphi_1^*, \varphi_2^*\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \left(\varphi_1^* \varphi_2 + \varphi_2^* \varphi_1 \right); \\ x_2 =& \left(\varphi_1^*, \varphi_2^*\right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = i \left(-\varphi_1^* \varphi_2 + \varphi_2^* \varphi_1 \right); \\ x_3 =& \left(\varphi_1^*, \varphi_2^*\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \left(\varphi_1^* \varphi_1 + \varphi_2^* \varphi_2 \right); \end{aligned} \tag{44}$$

which is further reduced to

$$\begin{aligned} \left(\varphi_1^* \varphi_2\right) =& \frac{x_1 + i x_2}{2}, \\ \left(\varphi_1 \varphi_2^*\right) =& \frac{x_1 - i x_2}{2}, \\ \left(\varphi_1^* \varphi_1\right) =& \frac{1 + x_3}{2}, \\ \left(\varphi_2^* \varphi_2\right) =& \frac{1 - x_3}{2}; \end{aligned} \tag{45}$$

where we have the condition $\varphi^\dagger \varphi = \mid \varphi_1 \mid^2 + \mid \varphi_2 \mid^2 = 1$ in (45) then the first Hopf spinor is written as

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \frac{1}{\sqrt{2\left(1 + x_3\right)}} \begin{pmatrix} 1 + x_3 \\ x_1 + i x_2 \end{pmatrix}; \tag{46}$$

Substituting $e_i = e$; $g_j = 0$; $e_j = 0$ and $g_j = g$ in the equation (18) we have system composed by two dyons with charges $(e, 0)$and $(0, g)$ which satisfies immediately the Dirac quantization condition condition.$eg = I$ given by equation (4).

Consequently the equation (24, 29) represent the Hamiltonian and Energy eigen values of two dimensional ($2d$) quantum hall effect on S^2, associated with dyons. Then the normalized eigenfunctions in the Lowest Landau Level are just the algebraic products of the spinor coordinates

$$< \varphi \mid I, m >= \sqrt{\frac{(2I)!}{(I+m)!\,(I-m)!}} \varphi_1^{I+m} \varphi_2^{I-m},$$
$$= \sqrt{\frac{(2I)!}{(m_1)!\,(m_2)!}} \varphi_1^{m_1} \varphi_2^{m2}. \tag{47}$$

where $m_1 = I + m,\ m_2 = I - m$. If we set the value I is half integer then $m_1 + m_2 = I$.

The simplest case has been obtained for $N = d_0$, (N particle density and $d_0 = 2\mu_{ij} + 1 \approx 2\mu_{ij} = 2I$), when the lowest level is completely filled. In this case the filling factor is describes as $\nu \equiv \frac{N}{d_0} = 1$, the result corresponds to the integer quantum Hall effect, and in the case of fractional quantum Hall effect the many body wave function is written like this $\Phi_m = \Phi^m (x_1, ...x_N)$ with odd integer m, then the filling fraction $\nu = \frac{d_0}{d_0(m)} \approx \frac{1}{m}$. this result corresponds the fractional quantum Hall effect.

References

[1] R. E. Prange, and S. M. Girvin, **"The Quantum Hall Effect"**, 2nd edition, New York: Springer-Verlag, (1990).

[2] K. Von Klitzing, G. Dorda and M. Pepper, **"New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hal**l", Phys. Rev. Lett., **45** (1980), 494.

[3] D. Yoshioka, **"The Quantum Hall Effect"**, Springer, (2002).

[4] R. B. Laughlin, **Anomalous quantum Hall effect: An incompressible quantum fluid with fractionally charged excitations**, Phys. Rev. Lett., **50** (1983), 1395. **"Quantized Hall conductivity in two dimensions"**, Phys. Rev., **B23**, (1981), 5632.

[5] D. C. Tsui, H. L. Stormer, and A. C. Gossard, **"Two-dimensional magnetotransport inthe extreme quantum limit**", Phys. Rev. Lett., **48** (1982), 1559 . D. C. Tsui, **"Interplay of disorder and interaction in two-dimensional electron gas in intense magnetic fields"**, Rev. Mod. Phys.**,** **71** (1999), 891.

[6] R. B. Laughlin and D. Pines, **Proceedings of the National Academy of Sciences of the United States of America**, **97(1)** (2000), 28.

[7] F. D. M. Haldane, **"Fractional quantization of the Hall effect: A hierarchy of incompressible quantum fluid states"**, Phys.Rev.Lett., **51** (1983), 605. F. D. M. Haldane and E. H. Rezayi, **"Finite-size studies of the incompressible state of the fractionally quantized Hall effect and its excitations**", Phys. Rev. Lett. **54** (1985), 237.

[8] G. Fano, F. Ortolani, and E. Colombo, **"Configuration-interaction calculations on the fractional quantum Hall effect"**. Phys. Rev., **B 34** (1986), 2670.

[9] K. Hasebe, **"Hopf Maps, Lowest Landau Level, and fuzzy Spheres"**, hep-th/1009.1192v3. K. Hasebe and Y. Kimura, **"Dimensional Hierarchy in Quantum Hall Effects on Fuzzy Spheres"**, Phys.Lett. **B602** (2004), 255.

[10] P. A. M. Dirac,"**The Theory of Magnetic Poles**" Phys. Rev., **74** (1948), 817.

[11] J. Schwinger, **"Magnetic Charge and Quantum Field Theory"**, Phys. Rev., **144** (1966), 1087. J. **"A Magnetic Model of Matter"**, Science, **165** (1969), 757. **"Electric and Magnetic Charge Renormalization"**, Phys. Rev., **151** (1966), 1055.

[12] D. Zwanziger, **"Dirac Magnetic Poles Forbidden in S - Matrix Theory"**, Phys. Rev., **B137** (1965), 647. **"Quantum Field Theory of Particles with Both Electric and Magnetic Charges"**, Phys. Rev., **176** (1968), 1489.

[13] P. A. M. Dirac, **"Quantised Singularities in the Electromagnetic Field"**, Proc. Roy. Soc. London, **A133** (1931), 60.

[14] S. C. Zhang and J. P. HU, **"A Four-Dimensional Generalization of the Quantum Hall Effect"**, Science, **294** (2001), 823.

[15] R. Mössner and P. Schiffer, **"Magnetism: Monopoles on the move"**, NATURE PHYSICS, **5** (2009), 250.

[16] C. Castelnovo, R. Mössner and S. Sondhi, **"Magnetic monopoles in spin ice"**, NATURE, **451** (2007),42.

[17] J. Schwinger, **"Dyons Versus Quarks"**, Science, **166** (1969), 690.

[18] E. Witten, **"Dyons of charge e\theta/2\pi "**, Phys. Lett., **B86** (1979), 283.

[19] T. T. Wu and C. N. Yang, **"Concept of nonintegrable phase factors and global formulation of gauge fields"**, Phys. Rev., **D12** (1975), 3845.

[20] R. Jackiw and C. Rebbi, **"Solitons with fermion number 1/2"**, Phys. Rev., **D13** (1976), 3398.

[21] A. S. Goldhaber, **"Connection of Spin and Statistics for Charge-Monopole Composites"**, Phy. Rev. Lett., **36** (1976), 1122.

[22] P. S. Bisht and O. P. S. Negi, **"Revisiting Quaternionic Dual Electrodynamics"**, Inter. J. Theor. Phys., **47** (2008), 3108.

[23] H. Dehnen and O. P. S. Negi, **"Electromagnetic Duality, Quaternion and Supersymmetric gauge Theory of Dyons"**, hep-th/0608164.

[24] D. S. Bhakuni , O.P.S. Negi and B.S.Rajput "**Angular Momentum Operators for Dyons**", Lett. Al Nuovo Cimento **36 N** (1983),15.

[25] Pawan Ku. Joshi, P. S. Bisht and O. P. S. Negi, **"Electron Monopole Duality In Quantum Hall Effect"**, J. Electromagnetic Analysis & Applications, **3** (2011), 22.

[26] F. Gürsey and Chia-Hsiung Tze, **"On the Role of Division, Jordan and Related Algebras in Particle Physics"**, World Scientific, Singapore, (1996).

[27] E. Demler, S. C. Zhang, "**Non-Abelian Holonomy of BCS and SDW Quasiparticles**", Annals of Phys., **271**(1999), 83.