Noname manuscript No.
(will be inserted by the editor)

From boundary thermodynamics towards a quantum gravity approach

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Received: date / Accepted: date

Abstract We examine how a thermodynamic model of the boundary of 4d-manifolds can be used for an approach to quantum gravity, to keep the number of assumptions low and the quantum degrees of freedom manageable. We start with a boundary action leading to Einstein's Equations under a restriction due to additional information from the bulk. Optionally, a modified form with torsion can be obtained. From the thermodynamic perspective, the number of possible microscopic states is evaluated for every macroscopic configuration, and this allows to compute the transition probability between quantum states. The formalism does not depend on specific microscopic properties. The smoothness and the topological space condition of the manifold structure are viewed as a preferred representation of a macroscopic space on mathematical grounds. By construction, gravity may be interpreted as thermodynamic model which is forced to be out of equilibrium depending on the restrictions imposed by matter. Instead of an ill-behaved path integral description of gravity, we obtain a non-divergent concept of sums over microstates.

1 Introduction

In a recent article [1], 't Hooft has discussed several challenges that might have to be considered carefully in order to progress towards a quantum gravity approach on solid grounds. One particular aspect is about the fundamental degrees of freedom of gravity, i.e. what quanta we expect to find within elementary "volumes" or units of "space". This aspect might be better understood in conjunction with the thermodynamic interpretation of gravity, as we infer from black holes [2, 3]. Another, perhaps related challenge is the complexity of the underlying quantum description which 't Hooft suggests should be simpler than quantum mechanics.

This article is intended as a contribution to progress towards a quantum gravity with the above-mentioned properties. It addresses several issues to this end: 1. From how few fundamental assumptions can we construct a theory compatible with Einstein's general relativity (GR) (or an extension accounting for torsion)? 2. What is the interpretation of these fundamental assumptions? 3. To what extent is this interpretation related to a model of quantum gravity? The article is structured in the same order as these issues.

In the first issue, restricting the fundamental assumptions to the minimum allows us to prevent a possible loss of physically relevant configuration while keeping the formulation as simple as possible. We need at least one condition to restrict the solutions in the classical limit to be compatible with GR. However, this relation will have a direct thermodynamic interpretation. The thermodynamic interpretation will help us to find a quantum model, and its degrees of freedom will be encoded along 2d-surfaces or "thin layers", somewhat similarly to black hole thermodynamics. We hope to obtain a quantum model of especially low complexity.

The above-mentioned relation can be cast into a (classical) action principle. We will first check that we can recover GR starting from the following action on the orientable boundary of a 4-dimensional manifold $\mathscr{M}$:

$$I = \int_{\partial\mathscr{M}} \mathrm{d}^3x \, \sqrt{\gamma} \, \tau_K^{kn} \, e_k^K, \tag{1}$$

up to an irrelevant multiplicative and additive constant (cosmological constant), where γ and and e_k^K are the determinant of the Euclidean metric γ_{ij} and the triad, respectively, on $\partial\mathscr{M}$, k is a Lorentz- and K a Minkowski-index ranging from 1 to 3,

$$\tau_K^{kn} = \mathscr{L}_n e_K^k - e_K^k e_j^J \mathscr{L}_n e_J^j \tag{2}$$

with unit vector n^μ normal to $\partial\mathscr{M}$ and $\hbar = 1$ (the derivation is analogous to [5]). Additional restrictions will be shown to be directly related to the matter content and other specific knowledge about $\mathscr{M}$. In addition to the usual gauge freedom on the metric, some freedom remains also for allowing optional torsion and, for this reason, the GR-type model strictly requires one more assumption than the model with allowed torsion.

The second issue (interpretation) is important in view of a physical justification of the assumptions, revealing their low level of complexity. Concretely, we will obtain a thermodynamic interpretation for (1). We will show that this action corresponds to a statistical model of 2d-layers. The statistical description is proposed on the 3d-boundary of a closed manifold, as in [9], in order to obtain a simple formalism. The 3d-space is also the space that one quantises in canonical quantum gravity.

The third issue (relation to a model of quantum gravity) will be addressed by using the thermodynamic interpretation of (1) and clarifying the domain of validity, once for classical gravity and once for a model of quantum gravity which results. Interestingly, the thermodynamic model will not refer to a canonical ensemble. Rather, gravity is forced, depending on the constraining function on the bulk, to be out of equilibrium. Our interpretation of the thermodynamic approach is thus somewhat different from [4–6]. Besides the constraining function, the total number of possible quantum states is not restricted by any dynamical equations or by any equilibrium condition or by any microscopic graph structures. This means, that the statistical model is fundamentally different from the canonical quantisation procedure.

We can identify two important features following from our assumptions: a. It is not restrictive to consider only smooth manifolds, the specific mathematical space-time structure is not physically relevant. b. No explicit knowledge is required on the properties of the quanta of gravity.

2 How few assumptions do we need?

a. Quanta and macroscopic space-time

We start with a 3-dimensional space parameterised by thermodynamic variables.

A quantum model of space-time must not necessarily be equipped with commutator relations of the form $[\hat{A},\hat{B}] =$ constant with operators $\hat{A},\hat{B},\ldots$ (conventional quantisation method) in order for a "physical" state space to be defined. Although commutator relations induce dynamics and the technique has been successfully applied until now, they might nevertheless restrict the theory of the most elementary building blocks unnecessarily. Namely, there is yet an intriguing alternative for defining what is "physical", using a statistical model. A statistical model can be a good option whenever we deal with systems of many objects. The objects occur in a certain number of different microscopic states (or points of phase space) within the same macroscopic state. Depending on this number, every macroscopic state has a certain probability to occur. In order to define macroscopic states, we need macroscopic variables. The space-time structure gives us a set of macroscopic variables in the form of distances ore metric. From this perspective, all observable space-time quantities are the classical limit of the collective behaviour of a large number of quanta, and space-time is undefined at the quantum level. This concept is consistent with the Hawking-Bekenstein interpretation of the Schwarzschild or Kerr-Newman black hole horizon, the area of which represents the entropy - a macroscopic variable. In gravity theories like e.g. loop quantum gravity (LQG), space-time (or its boundary) also emerges from certain quantum states in the semi-classical or classical limit, as can be shown by e.g. spin-foam computations.

Using fairly general arguments, 't Hooft has argued that the elementary degrees of freedom of quantum gravity must live in a 2-dimensional space, and one more dimension would evolve their states [7]. This hint suggests that we start with a 3-dimensional space (thinking of the space-time boundary) and partition it into 2d-layers.

b. GR-compatible space-time structure and dimension

The macroscopic variables defining space-time must be consistent with general relativity (GR) when evaluated at large scales (and in the experimentally verified curvature regime). In other words, in this large-scale limit, there must be a 4-dimensional manifold structure $\mathscr{M}$ together with a gravitational field represented by tetrads e^I_a in a covariant formulation. For negligible curvature, this large-scale approximation still holds with high accuracy at scales for which the matter fields require a quantum treatment (QFT). The simplest assumption is therefore that the manifold structure be valid (in good approximation) at any scales significantly larger than the Planck length. We also assume that $\mathscr{M}$ is globally hyperbolic.

Assumptions a. and b. have dramatic consequences on the concept of space-time. Space-time is commonly described using a 4-dimensional C^∞-manifold. However, we would not really need a C^∞-manifold if it was not for reasons of mathematical preference. For example, we could also constuct a discontinuous structure $\mathscr{O}$ by discretising a C^∞-manifold (space-time triangulation at the Planck scale). Or we could construct a non-Hausdorff space with local charts sending points p to n-tuples (x^i) with the pseudo-metric $\tilde{g}(x^i,y^j) = L_p^2\ \mathrm{trunc}(x_i y^i/L_p^2)$, where trunc means truncation to the integer with next lower absolute value. Or we could also turn the (densitized) vielbeins $\mathbf{E^i}$ into operators $\hat{\mathbf{E}}^{\mathbf{i}}$ with the semi-classical limit given by a discretization of $\mathscr{M}$.

The good news is: For any space $\mathscr{O}$ representing space-time, we can find large-scale coarse-grainings $\mathscr{O}_c$ of $\mathscr{O}$ matching a C^∞-manifold $\mathscr{M}$, i.e. there is an isomorphism I: $\mathscr{O}_c \to \mathscr{M}_c \subset \mathscr{M}$) in the large scale regime ($\delta x \gg L_p$). Due to this map I, $\mathscr{M}$ and $\mathscr{O}$ can both be used as equivalent descriptions for the same large-scale behaviour. Therefore, a 4-dimensional C^∞-manifold is a preferred, but not unique mathematical model of space-time at large scales, and the small scale details are irrelevant.

We now check that we can recover Einstein's Equations (for vanishing torsion) from Equation (1) defined on the orientable boundary $\mathscr{T} = \partial\mathscr{M}$. We can use Gauss' Theorem to convert (1) into an integral over the Euclidean 4d-manifold $\mathscr{M}$. In local coordinates at every boundary point, we introduce an (outward) normal unit vector $(n^\gamma) = (0,\ldots,0,1)$. The boundary element can be written as $\mathrm{d}\sigma_\gamma = n_\gamma \mathrm{d}^d x$. To rewrite the boundary integral expression, we extend the notation according to

$$\tau^{kn}_K\, e^K_k = \tau^{k4}_K\, e^K_k = \tau^{k\gamma}_K\, e^K_k\, n_\gamma, \tag{3}$$

where the new components τ^{kj}_K, $j = 1\ldots3$ are free gauge components. Clearly, there can be many bulk space-times with the same boundary. We use the notation x^μ for the bulk vector components and the tetrads e^Δ_δ and the metric $g_{\mu\nu}$ (determinant g). Gauss' Theorem then yields:

$$I = \oint_{\mathscr{T}} \mathrm{d}^3x\, n_\gamma \sqrt{\gamma}\, [\tau^{k\gamma}_K\, e^K_k] = \int_{\mathscr{M}} \mathrm{d}^4x\, \sqrt{g}\, [\tilde{\nabla}_\gamma(\tau^{\delta\gamma}_\Delta\, e^\Delta_\delta)], \tag{4}$$

where $\tilde{\nabla}_\gamma$ is the torsionless covariant derivative and remembering that $\partial\mathscr{M}$ is orientable. The integrand of (4) can be expanded:

$$\begin{aligned} \rho &= \tilde{\nabla}_\gamma(\tau^{\delta\gamma}_\Delta\, e^\Delta_\delta) = \delta^\delta_\mu \tilde{\nabla}_\gamma(\tau^{\mu\gamma}_\Lambda\, e^\Lambda_\delta) \\ &= g_{\mu\nu} g^{\nu\delta} \tilde{\nabla}_\gamma(\tau^{\mu\gamma}_\Lambda\, e^\Lambda_\delta) \\ &= e^\Delta_\mu \eta_{\Delta\Gamma} e^\Gamma_\nu g^{\nu\delta} \tilde{\nabla}_\gamma(\tau^{\mu\gamma}_\Lambda\, e^\Lambda_\delta) \\ &= e^\Delta_\mu e^\Gamma_\nu \Phi^{\mu\nu}_{\Delta\Gamma}, \end{aligned} \tag{5}$$

where

$$\Phi^{\mu\nu}_{\Delta\Gamma} = \eta_{\Delta\Gamma} g^{\nu\delta} \tilde{\nabla}_\gamma [e^\Lambda_\delta\, \tau^{\mu\gamma}_\Lambda]. \tag{6}$$

In order to obtain the classical equations of motion, the standard procedure is to set the variation of the action to zero. If we consider the pure gravitational action, we obtain the vacuum space-time. This is precisely what happens if we vary I. In general, we we need more information about the space-time configuration. To encode additional information, we must impose certain restrictions in the set of possible functions $\rho(x^\mu)$. One method is to introduce a constraining function $\mu(x^\mu)$ which contains a certain number of free parameters besides the tetrads. If we work with the total action

$$I_t = \int_{\mathscr{M}} \mathrm{d}^4x\, \sqrt{g}\, [e^\Delta_\mu e^\Gamma_\nu \Phi^{\mu\nu}_{\Delta\Gamma} + \mu] \tag{7}$$

instead of I, this yields a space-time different from the vacuum. Because ρ is purely given by the geometry of space-time, μ must contain the Lagrangian of matter fields. Setting the variation of I_t to zero yields the equation of motion with matter content:

$$\delta I_t = \delta \int_{\mathscr{M}} \mathrm{d}^4x\, \sqrt{g}\, [e^\Delta_\mu e^\Gamma_\nu \Phi^{\mu\nu}_{\Delta\Gamma} + \mu] = 0. \tag{8}$$

The constraining function μ can be splitted into two parts. The first part is caused by imposing to ρ inhomogeneous mode patterns as caused by the Lagrangian of matter fields $\mathscr{L}_m(a^j, f_j)$ with a set of free constants a^j and fixed functions $f_j(e^\Delta_\mu, x^\mu)$. In the presence of torsion, one can easily add the connection to the list of variables. In the limit of negligible gravitational field, (8) reduces to $\delta\mu = 0$. This equation of motion determines the physically relevant wave functions of the matter fields on flat space-time.

The second part of μ arises from observable data. Typically, one performs a few local measurements and / or imposes conditions on part of the geometry or part of the free parameters of the matter fields, depending on the specific scientific need. This is given by constraint relations

$$C_l(e^\Delta_\mu, a^j, x^\mu) = 0. \tag{9}$$

Introducing Lagrange multipliers λ^l yields the full constraining function

$$\mu = \mathscr{L}_m + \lambda^l C_l, \tag{10}$$

where we sum over double indices l, and complete the variation problem with the constraint condition (9).

To obtain the full physical interpretation of (8), we need to perform a Wick rotation at every point of $\mathscr{M}$ to obtain the metric with Lorentzian signature, which is always possible for a globally hyperbolic manifold. Our relation (5) has the form as the Palatini action and $\Phi^{\mu\nu}_{\Delta\Gamma}$ has the form of the curvature two-form, and (2) is proportional to the momentum $\gamma^{-1/2}\delta I/\delta e^K_k$ on $\partial\mathscr{M}$ according to GR, while μ is the Lagrangian of the matter $\mathscr{L}_m$. We will check below that, indeed, (8) leads us to Einstein's field equations.

We consider the option of including the possibility of torsion as $\mathscr{L}_m$ may contain matter fields with spin. To this end, we define

$$\omega = \sqrt{\gamma}\tau_K^{k\gamma} e_k^K n_\gamma \, dx^1 \wedge dx^2 \wedge dx^3, \qquad \xi^\gamma = \tau_K^{k\gamma} e_k^K \tag{11}$$

and rewrite

$$\begin{aligned}
&\int_{\partial\mathscr{M}} \sqrt{\gamma}\tau_K^{k\gamma} n_\gamma d^3x = \int_{\partial\mathscr{M}} \omega = \int_{\mathscr{M}} d\omega \\
&= \int_{\mathscr{M}} \sum_\gamma (-1)^{\gamma-1} d(\sqrt{g}\xi^\gamma) \wedge dx^1 \wedge \ldots \tilde{dx}^\gamma \ldots \wedge dx^4 \\
&= \int_{\mathscr{M}} \sum_\gamma (-1)^{\gamma-1} \partial_\gamma(\sqrt{g}\xi^\gamma)\, d^\gamma x \wedge dx^1 \wedge \ldots \tilde{dx}^\gamma \ldots \wedge dx^4 \\
&= \int_{\mathscr{M}} \sqrt{g} \sum_\gamma \frac{1}{\sqrt{g}} \partial_\gamma(\sqrt{g}\xi^\gamma)\, dx^1 \wedge \ldots \wedge dx^4 \\
&= \int_{\mathscr{M}} \sqrt{g}\tilde{\nabla}_\gamma(\tau_K^{k\gamma} e_k^K)\, dx^1 \wedge \ldots \wedge dx^4,
\end{aligned} \tag{12}$$

where $\tilde{dx}^\gamma$ means skipping the differential, $\tilde{\nabla}_\gamma \xi^\gamma$ is the torsion-less divergence, and $\mathscr{M}$ is orientable.

The case of non-orientable manifold $\mathscr{M}$ must be expected to occur but can be managed by using a double cover of $\mathscr{M}$ which is orientable and splits into two isomorphic components.

2.1 GR

Under the condition of vanishing torsion, we can easily show that (8) leads to a field equation of the form of Einstein's Equations, up to the observationally relevant second order in the dimension of the derivatives of $g_{\mu\nu}$. We only sketch the proof as it is fully analogous to the procedure of standard text books on GR. We first ignore the cosmological constant for a while and use the Lorentzian signature for the metric.

Due to the definition of the tetrads and the metric, the metricity condition $\nabla_\gamma g_{\alpha\beta} = 0$ is satisfied. With $\tau^{\alpha\beta\gamma} = \tau_\Gamma^{\alpha\gamma}\, \eta^{\Gamma\Delta}\, e_\Delta^\beta$ and $\rho^{\alpha\beta} = \nabla_\gamma \tau^{\alpha\beta\gamma}$, we have

$$\rho = \nabla_\gamma(\tau^{\alpha\beta\gamma}\, g_{\alpha\beta}) = \rho^{\alpha\beta}\, g_{\alpha\beta}. \tag{13}$$

Then define the tensor

$$\chi_{\mu\nu} = \rho_{\mu\nu} - \rho g_{\mu\nu}/2 \tag{14}$$

and

$$\theta^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mu)}{\delta g_{\mu\nu}}. \tag{15}$$

With these preliminaries, it is straight-forward to show:

$$\begin{aligned}
&\delta \int_{\mathscr{M}} \mathrm{d}^{d+1}x\, \sqrt{-g}\, [\rho + \mu] \\
&= \int_{\mathscr{M}} \mathrm{d}^{d+1}x\, \sqrt{-g}\, [\chi_{\mu\nu} + \theta_{\mu\nu}]\, \delta g^{\mu\nu} \\
&+ \int_{\mathscr{M}} \mathrm{d}^{d+1}x\, \sqrt{-g}\, g^{\mu\nu}\, \delta\rho_{\mu\nu} = 0,
\end{aligned} \tag{16}$$

where we have dropped one contribution to the variation which contains a divergence form. This directly yields the field equations

$$\chi^{\mu\nu} = -\theta^{\mu\nu}. \tag{17}$$

Following a standard procedure, it is easy to show that $\theta^{\mu\nu}$ and therefore $\chi^{\mu\nu}$ is divergence-free if we express the variation $\delta g^{\mu\nu}$ induced by a transformation $x^\mu \to x^\mu + \varepsilon a^\mu(x^\nu)$ with infinitesimal ε, while fixing $\theta_\nu^\lambda a^\nu = 0$ on the boundary.

If we consider contributions no higher than quadratic in the dimension of derivatives of $g_{\mu\nu}$, it immediately follows from [8] (after restoring the cosmological constant) that the only possible form of $\chi_{\mu\nu}$ is

$$\chi_{\mu\nu} = AG_{\mu\nu} + Bg_{\mu\nu}, \tag{18}$$

where A and B are constants, $G_{\mu\nu}$ is the Einstein tensor and the second term is the cosmological constant term. With (18), we have checked that (8) with vanishing torsion leads to Einstein's Equations to quadratic order in the dimension of derivatives of $g_{\mu\nu}$ when choosing the constants A and B according to GR. By fixing A, we also fix the proportionality constant relating $\theta^{\mu\nu}$ to the stress tensor $T^{\mu\nu}$ while setting $\chi_{\mu\nu} = R_{\mu\nu}$.

3 Thermodynamic interpretation of the action I

According to assumption a, space-time and therefore the gravitation field is the macroscopic manifestation of many "atoms of gravity" living in the 3d-boundary $\partial\mathscr{M}$. But how many such atoms can we put into a given element of space? To find an answer, we propose a method of layer-statistics in 2+1 dimensions which is compatible with the structure of the action I in (1).

Consider a 3-volume ΔV small enough to resolve the variations of the gravitational field and yet much larger than the Planck scale. To define a standard length scale within ΔV, we choose local Minkowski coordinates for which $e_i^K = \delta_i^K$, i.e. we can use coordinates x^K instead of x^i. We now partition ΔV across the Kth coordinate into N_K 2d-layers in such a way that, microscopically, each layer can have p_K different microscopic states $|q_K\rangle$. In a thermodynamic context, all microscopic states shall occur with the same probability

$p(|q_K\rangle)$. Disregarding later restrictions by the constraining function μ, the number of states is then given by

$$\Omega_K(\Delta V) = p_K^{N_K} = \exp s_K, \tag{19}$$

where $s_K = N_K \ln p_K$ is the entropy per volume ΔV, i.e. the 3-density of the entropy S of $\partial\mathscr{M}$ at the location of ΔV. The typical thickness of each layer can be chosen near the limit of the smallest possible length scale, of the order of the Planck length L_P. N_K can be estimated using the relation $N_K \approx k_K \Delta x_K$, where Δx_K is the size of ΔV along the Kth coordinate and the "narrowness parameter" k_K is a proportionality constant which we can hold fixed. In the same way, we partition ΔV along the other two coordinates. The 3 partitions (across 3 different coordinates) are performed independently from each other. This means that the total number of states is the product $\Omega = \Omega_1\Omega_2\Omega_3$:

$$\Omega(\Delta V) = \prod_{K=1}^{3} \exp s_K = \exp \sum_{K=1}^{3} s_K. \tag{20}$$

This formulation of the total resulting entropy density $s(\Delta V)$ as a sum over s_K with fixed coordinates is rather unpleasant in view of a relativistic treatment of gravity. We thus rewrite the sum in a form which makes the entropy S of the 3d-space invariant under coordinate transformations $x^K \to x'^j$ with $\partial x^K/\partial x'^j = e^K_j$:

$$\begin{aligned} s(\Delta V) &= \sum_{K=1}^{3} s_K = \sum_{K=1}^{3} k_K \Delta x^K \ln p_K = \sum_{K=1}^{3} k_K \Delta x'^j e^K_j \ln p_K \\ &= \sqrt{\gamma}\kappa^j_K e^K_j, \end{aligned} \tag{21}$$

with $\sqrt{\gamma}\kappa^j_K = k_K \Delta x'^j \ln p_K$. The factor $\sqrt{\gamma}$ has been inserted in order to separate $\kappa^j_K e^K_j$ from the invariant 3-volume element $\sqrt{\gamma}\Delta V$. Indeed, the integral of (21) over $\partial\mathscr{M}$ coincides with (1) up to a constant factor if we impose

$$\kappa^j_K \sim \tau^j_K \tag{22}$$

i.e. if

$$\sqrt{\gamma}(\mathscr{L}_n e^k_K - e^k_K e^J_j \mathscr{L}_n e^j_J) \sim \Delta x'^k \ln p_K. \tag{23}$$

If we also fix the size $\Delta x'^k$, this condition determines the number of states p_k per layer.

We can test this condition for the special case of a non-extremal Schwarzschild black hole, for which the (only) "boundary" can be taken to be the 3d-surface just outside but very close to the event horizon. The normal vector is radial. The expression $(\mathscr{L}_r e^t_T)e^T_t$ is much larger than the corresponding expression for the other components, so that we only need to partition across the time-coordinate (T). We fix $\Delta x'^t = \sqrt{2ML_p}$, where $2M$ is the Schwarzschild radius. When the radial distance of our 3d-surface from the horizon approaches $\approx L_p$ (the smallest distance), we have

$$\Delta x^T = \Delta x'^t e^T_t \approx L_p \tag{24}$$

with $e^T_t = \sqrt{1-2M/r}$. Therefore, we have at most about one layer across the time. Then, p_t equals the number of states Ω_t in the layer of the horizon and $S \sim \ln p_t$ is proportional to the horizon area ($\sim M^2$), which is the expected black hole entropy up to a constant factor.

To summarise, under the restriction (23), the Euclidean 3d-action I is proportional to the total entropy S of the gravitational degrees of freedom on $\partial\mathscr{M}$. The action I can be interpreted as an entropy of layers of external curvature states, up to a constant factor. At this point, we skip and postpone the contribution from the constraining function μ. In order to obtain the macroscopic state with the highest probability, we must maximise S, i.e. we set the variation of S to zero. It is thus not surprising that we apply Hamilton's variation principle to the action to obtain the classical dynamics, as the action is related to an entropy. This is true because we have been using the method of layer-statistics where ΔV is partitioned only across one coordinate at the time. Had we partitioned each layer into bars and each bar into pieces as well, the expression for the entropy would have been proportional to the product $\prod_K \Delta x^K$ and this would have been incompatible with (1). Notice that some kind of correspondence between the gravitational action and the entropy has been proposed earlier [9, 10]. For quantum mechanics, the correspondence has been conjectured earlier [11].

We can now introduce the constraining function μ as in section 2, by equating:

$$S_t = S + \int_{\mathscr{M}} \mathrm{d}^4x \, \sqrt{g}\, \mu \tag{25}$$

This lowers the entropy of macrostates (despite the plus sign) by reducing the number of their microstates. In general, the point of maximum entropy (or the saddle point of the action) is shifted. Besides the (not quite specific) partitioning procedure, there is absolutely no microscopic properties required in order to describe the microscopic degrees of freedom and obtain a complete theory with a GR-compatible macroscopic limit.

From (21) and (22), we immediatly recognize two universal constants of the quantum theory as one should expect. Firstly, if the layers of s_K are so small that their number of possible states p_K is of the order of e, we find that $\Delta x^K/N^K = k_k^{-1}$ must be of the order of the smallest length, i.e. of the order of the Planck length L_P. This gives an estimate of the smallest length (the first constant). Secondly, by restoring the Planck constant $\hbar$ in (1),

$$I = \oint_{\partial\mathscr{M}} \mathrm{d}^3x \, \sqrt{|\gamma|}\, \tau^k_K \, e^K_k/\hbar, \tag{26}$$

we immediately recognize that each unit of action $\hbar$ is just one quantum degree of freedom in the thermodynamic interpretation of layer statistics. This confirms that our microstates can be interpreted as quantum states and the partitions of 3d-volumes reveal the quantum degrees of freedom in every layer.

4 To what extent is this interpretation related to a model of quantum gravity?

Using the thermodynamic interpretation of Section 3, we can rewrite (1) as

$$I = \int_{\partial\mathcal{M}} \mathrm{d}^3x\, \sqrt{\gamma}\, \tilde{\tau}^k_K\, e^K_k \tag{27}$$

with

$$\tilde{\tau}^k_K \sim N^K e^j_K \ln p_K / \sqrt{\gamma}, \tag{28}$$

where we do not sum over double indices K. The notation $\tilde{\tau}^k_K$ is used to emphasize that its values are discrete (in contrast to τ^k_K). The classical limit is $N_K, p_k \to \infty$ and the classical dynamics are obtained by varying the total action including the constraining function μ, i.e. by maximising the entropy after having removed the states which are incompatible with μ,

$$\delta I_t = \delta(I + \int d^4x \sqrt{|g|}\mu) = 0. \tag{29}$$

In the classical limit, (29) determines a sharp maximum of the entropy.

Unlike the classical case, the quantum treatment deals with small numbers of quanta or states (i.e. not much greater than 1). For such small numbers, the entropy does not have a sharp maximum at one point of the parameter space. Neighbouring points have similar magnitudes of S. Each point is given by a set of (discretised) parameter functions

$$\mathcal{G} = (e^K_k(x^\mu), N^K(x^\mu), p^K(x^\mu)) \tag{30}$$

To include torsion, we would add the connection to the list (30). At a point $\mathcal{G}$ for which a parameter function π differs at some location x^μ by a small difference D_π from the maximum point $\mathcal{G}_{\max}$, we have

$$I_t(\mathcal{G}) - I_t(\mathcal{G}_{\max}) \sim -D^2_\pi + \mathcal{O}(D^3_\pi). \tag{31}$$

We can thus approximate the distribution $\Omega(\mathcal{G})$ around the maximum by a Gaussian distribution, and the standard deviation of π determines the uncertainty of π. Because of (23), e^K_k itself and thus space-time distances are affected by an uncertainty. In this way, the statistical interpretation of gravity leads to a quantum uncertainty of the geometry.

Similarly to the n-point function computations of quantum field theory, we can evaluate the transition probability between n local quantum geometries $\mathcal{G}_m$ at (mean) locations P_m, $m = 1 \ldots n$. For example, let every number p^m_k of states per layer of $\mathcal{G}_m$ at P_m be restricted by the quantum channel

$$p_k^{m \pmod{n} + 1} = p^m_k + \Delta p_k^{m \pmod{n} + 1} \tag{32}$$

with Δp^m_k being integers not far from 0 (given for this channel) and $\sum_m \Delta p^m_k = 0$. Then, considering all possible configurations $\mathcal{G}_k$, we can compute the transition probability as

$$p(P_m, \Delta p^m_k) = \frac{\sum_k \Omega(\mathcal{G}_k)\big|_{p^m_k}}{\sum_k \Omega(\mathcal{G}_k)}, \tag{33}$$

where the terms in the numerator are restricted by (32), and

$$\Omega(\mathcal{G}) = exp I_t\big|_{\mathcal{G}} \tag{34}$$

in terms of Euclidian coordinates. Remember that, in the thermodynamic interpretation, all integrals over space-time are replaced by sums over volumes ΔV:

$$I_t = \sum_{\Delta V_l \subset \partial\mathcal{M}} \Delta V_l\, \sqrt{\gamma}\, \tilde{\tau}^k_K\, e^K_k + \sum_{\Delta V_l \subset \mathcal{M}} \Delta V_l \sqrt{|g|}\mu. \tag{35}$$

Because the size of ΔV cannot be arbitrarily small (Planck scale limit) and $\mathcal{M}$ is assumed to have a (closed) boundary, the sum over volumes is finite. Moreover, the parameter functions e^K_k, N^K, p^K are discrete-valued and bouned from above (black hole limit). Therefore, the sum over geometries ($\sum_k$) is always finite, too. Therefore, (33) never diverges for non-singular manifolds.

With (33), we can, in principle, make predictions on outcome rates of arbitrary quantum processes. However, the μ-term is non-trivial. Imagine i.e. that, near the location P_n, we have a macroscopic cloud of matter which remains stable in its future for some value of p^n_k but collapses to a much more compact body if we change p^n_k by 1 unit. This causes μ to depend dramatically on the selected channels of quantum processes, and this in turn has a dramatic impact on the probability distribution of (33).

Moreover, μ is partly determined by e.g. data from measurements and these may be strongly affected by rapid spatial changes of the macroscopic mass density, as for example when considering the transition across a sharp spatial boundary of a condensed matter phase, if this transition occurs within a few Planck lengths, as could happen during the collapse of a star. The entropy density will thus also change rapidly. Therefore, the volumes ΔV cannot be in equilibrium, and the denominator of (33) cannot be interpreted as a "partition function". Notice that GR does not, in general, provide us with a conserved quantity like "energy". Such a quantity would be available if space-time was forced to have a time-translation symmetry. In order for the notion of a canonical ensemble (or partition function) to make sense, we would expect to have a conserved quantity like ”energy” E at our disposal,

$$\sum_{\Delta V_l} E_l = E_t = \text{constant}, \tag{36}$$

so that a given volume could exchange E with a thermal bath (the other volumes) and the constraint (36) would lead to the usual ”partition function” for the canonical ensemble and to the notion of temperature, via the Lagrange multiplier method.

5 Conclusions

By using a thermodynamic interpretation of gravity on the boundary of a 4d-manifold, a connection between the classical and a quantum formulation of gravity has been identified. Due to the thermodynamic treatment and the concept of macroscopic space-time, the number of assumptions is kept to a minimum while the number of quantum degrees of freedom per volume is technically under control. In this frame-work, neither the specific analytic or topological properties of space-time nor the specific properties of the microscopic degrees of freedom are relevant to determine the quantum behaviour of gravity. This is in contrast to theories with canonical quantisation. In the quantum regime, the geometric parameters acquire an uncertainty, and the transition probabilities of quantum states can be computed using a well-behaved, non-divergent formula. The normalisation factor of the transition probability is a finite sum over all possible states which, in the classical limit, looks somehow similar to a path integral. However, the meaning of this sum is very different from a partition function which is used for canonical ensembles. Because of the matter term and because of the lack of symmetries, gravity is a theory of non-equilibrium.

Acknowledgements I would like to thank Gino Isidori for fruitful discussions and Philippe Jetzer for hospitality at University of Zurich.

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