Lagrangian Analysis of a Class of Quadratic Liénard-Type Oscillator Equations with Exponential-Type Restoring Force function

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Abstract

This research work proposes a Lagrangian and Hamiltonian analysis for the unique class of position-dependent mass oscillator characterized by a harmonic periodic solution and parabolic potential energy and its inverted version admitting a position-dependent mass dynamics.

1. Analysis of the class of quadratic Liénard-type harmonic nonlinear oscillator equations

This section is devoted to the analysis of a class of quadratic Liénard-type nonlinear dissipative oscillator equations that admits exact analytical harmonic periodic solutions. Consider the equation [1, 2]

\[ \ddot{x} - \gamma \varphi(x)x^2 + \omega^2 x e^{2\varphi(x)} = 0 \]

that represents the class of equations under analysis. \( \gamma \) and \( \omega \) are arbitrary parameters, and \( \varphi(x) \) is an arbitrary function of \( x \). The dot over a symbol means differentiation with respect to time, and prime holds for differentiation with respect to \( x \). By restriction of \( \varphi(x) = \ln f(x) \) and \( \gamma = -\frac{1}{2} \), the equation (1), yields

\[ \ddot{x} + \frac{1}{2} \frac{f'(x)}{f(x)} \dot{x}^2 + \frac{\omega^2 x}{f(x)} = 0 \]  

(2)

where \( f(x) \neq 0 \), is an arbitrary function of \( x \). The equation (1) is of the general form

\[ \ddot{x} + F(x)\dot{x}^2 + G(x) = 0 \]

(3)

for which the Lagrangian is given by [3,4]

\[ L(\dot{x}, x) = \frac{1}{2} \dot{x}^2 M(x) - V(x) \]

(4)

where

\[ M(x) = e^{\int f(x) \, dx} \]

(5)

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and

\[ V(x) = \int M(x)G(x)dx \]  \hspace{1cm} (6)

designate the position dependent mass and the potential function respectively.

The Lagrangian of the equation (1) becomes

\[ L(\dot{x}, x) = \frac{1}{2} x^2 e^{-2\gamma \phi(x)} - \frac{1}{2} \omega^2 x^2 \]  \hspace{1cm} (7)

Applying the Euler-Lagrange equation formula in [4]

\[ \ddot{x} + \frac{1}{2} \frac{M'(x)}{M(x)} x^2 + \frac{1}{M(x)} \frac{\partial V(x)}{\partial x} = 0 \]  \hspace{1cm} (8)

to the equation (7), gives the equation (1). By restricting \( V(x) \) to the harmonic potential, that is \( V(x) = \frac{1}{2} m_0 \omega^2 x^2 \), with unit mass, \( m_0 = 1 \), the equation (8) becomes identical to the equation (2), with the position-dependent mass function \( M(x) = f(x) \). In this regard, the equation (1) represents the unique class of position-dependent mass oscillators exhibiting not only exact harmonic periodic solution but also a harmonic potential function.

Now, using [3]

\[ H(p, x) = \frac{p^2}{2M(x)} + V(x) \]  \hspace{1cm} (9)

one may deduce from (5) and (6) the Hamiltonian

\[ H(p, x) = \frac{p^2}{2} e^{-2\gamma \phi(x)} + \frac{1}{2} \omega^2 x^2 \]  \hspace{1cm} (10)

Let us now consider, as illustration, some specific examples of (1). Let \( \phi(x) = x \). Then (1) becomes

\[ \ddot{x} - \gamma^2 x^2 + \omega^2 xe^{2\gamma x} = 0 \]  \hspace{1cm} (11)

The equation (10) admits the position dependent mass and the potential

\[ M(x) = e^{-2\gamma x} \], and \( V(x) = \frac{1}{2} \omega^2 x^2 \]  \hspace{1cm} (12)

respectively, which provides the Lagrangian function.
\[ L(\dot{x}, x) = \frac{1}{2} \dot{x}^2 e^{-2\gamma x} - \frac{1}{2} \omega^2 x^2 \]  

(13)

The application of the Euler-Lagrange equation (8) to (13) gives, as expected, (11). In this regard the Hamiltonian associated to (11) takes the form

\[ H(p, x) = \frac{p^2}{2} e^{2\gamma x} - \omega^2 x^2 \]  

(14)

So, the Hamilton equations

\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial p} \\
\dot{p} &= -\frac{\partial H}{\partial x}
\end{align*}
\]  

(15)

yield for (14)

\[
\begin{align*}
\dot{x} &= p e^{2\gamma x} \\
\dot{p} &= -\gamma p^2 e^{2\gamma x} - \omega^2 x
\end{align*}
\]  

(16)

The explicit expression for the conjugate momentum \( p \), as a function of \( x \) and \( \dot{x} \) takes then the form

\[ \dot{p} = -e^{2\gamma x} \left( x^2 + \omega^2 x e^{2\gamma x} \right) \]  

(17)

Putting now \( \varphi(x) = \frac{1}{2} x^2 \), into (1), one may obtain as equation

\[ \ddot{x} - \gamma xx^2 + \omega^2 x e^{\gamma x} = 0 \]  

(18)

The position dependent mass and the potential of (18) take then the form

\[ M(x) = e^{-\gamma x^2} \quad \text{and} \quad V(x) = \frac{1}{2} \omega^2 x^2 \]  

(19)

respectively.

The associated Lagrangian becomes

\[ L(\dot{x}, x) = \frac{1}{2} \dot{x}^2 e^{-\gamma x^2} - \frac{1}{2} \omega^2 x^2 \]  

(20)

The application of the Euler-Lagrange equation (8) to (20) gives with satisfaction (18). So, the associated Hamiltonian may be written as
\[ H(p, x) = \frac{p^2}{2} e^{-\gamma x} + \frac{1}{2} \omega^2 x^2 \]  
\hspace{2cm} (21)

such that the Hamilton equations take the form
\[
\begin{align*}
\dot{x} &= pe^{\gamma x} \\
\dot{p} &= -\gamma p^2 e^{\gamma x} - \omega^2 x
\end{align*}
\hspace{2cm} (22)
\]

The relation between \( \dot{x} \) and \( \dot{p} \) reads in this perspective
\[
\dot{p} = -xe^{-\gamma x}(\gamma \dot{x}^2 + \omega^2 e^{\gamma x})
\hspace{2cm} (23)
\]

2. Analysis of inverted versions

Consider now the inverted version of (1)
\[
\ddot{x} + \gamma \varphi'(x) \dot{x}^2 + \omega^2 x e^{2\gamma \varphi(x)} = 0
\hspace{2cm} (24)
\]

which gives for \( \varphi(x) = x \), the following equation
\[
\ddot{x} + p x^2 + \omega^2 x e^{2\gamma x} = 0
\hspace{2cm} (25)
\]

The position dependent mass and potential function of (25) may be then deduced from (4) as
\[
M(x) = e^{2\gamma x} \text{ and } V(x) = \frac{\omega^2}{4 \gamma} x e^{4\gamma x} - \frac{\omega^2}{16 \gamma^2} e^{4\gamma x}
\hspace{2cm} (26)
\]

respectively.

Therefore, the Lagrangian for (25) may be written in the form
\[
L(\dot{x}, x) = \frac{1}{2} \dot{x}^2 e^{2\gamma x} + \frac{\omega^2}{4 \gamma} e^{4\gamma x} - \frac{\omega^2}{16 \gamma^2} e^{4\gamma x}
\hspace{2cm} (27)
\]

In this perspective, it may be verified that the application of the Euler-Lagrange equation (8) to (27) yields, as expected, (25). The Hamiltonian for (25) may also be computed as
\[
H(p, x) = \frac{p^2}{2} e^{-2\gamma x} + \frac{\omega^2}{4 \gamma} x e^{4\gamma x} - \frac{\omega^2}{16 \gamma^2} e^{4\gamma x}
\hspace{2cm} (28)
\]

which gives the Hamiltonian equations
\[
\begin{align*}
\dot{x} &= p e^{-2\gamma x} \\
\dot{p} &= \gamma p^2 e^{-2\gamma x} - \omega^2 x e^{4\gamma x}
\end{align*}
\hspace{2cm} (29)
\]

from which the conjugate momentum becomes
\[ \dot{p} = e^{2\pi} \left( \dot{x}^2 - \omega^2 xe^{2\pi} \right) \]  

(30)

By analysis, other forms of equations are also suggested by the previous studied equations. So, the following equations may also be considered in the perspective of this study, that is

\[ \ddot{x} + \gamma x \dot{x}^2 + \omega^2 xe^{\gamma \theta} = 0 \]  

(31)

or in general

\[ \ddot{x} + \gamma \varphi'(x) \dot{x}^2 + \omega^2 xe^{\gamma \varphi(x)} = 0 \]  

(32)

\[ \ddot{x} - \gamma \varphi'(x) \dot{x}^2 + \omega^2 xe^{\gamma \varphi(x)} = 0 \]  

(33)

Finally one may consider the following more generalizations

\[ \ddot{x} + \gamma \varphi'(x) \dot{x}^2 + \omega^2 h(x) e^{\gamma \varphi(x)} = 0 \]  

(34)

\[ \ddot{x} - \gamma \varphi'(x) \dot{x}^2 + \omega^2 h(x) e^{\gamma \varphi(x)} = 0 \]  

(35)

\[ \ddot{x} + \gamma \varphi'(x) \dot{x}^2 + \omega^2 h(x) e^{2\gamma \varphi(x)} = 0 \]  

(36)

\[ \ddot{x} - \gamma \varphi'(x) \dot{x}^2 + \omega^2 h(x) e^{2\gamma \varphi(x)} = 0 \]  

(37)

\[ \ddot{x} + \gamma \varphi'(x) \dot{x}^2 + \omega^2 h(x) e^{-\gamma \varphi(x)} = 0 \]  

(38)

\[ \ddot{x} + \gamma \varphi'(x) \dot{x}^2 + \omega^2 h(x) e^{-\gamma \varphi(x)} = 0 \]  

(39)

\[ \ddot{x} + \gamma \varphi'(x) \dot{x}^2 + \omega^2 h(x) e^{-2\gamma \varphi(x)} = 0 \]  

(40)

These equations will be investigated in a subsequent work.

References


