General Solution For Navier-Stokes Equations With Any Smooth Initial Data

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Abstract – We present a solution for the Navier-Stokes equations for incompressible case with any smooth (C^∞) initial velocity given a pressure and external force in $n = 3$ spatial dimensions, based on expansion in Taylor's series of time. Without major difficulties, it can be adapted to any spatial dimension, $n \geq 1$.

Keywords – Lagrange, Mécanique Analitique, exact differential, Euler's equations, Navier-Stokes equations, Taylor's series, Cauchy, Mémoire sur la Théorie des Ondes, Lagrange's theorem, Bernoulli's law.

Let p, q, r be the three components of velocity of an element of fluid in the 3-D orthogonal Euclidean system of spatial coordinates (x, y, z) and t the time in this system.

Lagrange in his *Mécanique Analitique*, firstly published in 1788, proved that if the quantity $(p\,dx + q\,dy + r\,dz)$ is an exact differential when $t = 0$ it will also be an exact differential when t has any other value. If the quantity $(p\,dx + q\,dy + r\,dz)$ is an exact differential at an arbitrary instant, it should be such for all other instants. Consequently, if there is one instant during the motion for which it is not an exact differential, it cannot be exact for the entire period of motion. If it were exact at another arbitrary instant, it should also be exact at the first instant.[1]

To prove it Lagrange used

$$(1) \qquad \begin{cases} p = p^I + p^{II}t + p^{III}t^2 + p^{IV}t^3 + \cdots \\ q = q^I + q^{II}t + q^{III}t^2 + q^{IV}t^3 + \cdots \\ r = r^I + r^{II}t + r^{III}t^2 + r^{IV}t^3 + \cdots \end{cases}$$

in which the quantities p^I, p^{II}, p^{III}, etc., q^I, q^{II}, q^{III}, etc., r^I, r^{II}, r^{III}, etc., are functions of x, y, z but without t.

Here we will finally solve the equations of Euler and Navier-Stokes using this representation of the velocity components in infinite series, as pointed by Lagrange. We assume satisfied the condition of incompressibility, for brevity. Without it the resulting equations are more complicated, as we know, but the method of solution is essentially the same in both cases.

To facilitate and abbreviate our writing, we represent the fluid velocity by its three components in indicial notation, i.e., $u = (u_1, u_2, u_3)$, as well as the external force will be $f = (f_1, f_2, f_3)$ and the spatial coordinates $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$. The pressure, a scalar function, will be represented as p.

The representation (1) is as the expansion of the velocity in a Taylor´s series in relation to time around $t = 0$, considering x, y, z as constant, i.e., for $1 \leq i \leq 3$,

(2) $$u_i = u_i|_{t=0} + \frac{\partial u_i}{\partial t}\Big|_{t=0}\, t + \frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0}\, \frac{t^2}{2} + \frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0}\, \frac{t^3}{6} + \cdots + \frac{\partial^k u_i}{\partial t^k}\Big|_{t=0}\, \frac{t^k}{k!} + \cdots$$

or

(3) $$u_i = u_i^0 + \sum_{k=1}^{\infty} \frac{\partial^k u_i}{\partial t^k}\Big|_{t=0}\, \frac{t^k}{k!}.$$

For the calculation of $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \frac{\partial^3 u_i}{\partial t^3}, \ldots$ we use the values that are obtained directly from the Navier-Stokes equations and its derivatives in relation to time, i.e.,

(4) $$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} - \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} + \nu\, \nabla^2 u_i + f_i,$$

and therefore

(5) $$\frac{\partial^2 u_i}{\partial t^2} = -\frac{\partial^2 p}{\partial t\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial u_j}{\partial t} \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} \right) + \nu\, \nabla^2 \frac{\partial u_i}{\partial t} + \frac{\partial f_i}{\partial t},$$

(6) $$\frac{\partial^3 u_i}{\partial t^3} = -\frac{\partial^3 p}{\partial t^2\, \partial x_i} - \sum_{j=1}^{3} \left(\frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_i}{\partial x_j} + 2 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2} \right) + \nu\, \nabla^2 \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial^2 f_i}{\partial t^2},$$

(7) $$\frac{\partial^4 u_i}{\partial t^4} = -\frac{\partial^4 p}{\partial t^3\, \partial x_i} - \sum_{j=1}^{3} N_j^3 + \nu\, \nabla^2 \frac{\partial^3 u_i}{\partial t^3} + \frac{\partial^3 f_i}{\partial t^3},$$

$$N_j^3 = \frac{\partial}{\partial t} N_j^2, \quad N_j^2 = \frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_i}{\partial x_j} + 2 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2},$$

$$N_j^3 = \frac{\partial^3 u_j}{\partial t^3} \frac{\partial u_i}{\partial x_j} + 3 \frac{\partial^2 u_j}{\partial t^2} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + 3 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2} + u_j \frac{\partial}{\partial x_j} \frac{\partial^3 u_i}{\partial t^3},$$

(8) $$\frac{\partial^5 u_i}{\partial t^5} = -\frac{\partial^5 p}{\partial t^4\, \partial x_i} - \sum_{j=1}^{3} N_j^4 + \nu\, \nabla^2 \frac{\partial^4 u_i}{\partial t^4} + \frac{\partial^4 f_i}{\partial t^4},$$

$$N_j^4 = \frac{\partial}{\partial t} N_j^3 = \frac{\partial^4 u_j}{\partial t^4} \frac{\partial u_i}{\partial x_j} + 4 \frac{\partial^3 u_j}{\partial t^3} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} + 6 \frac{\partial^2 u_j}{\partial t^2} \frac{\partial}{\partial x_j} \frac{\partial^2 u_i}{\partial t^2} + 4 \frac{\partial u_j}{\partial t} \frac{\partial}{\partial x_j} \frac{\partial^3 u_i}{\partial t^3} + u_j \frac{\partial}{\partial x_j} \frac{\partial^4 u_i}{\partial t^4},$$

and using induction we come to

(9) $$\frac{\partial^k u_i}{\partial t^k} = -\frac{\partial^k p}{\partial t^{k-1}\,\partial x_i} - \sum_{j=1}^{3} N_j^{k-1} + \nu\,\nabla^2 \frac{\partial^{k-1} u_i}{\partial t^{k-1}} + \frac{\partial^{k-1} f_i}{\partial t^{k-1}},$$
$$N_j^{k-1} = \frac{\partial}{\partial t} N_j^{k-2} = \sum_{l=0}^{k-1} \binom{k-1}{l} \partial_t^{k-1-l} u_j \,\frac{\partial}{\partial x_j} \partial_t^l u_i,$$
$$\partial_t^0 u_n = u_n,\ \partial_t^m u_n = \frac{\partial^m u_n}{\partial t^m},\ \binom{k-1}{l} = \frac{(k-1)!}{(k-1-l)!\,l!}.$$

In (2) and (3) it is necessary to know the values of the derivatives $\frac{\partial u_i}{\partial t}, \frac{\partial^2 u_i}{\partial t^2}, \dots, \frac{\partial^k u_i}{\partial t^k}$ in $t = 0$ then we must to calculate, from (4) to (9),

(10) $$\frac{\partial u_i}{\partial t}\Big|_{t=0} = -\frac{\partial p^0}{\partial x_i} - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j} + \nu\,\nabla^2 u_i^0 + f_i^0,$$

the superior index 0 meaning the value of the respective function at $t = 0$, and

(11) $$\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} = -\frac{\partial^2 p}{\partial t\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^{3} N_j^1\Big|_{t=0} +$$
$$+\nu\,\nabla^2 \frac{\partial u_i}{\partial t}\Big|_{t=0} + \frac{\partial f_i}{\partial t}\Big|_{t=0},$$
$$N_j^1\Big|_{t=0} = \sum_{j=1}^{3}\left(\frac{\partial u_j}{\partial t}\Big|_{t=0} \frac{\partial u_i^0}{\partial x_j} + u_j^0 \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t}\Big|_{t=0}\right),$$

(12) $$\frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0} = -\frac{\partial^3 p}{\partial t^2\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^{3} N_j^2\Big|_{t=0} +$$
$$+\nu\,\nabla^2 \frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} + \frac{\partial^2 f_i}{\partial t^2}\Big|_{t=0},$$
$$N_j^2\Big|_{t=0} = \frac{\partial^2 u_j}{\partial t^2}\Big|_{t=0} \frac{\partial u_i^0}{\partial x_j} + 2\frac{\partial u_j}{\partial t}\Big|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0} +$$
$$+ u_j^0 \frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0},$$

(13) $$\frac{\partial^4 u_i}{\partial t^4}\Big|_{t=0} = -\frac{\partial^4 p}{\partial t^3\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^{3} N_j^3\Big|_{t=0} +$$
$$+\nu\,\nabla^2 \frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0} + \frac{\partial^3 f_i}{\partial t^3}\Big|_{t=0},$$
$$N_j^3\Big|_{t=0} = \frac{\partial^3 u_j}{\partial t^3}\Big|_{t=0} \frac{\partial u_i^0}{\partial x_j} + 3\frac{\partial^2 u_j}{\partial t^2}\Big|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t}\Big|_{t=0} +$$
$$+ 3\frac{\partial u_j}{\partial t}\Big|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}\Big|_{t=0} + u_j^0 \frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3}\Big|_{t=0},$$

(14) $$\frac{\partial^5 u_i}{\partial t^5}\Big|_{t=0} = -\frac{\partial^5 p}{\partial t^4\,\partial x_i}\Big|_{t=0} - \sum_{j=1}^{3} N_j^4\Big|_{t=0} +$$
$$+\nu\,\nabla^2 \frac{\partial^4 u_i}{\partial t^4}\Big|_{t=0} + \frac{\partial^4 f_i}{\partial t^4}\Big|_{t=0},$$

$$N_j^4|_{t=0} = \frac{\partial^4 u_j}{\partial t^4}|_{t=0} \frac{\partial u_i^0}{\partial x_j} + 4\frac{\partial^3 u_j}{\partial t^3}|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial t} +$$
$$+ 6\frac{\partial^2 u_j}{\partial t^2}|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial^2 u_i}{\partial t^2}|_{t=0} + 4\frac{\partial u_j}{\partial t}|_{t=0} \frac{\partial}{\partial x_j}\frac{\partial^3 u_i}{\partial t^3}|_{t=0} +$$
$$+ u_j^0 \frac{\partial}{\partial x_j}\frac{\partial^4 u_i}{\partial t^4}|_{t=0},$$

and of generic form,

(15) $$\frac{\partial^k u_i}{\partial t^k}|_{t=0} = -\frac{\partial^k p}{\partial t^{k-1}\partial x_i}|_{t=0} - \sum_{j=1}^3 N_j^{k-1}|_{t=0} +$$
$$+ \nu\,\nabla^2 \frac{\partial^{k-1} u_i}{\partial t^{k-1}}|_{t=0} + \frac{\partial^{k-1} f_i}{\partial t^{k-1}}|_{t=0},$$
$$N_j^{k-1}|_{t=0} = \sum_{l=0}^{k-1} \binom{k-1}{l} \partial_t^{k-1-l} u_j|_{t=0} \frac{\partial}{\partial x_j} \partial_t^l u_i|_{t=0},$$
$$\partial_t^0 u_n|_{t=0} = u_n^0,\ \partial_t^m u_n|_{t=0} = \frac{\partial^m u_n}{\partial t^m}|_{t=0}.$$

If the external force is conservative there is a scalar potential U such as $f = \nabla U$ and the pressure can be calculated from this potential U, i.e.,

(16) $$\frac{\partial p}{\partial x_i} = f_i = \frac{\partial U}{\partial x_i},$$

and then

(17) $$p = U + \theta(t),$$

$\theta(t)$ a generic function of time of class C^∞, so it is not necessary the use of the pressure p and external force f, and respective derivatives, in (4) to (15) if the external force is conservative. In this case, the velocity can be independent of the both pressure and external force, otherwise it will be necessary to use both the pressure and external force derivatives to calculate the velocity in powers of time.

The result that we obtain here in this development in Taylor's series seems to me a great advance in the search of the solutions of the Euler's and Navier-Stokes equations. It is possible now to know on the possibility of non-uniqueness solutions as well as breakdown solution respect to unbounded energy of another manner.

We now can choose previously an infinity of different pressures such that the calculation of $\frac{\partial u}{\partial t}$ and derivatives can be done, for a given initial velocity and external force, although such calculation can be very hard.

It is convenient say that Cauchy[2] in his memorable and admirable *Mémoire sur la Théorie des Ondes*, winner of the Mathematical Analysis award, year 1815, firstly does a study on the equations to be obeyed by three-dimensional molecules in a homogeneous fluid in the initial instant $t = 0$, coming to the conclusion which the initial velocity must be irrotational, i.e., a potential flow. Of this manner, after, he comes to conclusion that the velocity is always irrotational, potential flow, if the external force is conservative, which is essentially the Lagrange's theorem described in the begin of this article, but it is shown without the use of series expansion (a possible exception occurs if one or two components of velocity are identically zero, when the reasonings on 3-D molecular volume are not valid). The solution obtained by Cauchy for Euler's equations is the Bernoulli's law, as almost always happens. Perhaps a solution in Eulerian description not always corresponds to some solution in Lagrangian description, and vice-versa, I yet don't know for sure. There can be no contradiction in science, particularly in mathematics.

I began my study of the Navier-Stokes equations verifying the lack (inexistence, I called *breakdown*) of solutions, but realizing that given the pressure and initial velocity there would be no problem about not being possible to integrate the equations of Navier-Stokes and find the velocity, in general case. Now with more clarity and conviction I realize that, given only the velocity may not be possible to find the corresponding pressure, but given the pressure we can find the velocity, in special using the expansion in Taylor's series, as we see here.

If the mentioned series is divergent may be an indicative of that the correspondent velocity and its square diverge, again going to the case of breakdown solution due to unbounded energy. Without pressure and with initial velocity and external force both belonging to Schwartz Space is expected that the solution for velocity also belonging to Schwartz Space, obtaining physically reasonable and well-behaved solutions throughout the space.

The method presented here can also be applied in other equations, of course, for example in the heat equation. Always will be necessary that the remainder in the Taylor's series goes to zero when the order k of the derivative tends to infinity.[3] Applying this concept in (3) and (9), substituting t by τ, the remainder $R_{i,k}$ of order k for velocity component i is

(18) $$R_{i,k} = \frac{1}{k!}\int_0^t (t-\tau)^k \frac{\partial^{k+1} u_i}{\partial t^{k+1}} d\tau,$$

which can be estimated by Lagrange's remainder,

(19) $$R_{i,k} = \frac{t^{k+1}}{(k+1)!}\frac{\partial^{k+1} u_i}{\partial t^{k+1}}(\xi),$$

or by Cauchy's remainder,

(20) $$R_{i,k} = \frac{t^{k+1}}{k!}(1-\theta)^k \frac{\partial^{k+1} u_i}{\partial t^{k+1}}(\xi),$$

with $0 \leq \xi \leq t$ and $0 \leq \theta \leq 1$.

To Jean-Christophe Yoccoz, *in memorian*. I have just know of his premature death, great friend of mathematicians of IMPA. I'm not one successful, I do not have fame, I did not win any awards. In common we have only a great love for mathematics. He was a genius man who now leaves the Earth, but I know that even in heaven there are math and science to be done. He did an excellent job.

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References

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