General Solution for Navier-Stokes Equations with Conservative External Force

Valdir Monteiro dos Santos Godoi
valdir.msgodoi@gmail.com

Abstract – We present two proofs of theorems on solutions of the Navier-Stokes equations for incompressible case with a conservative external force in n = 3 spatial dimensions. Without major difficulties, it can be adapted to any spatial dimension, $n >= 1$.

Keywords – Navier-Stokes equations, velocity, pressure, Eulerian description, formulation, conservative external force, equivalent equations, exact solutions, existence, inexistence, Cauchy, irrotational, potential flow, Bernouilli's law.

We find previously[1] a general solution for Navier-Stokes Equations, supposing that there is a solution for initial instant $t = 0$ and applying an additional initial condition $\frac{\partial u_i(X,t)}{\partial t}|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j}$, $1 \le i \le 3$, in the case on what the external force is zero. We will now generalize that solution to the case where there is a conservative external force, $f = \nabla U$, being applied in the fluid, for example, gravity. The problem is resolved dividing the original pressure in two parts, $p = p_f + p_u$, one of them (p_f) depending exclusively of the potential of f and another (p_u) as the obtained previously, depending exclusively of the velocity u (and therefore u^0). The influence of the conservative external force is only change the total pressure, without influence in the velocity, as happens in the Bernouilli´s law.

Firstly, we will prove theorems without external force, using $p = p_u, p_f = 0$, the identical proofs of [1].

Let $u^0(x,y,z)$ and $p^0(x,y,z)$ be respectively the initial velocity and initial pressure of the three-dimensional incompressible ($\nabla \cdot u = \nabla \cdot u^0 = 0$) Navier-Stokes equations without external force and with mass density equal to 1,

$$(1) \qquad \frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t) \frac{\partial u_i(X,t)}{\partial x_j} = \nu \nabla^2 u_i(X,t),$$

$1 \le i \le 3,\ X = (x_1, x_2, x_3) \in \mathbb{R}^3,\ x_1 \equiv x,\ x_2 \equiv y,\ x_3 \equiv z,\ x_i, t \in \mathbb{R},\ t \ge 0.$

Then in $t = 0$ is valid, for each integer i belongs to $1 \le i \le 3$,

$$(2) \qquad \frac{\partial p^0(X)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t}|_{t=0} + \sum_{j=1}^{3} u_j^0(X) \frac{\partial u_i^0(X)}{\partial x_j} = \nu \nabla^2 u_i^0(X).$$

Supposing that $u(x,y,z,t) = u^0(x+t, y+t, z+t)$ and $p(x,y,z,t) = p^0(x+t, y+t, z+t)$ is a solution (u,p) for (1), we have

(3) $$\frac{\partial p^0(\xi)}{\partial x_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial x_j} = \nu \nabla^2 u_i^0(\xi),$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ and $\xi_i = \xi_i(X,t) = x_i + t$, $1 \le i \le 3$.

For $t = 0$ the equations (2) and (3) are equals, because in $t = 0$ we have $\xi_i = x_i$ and therefore $\xi = (\xi_1, \xi_2, \xi_3) = (x_1, x_2, x_3) = X$.

For $t > 0$, if (2) is valid for any $X = (x,y,z) \in \mathbb{R}^3$ then (3) is valid for any $\xi \in \mathbb{R}^3$ substituting $x \mapsto \xi_1 = x + t,\ y \mapsto \xi_2 = y + t,\ z \mapsto \xi_3 = z + t,\ x,y,z \in \mathbb{R}, t \ge 0$, so $u(x,y,z,t) = u^0(x+t, y+t, z+t)$ and $p(x,y,z,t) = p^0(x+t, y+t, z+t)$, i.e., $u(X,t) = u^0(\xi)$ and $p(X,t) = p^0(\xi)$, solve equation (3) and therefore the Navier-Stokes equation (1).

The initial motivation to prove it is as follows. Let $A(x), B(x), C(x)$ and $D(x)$ functions such that is always valid, for any $x \in \mathbb{R}$, the relation

(4) $$A(x) + B(x) + C(x) = D(x).$$

Then, as $(x+t) \in \mathbb{R}, x,t \in \mathbb{R}, t \ge 0$, need be valid too the relation

(5) $$A(x+t) + B(x+t) + C(x+t) = D(x+t).$$

The same argument can be used for functions of two and three spatial dimensions (or better, for n spatial dimensions), for example, $\forall x,y,z,t \in \mathbb{R}, t \ge 0$,

(6) $$\begin{aligned} &A_i(x,y,z) + B_i(x,y,z) + C_i(x,y,z) = D_i(x,y,z) \\ &\Longrightarrow A_i(x+t,\ y+t,\ z+t) + B_i(x+t,\ y+t,\ z+t) + \\ &\quad + C_i(x+t,\ y+t,\ z+t) = D_i(x+t,\ y+t,\ z+t). \end{aligned}$$

Applying the previous relation (6) to the Navier-Stokes equations (2) for $t = 0$, if

(7.1) $$A_i(x,y,z) = \frac{\partial p^0(X)}{\partial x_i},$$

(7.2) $$B_i(x,y,z) = \frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0},$$

(7.3) $$C_i(x,y,z) = \sum_{j=1}^{3} u_j^0(X) \frac{\partial u_i^0(X)}{\partial x_j},$$

(7.4) $$D_i(x,y,z) = \nu \nabla^2 u_i^0(X),$$

(7.5) $$A_i(x,y,z) + B_i(x,y,z) + C_i(x,y,z) = D_i(x,y,z),$$

$X = (x, y, z)$, then, using $\xi = \xi(X, t) = (x + t, y + t, z + t)$, need be valid too the equalities

(8.1) $$A_i(x + t,\ y + t,\ z + t) = \frac{\partial p^0(\xi)}{\partial x_i},$$

(8.2) $$B_i(x + t,\ y + t,\ z + t) = \left(\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0}\right)(\xi),$$

(8.3) $$C_i(x + t,\ y + t,\ z + t) = \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j},$$

(8.4) $$D_i(x + t,\ y + t,\ z + t) = \nu\nabla^2 u_i^0(\xi),$$

(8.5) $$A_i(x + t,\ y + t,\ z + t) + B_i(x + t,\ y + t,\ z + t) + \\ + C_i(x + t,\ y + t,\ z + t) = D_i(x + t,\ y + t,\ z + t).$$

The expression $\left(\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0}\right)(\xi)$ in (8.2) means that first is calculated the value of $\frac{\partial u_i(X,t)}{\partial t}$, next we assign the value $t = 0$ in this result and then we substitute $x \mapsto \xi_1 = x + t,\ y \mapsto \xi_2 = y + t,\ z \mapsto \xi_3 = z + t$, i.e., $X \mapsto \xi$.

Note that the right side of the relations (8.1) to (8.4) corresponds to each parcel of the Navier-Stokes equations (8.5) with the solution (u, p) such that

(9.1) $$u(X, t) = u^0(\xi),$$
(9.2) $$p(X, t) = p^0(\xi),$$

$X = (x, y, z),\ \xi = \xi(X, t) = (x + t,\ y + t,\ z + t)$, then (9) is a solution for (1) if $u^0(X)$ and $p^0(X)$ are initial conditions.

We will now prove that if the variables (9.1) and (9.2) solve (1) for $t \geq 0$ then $u^0(x, y, z)$ and $p^0(x, y, z)$ solve (1) for $t = 0$, i.e., then both $u^0(x, y, z)$ and $p^0(x, y, z)$ solve (2). This is an important result of this paper. We'll use the chain rule[2].

Proof: Starting from (1), the three-dimensional incompressible Navier-Stokes equations, where $\nabla \cdot u = \nabla \cdot u^0 = 0$,

(10) $$\frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X, t)\frac{\partial u_i(X,t)}{\partial x_j} = \nu\nabla^2 u_i(X, t),$$

$1 \leq i \leq 3,\ X = (x, y, z)$, if a solution (u, p) for them is (9), i.e.,

(11.1) $$u(X, t) = u^0(\xi),$$
(11.2) $$p(X, t) = p^0(\xi),$$

$\xi = \xi(X, t) = (x + t,\ y + t,\ z + t)$, then we have, according (3),

(12) $$\frac{\partial p^0(\xi)}{\partial x_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j} = \nu\nabla^2 u_i^0(\xi).$$

How $\xi_i = x_i + t$ then $\frac{\partial \xi_i}{\partial x_i} = \frac{\partial \xi_i}{\partial t} = 1$ and $\frac{\partial \xi_i}{\partial x_j} = 0$ if $i \neq j$, so using the chain rule[1] we have, for each parcel in (10) and (12),

(13.1) $$\frac{\partial p(X,t)}{\partial x_i} = \frac{\partial p^0(\xi)}{\partial x_i} = \sum_{j=1}^{3}\frac{\partial p^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_i} = \frac{\partial p^0(\xi)}{\partial \xi_i}$$

(13.2) $$\frac{\partial u_i(X,t)}{\partial t} = \frac{\partial u_i^0(\xi)}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

(13.3) $$u_j(X,t)\frac{\partial u_i(X,t)}{\partial x_j} = u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j} = u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j} =$$
$$= u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

(13.4) $$\nabla^2 u_i(X,t) = \nabla^2 u_i^0(\xi) = \left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_3}\right)u_i^0(\xi) =$$
$$= \sum_{j=1}^{3}\left(\frac{\partial}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j}\frac{\partial}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j}\right)u_i^0(\xi) = \sum_{j=1}^{3}\left(\frac{\partial}{\partial \xi_j}\frac{\partial}{\partial \xi_j}\right)u_i^0(\xi) =$$
$$= \nabla_\xi^2 u_i^0(\xi)$$

Adding the parcels (13), with (13.3) for each integer $j = 1, 2, 3$ and the multiplication of (13.4) by viscosity coefficient ν, we come to

(14) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu\nabla_\xi^2 u_i^0(\xi),$$

which is equivalent to previous Navier-Stokes equations (10) and (12) with the solution (11), although it is not a conventional Navier-Stokes equation because the time derivative disappears, i.e.,

(15) $$\frac{\partial u_i(X,t)}{\partial t} \mapsto \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}.$$

Note that the right side of (15) is not $\frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}$, because here u_i^0 is, initially, a function only of $\xi = (\xi_1, \xi_2, \xi_3)$, not including t, but each ξ_i is a function of t and for this reason here is $\frac{\partial u_i(X,t)}{\partial t} = \frac{\partial u_i^0(\xi)}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}$, with $\xi_j = x_j + t$, $\frac{\partial \xi_j}{\partial t} = 1$.

In $t = 0$, when $\xi_i = x_i$, the equation (14) became

(16) $$\frac{\partial p^0(X)}{\partial x_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j} + \sum_{j=1}^{3} u_j^0(X) \frac{\partial u_i^0(X)}{\partial x_j} = \nu \nabla^2 u_i^0(X).$$

If this equation is equivalent to (2) then

(17) $$\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j},$$

which is thereby a good manner of define or choose the temporal derivative of velocity at $t = 0$ when the solution for velocity is $u(X,t) = u^0(\xi)$.

Similarly, for $t > 0$ we have

(18) $$\frac{\partial u_i(X,t)}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j},$$

$X = (x,y,z),\ \xi = (\xi_1, \xi_2, \xi_3),\ \xi_i = \xi_i(X,t) = x_i + t,\ 1 \le i \le 3.$

Concluding, assuming that (9), identical to (11), is a solution for (1), identical to (10), we come to (16) for $t = 0$, which is equivalent to (2) with the additional initial condition (17) and it has a solution $(u^0(X), p^0(X))$. This is what we wanted to prove. □

Next, we will prove the opposite way of the previous demonstration: if $u^0(x,y,z)$ and $p^0(x,y,z)$ solve (1) for $t = 0$, i.e., if both $u^0(x,y,z)$ and $p^0(x,y,z)$ solve (2), then the variables (u,p) given in (9.1) and (9.2) solve (1) for $t \ge 0$. This is the fundamental result of this paper. The proof basically follows what we write from beginning of this paper until the equations (9), increasing the transformations (13) and the conditions (17) and (18). We'll use the chain rule[2] again.

Proof: If $u^0(x,y,z)$ and $p^0(x,y,z)$ solve the three-dimensional incompressible ($\nabla \cdot u = \nabla \cdot u^0 = 0$) Navier-Stokes equations

(19) $$\frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t) \frac{\partial u_i(X,t)}{\partial x_j} = \nu \nabla^2 u_i(X,t)$$

for $t = 0$, with $1 \le i \le 3$, $X = (x_1, x_2, x_3) \in \mathbb{R}^3$, $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, $x_i, t \in \mathbb{R}$, $t \ge 0$, then in $t = 0$ is valid, for each integer i belongs to $1 \le i \le 3$,

(20) $$\frac{\partial p^0(X)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} + \sum_{j=1}^{3} u_j^0(X) \frac{\partial u_i^0(X)}{\partial x_j} = \nu \nabla^2 u_i^0(X).$$

Supposing that $u(x,y,z,t) = u^0(x+t, y+t, z+t)$ and $p(x,y,z,t) = p^0(x+t, y+t, z+t)$ is a solution (u,p) for (19), we have

(21) $$\frac{\partial p^0(\xi)}{\partial x_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial x_j} = \nu \nabla^2 u_i^0(\xi),$$

using $\xi = (\xi_1, \xi_2, \xi_3)$ and $\xi_i = \xi_i(X,t) = x_i + t$, $1 \le i \le 3$.

For $t = 0$ the equations (20) and (21) are equals, because in $t = 0$ we have $\xi_i = x_i$ and therefore $\xi = (\xi_1, \xi_2, \xi_3) = (x_1, x_2, x_3) = X$.

For $t > 0$, if (20) is valid for any $X = (x, y, z) \in \mathbb{R}^3$ then (21) is valid for any $\xi \in \mathbb{R}^3$ substituting $x \mapsto \xi_1 = x + t,\ y \mapsto \xi_2 = y + t,\ z \mapsto \xi_3 = z + t,\ x, y, z \in \mathbb{R}, t \ge 0$, according transformations (22) below, so $u(x, y, z, t) = u^0(x + t, y + t, z + t)$ and $p(x, y, z, t) = p^0(x + t, y + t, z + t)$, i.e., $u(X, t) = u^0(\xi)$ and $p(X, t) = p^0(\xi)$, solve equation (21) and therefore the Navier-Stokes equation (19).

How $\xi_i = x_i + t$ then $\frac{\partial \xi_i}{\partial x_i} = \frac{\partial \xi_i}{\partial t} = 1$ and $\frac{\partial \xi_i}{\partial x_j} = 0$ if $i \ne j$, so using the chain rule[2] we have, for each parcel in (21),

(22.1) $$\frac{\partial p^0(\xi)}{\partial x_i} = \frac{\partial p(\xi(X,t))}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial p^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_i} = \frac{\partial p^0(\xi)}{\partial \xi_i}$$

(22.2) $$\frac{\partial u_i^0(\xi)}{\partial t} = \frac{\partial u_i(\xi(X,t))}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

(22.3) $$u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial x_j} = u_j(\xi(X,t)) \frac{\partial u_i(\xi(X,t))}{\partial x_j} = u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_j} =$$
$$= u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

(22.4) $$\nabla^2 u_i^0(\xi) = \nabla^2 u_i(\xi(X,t)) = \sum_{j=1}^{3} \left(\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j}\right) u_i^0(\xi(X,t)) =$$
$$= \sum_{j=1}^{3} \left(\frac{\partial}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_j} \frac{\partial}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_j}\right) u_i^0(\xi) = \sum_{j=1}^{3} \left(\frac{\partial}{\partial \xi_j} \frac{\partial}{\partial \xi_j}\right) u_i^0(\xi) =$$
$$= \nabla_\xi^2 u_i^0(\xi)$$

The equation (21) transformed through by (22) gives

(23) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

that is, we transform $X \longmapsto \xi$ and from $\xi_i = x_i + t$ we have $\frac{\partial \xi_i}{\partial x_i} = 1$ and therefore $\partial x_i = \partial \xi_i$.

The unexpected transformation is

(24) $$\frac{\partial u_i^0(\xi)}{\partial t} = \frac{\partial u_i(\xi(X,t))}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j},$$

making (23) not be in the form of a standard Navier-Stokes equation. In $t = 0$ the transformation (24) becomes

(25) $$\frac{\partial u_i^0(\xi)}{\partial t}|_{t=0} = \frac{\partial u_i(\xi(X,t))}{\partial t}|_{t=0} = \frac{\partial u_i(X,t)}{\partial t}|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j},$$

$\xi_j = x_j,\ \xi = X$, for $t = 0$, thus we need to assume the additional initial condition

(26) $$\frac{\partial u_i(X,t)}{\partial t}|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j}$$

when the solution for Navier-Stokes equation (1), identical to (19), is given by (9), i.e.,

(27.1) $u(X,t) = u^0(\xi)$,
(27.2) $p(X,t) = p^0(\xi)$,

$X = (x,y,z),\ \xi = \xi(X,t) = (x+t,\ y+t,\ z+t)$.

Concluding, if $(u^0(X), p^0(X))$ solve (2), identical to (20), substituting in (20) the transformation $X \mapsto \xi,\ X = (x,y,z),\ \xi = (\xi_1, \xi_2, \xi_3),\ \xi_i = x_i + t$, we come to (23),

(28) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

assuming the additional initial condition (26)

(29) $$\frac{\partial u_i(X,t)}{\partial t}|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j}$$

due to transformation (24),

(30) $$\frac{\partial u_i^0(\xi)}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j}.$$

Using (30) in (28) we come to

(31) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

the Navier-Stokes equations with the solution $(u^0(\xi), p^0(\xi))$, i.e., $(u(X,t), p(X,t))$, according (27), identical to (9).

Using (27) and $\partial \xi_i = \partial x_i$ in (31) we come finally to

(32) $$\frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t) \frac{\partial u_i(X,t)}{\partial x_j} = \nu \nabla_X^2 u_i(X,t),$$

the Navier-Stokes equations (1) with the solution $(u(X,t), p(X,t))$. This is what we wanted to prove. ◻

What we see in the two previous proofs can be applied, with the obvious adaptations, to solutions of the form

(33.1) $u(X,t) = u^0(\xi),$

(33.2) $p(X,t) = p^0(\xi),$

$X = (x,y,z),\ \xi = (\xi_1, \xi_2, \xi_3),\ \xi_i = x_i + T_i(t),\ T_i(0) = 0,\ 1 \le i \le 3,$

with the conditions

(34) $$\frac{\partial u_i(X,t)}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(t),$$

and

(35) $$\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(0) = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j} T_j'(0),$$

being the functions $T_i(t)$ differentiable of class $C^1([0,\infty))$. In this case the equations (23) and (28) are

(36) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(t) + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} =$$

$$= \frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} \left[T_j'(t) + u_j^0(\xi)\right] = \nu \nabla_\xi^2 u_i^0(\xi).$$

Note that the equation (34) implies

(37) $$u_i(X,t) = u_i^0(X) + \int_0^t \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(t)\, dt =$$

$$= u_i^0(\xi_1, \xi_2, \xi_3) = u_i^0\big(x_1 + T_1(t),\ x_2 + T_2(t),\ x_3 + T_3(t)\big),$$

that must be true for all differentiable function $u_i^0(\xi)$ with $\xi_i = x_i + T_i(t)$, $T_i(t)$ differentiable, $T_i(0) = 0,\ 1 \le i \le 3$.

Also it is not hard see that, without major difficulties, it can be adapted to any integer spatial dimension, $n >= 1$.

Including in the system a conservative external force $f = (f_1, f_2, f_3)$ whose potential is U, $f = \nabla U$, we can separate the total pressure p in two parts, p_f and p_u, such that $p = p_f + p_u$. In this case, the more complete equations for incompressible Navier-Stokes equations are, for $1 \le i \le 3$,

(38) $$\frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t)\frac{\partial u_i(X,t)}{\partial x_j} = \nu\nabla^2 u_i(X,t) + f_i,$$

with

(39) $$\nabla \cdot u = \nabla \cdot u^0 = 0.$$

Defining

(40) $$p(X,t) = p_f(X,t) + p_u(X,t)$$

and the respective initial pressures

(41) $$p^0(X) = p_f^0(X) + p_u^0(X),$$

the obtained results in equations (1) and (2) for the pressure without external force will be attributed to p_u and p_u^0, respectively, while $p_f(X,t)$ is equal to force-potential U, i.e.,

(42.1) $$\nabla p_f = f = \nabla U$$

(42.2) $$p_f = U + \theta_f(t),$$

$\theta_f(t)$ a generic physically and mathematically reasonable function of time, as we already know.

Of this manner, the introduction of an external force do not change the velocity, but only the total pressure, such that

(43) $$p = p_f + p_u.$$

Then, the velocity can be calculated without the use of external force, in case of a conservative external force $f = \nabla U$.

It is clear that in the Eulerian description[3] the computational and analytical challenges will be, more than solving the Navier-Stokes equations for $t > 0$, solve these equations for $t = 0$, the initial instant. Unfortunately, it is not for all pair of values (u^0, p^0) that exists solution to the equation (28) and related equations, so or u^0 is a function of p^0, or p^0 is a function of u^0, or both u^0 and p^0 are functions of another functions, for example, a potential function ϕ such that $u^0 = \nabla\phi(t = 0)$, $u = \nabla\phi$, resulting in the known Bernouilli's law.

It is convenient say that Cauchy in his memorable and admirable *Mémoire sur la Théorie des Ondes*, firstly does a study on the equations to be obeyed by three-dimensional molecules in a homogeneous fluid in the initial instant $t = 0$, coming to the conclusion which the initial velocity must be irrotational, i.e., a potential flow. Of this manner, after, he comes to conclusion that the velocity is always irrotational, potential flow, if the external force is conservative (a possible exception occurs if one or two components of velocity are identically zero, when the reasons on the molecular volume are not valid). The solution obtained by Cauchy for Euler's equations is the Bernouilli's law, as almost always happens.

References

[1] Godoi, Valdir M.S., *Two Theorems on Solutions in Eulerian Description*, available in http://www.vixra.org/abs/1605.0236 (2016).

[2] Apostol, Tom M., *Calculus*, vol. II, chap. 8. New York: John Wiley & Sons (1969).

[3] Childress, Stephen, *An Introduction to Theoretical Fluid Mechanics*, Courant Lecture Notes in Mathematics, 19. New York: Courant Institute of Mathematical Sciences (2009).