

# The topology on a complete semilattice

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## Abstract

We define the topology  $atop(\chi)$  on a complete upper semilattice  $\chi = (M, \leq)$ . The limit points are determined by the formula

$$\lim_D(X) = \sup\{a \in M \mid \{x \in X \mid a \leq x\} \in D\},$$

where  $X \subseteq M$  is an arbitrary set,  $D$  is an arbitrary non-principal ultrafilter on  $X$ . We investigate  $\lim_D(X)$  and topology  $atop(\chi)$  properties. In particular, we prove the compactness of the topology  $atop(\chi)$ .

## 1. Preliminaries

For any set  $X$  we use  $P(X)$  to denote the set of all subsets of  $X$ . For an arbitrary collection  $S$  of sets we use  $\cup S$  and  $\cap S$  to denote the union and the intersection of all sets of  $S$  respectively.

A cardinal will be identified with the corresponding lowest ordinal. The cardinality (size) of a set will be identified with the corresponding cardinal. Example:  $|\omega| = \omega = \omega_0$ . We assume the axiom of choice.

Let  $D$  be an ultrafilter on  $X$  and  $Y \in D$ .  $D|_Y$  is the ultrafilter on  $Y$ , where  $D|_Y = \{Z \cap Y \mid Z \in D\}$ .

A principal ultrafilter is an ultrafilter containing a least element. An ultrafilter is a non-principal, if it does not contain finite sets.

A complete upper semilattice is a partially ordered set in which every subset has a least upper bound (sup).

A complete lattice is a partially ordered set in which every subset has a least upper bound (sup) and a greatest lower bound (inf).

## 2. The limit $\lim_D(X)$ and the associated topology

**Definition 1.** Let  $\chi = (M, \leq)$  be a complete upper semilattice,  $X \subseteq M$ ,  $X \neq \emptyset$  and let  $D$  be an arbitrary ultrafilter on  $X$ . We denote

$$\lim_D(X) = \sup\{a \in M \mid \{x \in X \mid a \leq x\} \in D\},$$

**Definition 2.** Let  $\chi = (M, \leq)$  be a complete upper semilattice. A set  $\Delta \subseteq M$  is an **approximation base**, if for every  $x \in M$  we have

$$X = \sup\{\alpha \in \Delta \mid \alpha \leq x\}.$$

**Definition 3.** Let  $\chi = (M, \leq)$  be a complete upper semilattice, let  $\Delta$  be an approximation base,  $X \subseteq M$ ,  $X \neq \emptyset$  and let  $D$  be an arbitrary ultrafilter on  $X$ . We denote

$$\lim_1(\Delta, D, X) = \sup\{\alpha \in \Delta \mid \{x \in X \mid \alpha \leq x\} \in D\}$$

**Remark 1.** The approximation base  $\Delta = M$  is associated with every complete upper semilattice  $\chi = (M, \leq)$ . In this case we have

$$\lim_D(X) \equiv \lim_1(\Delta, D, X)$$

**Definition 4.** Let  $\chi = (M, \leq)$  be a complete lattice,  $X \subseteq M$ ,  $X \neq \emptyset$  and let  $D$  be an arbitrary ultrafilter on  $X$ . We denote

$$\lim_2(D, X) = \sup\{\inf(Y) \mid Y \in D\}.$$

**Proposition 1.** Let  $\chi = (M, \leq)$  be a complete upper semilattice, let  $\Delta$  be an approximation base,  $X \subseteq M$ ,  $X \neq \emptyset$  and let  $D$  be an ultrafilter on  $X$ . We have

$$\lim_D(X) = \lim_1(\Delta, D, X).$$

**Proof.** Since  $\lim_D(X) \equiv \lim_1(M, D, X)$ , it is sufficiently to prove that  $\lim_1(\Delta, D, X) = \lim_1(M, D, X)$ . It is easy to see that

$$\lim_1(\Delta, D, X) \leq \lim_1(M, D, X).$$

Let  $a_0 \in \{a \in M \mid \{x \in X \mid a \leq x\} \in D\}$ . Since  $\Delta$  is an approximation base,

$$a_0 = \sup\{\delta \in \Delta \mid \delta \leq a_0\}.$$

Let  $\delta_0 \in \{\delta \in \Delta \mid \delta \leq a_0\}$ , i.e.  $\delta_0 \leq a_0$  and

$$\{x \in X \mid a_0 \leq x\} \subseteq \{x \in X \mid \delta_0 \leq x\}.$$

Since  $\{x \in X \mid a_0 \leq x\} \in D$ , we have  $\{x \in X \mid \delta_0 \leq x\} \in D$  and

$$\delta_0 \in \{\delta \in \Delta \mid \{x \in X \mid \delta \leq x\} \in D\},$$

$$\{\delta \in \Delta \mid \delta \leq a_0\} \subseteq \{\delta \in \Delta \mid \{x \in X \mid \delta \leq x\} \in D\},$$

$$\sup\{\delta \in \Delta \mid \delta \leq a_0\} \leq \sup\{\delta \in \Delta \mid \{x \in X \mid \delta \leq x\} \in D\},$$

$$a_0 \leq \sup\{\delta \in \Delta \mid \{x \in X \mid \delta \leq x\} \in D\}.$$

Since  $a_0 \in \{a \in M \mid \{x \in X \mid a \leq x\} \in D\}$  is arbitrary, we have

$$\sup\{a \in M \mid \{x \in X \mid a \leq x\} \in D\} \leq \sup\{\delta \in \Delta \mid \{x \in X \mid \delta \leq x\} \in D\},$$

$$\lim_1(M, D, X) \leq \lim_1(\Delta, D, X).$$

Proposition 1 is proved.

**Remark 2.** From Proposition 1 it follows that  $\lim_1(\Delta, D, X)$  does not depend on  $\Delta$  actually. But we will use it in the future, because it is easier to prove certain properties associated with  $\lim_D(X)$ .

**Proposition 2.** Let  $\chi = (M, \leq)$  be a complete lattice, let  $\Delta$  be an approximation base,  $X \subseteq M$ ,  $X \neq \emptyset$  and let  $D$  be an ultrafilter on  $X$ . We have

$$\lim_D(X) = \lim_1(\Delta, D, X) = \lim_2(D, X).$$

**Proof.** First equation  $\lim_D(X) = \lim_1(\Delta, D, X)$  follows from Proposition 1. It is necessary to prove that

$$\sup\{\inf(Y) \mid Y \in D\} = \sup\{\alpha \in \Delta \mid \{x \in X \mid \alpha \leq x\} \in D\}.$$

Let  $Y \in D$ . Since  $\Delta$  is an approximation base,

$$\inf(Y) = \sup\{\alpha \in \Delta \mid \alpha \leq \inf(Y)\}.$$

Let  $\alpha_0 \in \{\alpha \in \Delta \mid \alpha \leq \inf(Y)\}$ ,  $X_0 = \{x \in X \mid \alpha_0 \leq x\}$  then  $\alpha_0 \leq \inf(Y)$  and  $Y = \{y \in Y \mid \alpha_0 \leq y\} \subseteq \{x \in X \mid \alpha_0 \leq x\} = X_0$ . Since  $Y \in D$  and  $Y \subseteq X_0$  then  $X_0 \in D$ . We see that

$$\alpha_0 \in \{\alpha \in \Delta \mid \{x \in X \mid \alpha \leq x\} \in D\},$$

$$\{\alpha \in \Delta \mid \alpha \leq \inf(Y)\} \subseteq \{\alpha \in \Delta \mid \{x \in X \mid \alpha \leq x\} \in D\},$$

$$\inf(Y) = \sup\{\alpha \in \Delta \mid \alpha \leq \inf(Y)\} \leq \sup\{\alpha \in \Delta \mid \{x \in X \mid \alpha \leq x\} \in D\},$$

$$\sup\{\inf(Y)|Y \in D\} \leq \sup\{\alpha \in \Delta | \{x \in X | \alpha \leq x\} \in D\}.$$

Let us prove the opposite direction.

Let  $\alpha_0 \in \{\alpha \in \Delta | \{x \in X | \alpha \leq x\} \in D\}$  then

$$\{x \in X | \alpha_0 \leq x\} \in D,$$

$$\alpha_0 \leq \inf\{x \in X | \alpha_0 \leq x\} \leq \sup\{\inf(Y)|Y \in D\},$$

$$\sup\{\alpha \in \Delta | \{x \in X | \alpha \leq x\} \in D\} \leq \sup\{\inf(Y)|Y \in D\}.$$

Proposition 2 is proved.

**Lemma 1.** Let  $\chi = (M, \leq)$  be a complete upper semilattice,  $X \subseteq M$ ,  $X \neq \emptyset$ , let  $D$  be an ultrafilter on  $X$ . The following are

1) if  $\{a\} \in D$  then  $\lim_D(X) = a$ ;

2) if  $Y \in D$  then  $\lim_D(X) = \lim_F(Y)$ , where  $F = D|_Y$ ,

$D$  is a principal ultrafilter  $\Leftrightarrow F$  is a principal ultrafilter;

3) if  $Z \subseteq M$  and  $X \subseteq Z$  then exist an ultrafilter  $G$  on  $Z$  that  $X \in G$ ,  $G|_X = D$ ,  $\lim_D(X) = \lim_G(Z)$ ,

$D$  is a principal ultrafilter  $\Leftrightarrow G$  is a principal ultrafilter;

4) if  $X \subseteq M$  is an infinite set,  $D$  is a non-principal ultrafilter on  $X$  then for any finite set  $X' \subseteq X$  we have  $\lim_F(X \setminus X') = \lim_D(X)$ , where  $F = D|_{X \setminus X'}$  is a non-principal ultrafilter.

**Proof.** Let  $\Delta = M$ . We prove 1). If  $\{a\} \in D$  then for any  $Y \in D$  we have  $\{a\} \cap Y \neq \emptyset$ ,  $a \in Y$ . Hence for any  $\delta \in \Delta$  it is

$$\delta \leq a \Leftrightarrow \{x \in X | \delta \leq x\} \in D,$$

i.e.

$$\begin{aligned} a &= \sup\{\delta \in \Delta | \delta \leq a\} = \sup\{\delta \in \Delta | \{x \in X | \delta \leq x\} \in D\} = \\ &= \lim_1(\Delta, D, X) = \lim_D(X). \end{aligned}$$

The last equation follows from Proposition 1.

We prove 2). We prove that for every

$\alpha \in \Delta$  it is

$$\{x \in X | \alpha \leq x\} \in D \Leftrightarrow \{y \in Y | \alpha \leq y\} \in F. \quad (1)$$

Let  $\{x \in X | \alpha \leq x\} \in D$ . Since  $Y \subseteq X$ ,

$$\{y \in Y | \alpha \leq y\} = Y \cap \{x \in X | \alpha \leq x\} \in D|_Y = F.$$

We prove in the opposite direction. Let  $Y_0 = \{y \in Y | \alpha \leq y\} \in F$ . We suppose the opposite that  $\{x \in X | \alpha \leq x\} \notin D$ , i.e.  $\{x \in X | \alpha \not\leq x\} \in D$  and

$$Y \setminus Y_0 = \{y \in Y | \alpha \not\leq y\} = Y \cap \{x \in X | \alpha \not\leq x\} \in D|_Y = F.$$

We obtain the contradictory  $Y_0 \in F, Y \setminus Y_0 \in F$ . We conclude that  $\{x \in X \mid \alpha \leq x\} \in D$ . We have proved the statement (1). From (1) it follows

$$\begin{aligned} \{\alpha \in \Delta \mid \{x \in X \mid \alpha \leq x\} \in D\} &= \{\alpha \in \Delta \mid \{y \in Y \mid \alpha \leq y\} \in F\}, \\ \sup\{\alpha \in \Delta \mid \{x \in X \mid \alpha \leq x\} \in D\} &= \sup\{\alpha \in \Delta \mid \{y \in Y \mid \alpha \leq y\} \in F\}, \\ \lim_1(\Delta, D, X) &= \lim_1(\Delta, F, Y), \\ \lim_D(X) &= \lim_F(Y). \end{aligned}$$

As  $F \subseteq D$ , if  $F$  contains a finite set then  $D$  also contains a finite set. On the other hand, if  $D$  contains a finite set  $Z$  then ultrafilter  $F$  contains a finite set  $Z \cap Y$ . This implies that  $D$  is a principal ultrafilter  $\Leftrightarrow F$  is a principal ultrafilter.

We prove 3). Let  $G = \{Y \subseteq Z \mid Y \cap X \in D\}$ .  $G$  is an ultrafilter obviously,  $G|_X = D$ ,  $X \in G$ . The remaining assertions of the item 3) follow from the item 2).

Let us prove 4). Let  $D$  be a non-principal ultrafilter, i.e.  $X \setminus X' \in D$ . Let  $F = D|_{X \setminus X'}$ . The ultrafilter  $F$  with respect to the item 2) is a non-principal. According to 2),

$$\lim_F(X \setminus X') = \lim_D(X).$$

**Definition 3.** Let  $\chi = (M, \leq)$  be a complete upper semilattice. We define the operation  $C()$  on the subsets of  $M$ . Let  $X \subseteq M$ , we define

$$C(X) = X \cup \{\lim_D(X) \mid D \text{ is a non-principal ultrafilter on } X\}. \quad (2)$$

**Lemma 2.** Let  $\chi = (M, \leq)$  be a complete upper semilattice. The operation  $C()$  defined by (2) has the following properties:

- 1)  $C(X_1 \cup X_2) = C(X_1) \cup C(X_2)$ , where  $X_1, X_2 \subseteq M$ ;
- 2)  $X \subseteq C(X)$ , where  $X \subseteq M$ ;
- 3) if  $X$  is finite then  $C(X) = X$ , where  $X \subseteq M$  (particularly,  $C(\emptyset) = \emptyset$ );
- 4) if  $X \subseteq Y$  then  $C(X) \subseteq C(Y)$  for all  $X, Y \subseteq M$ .

**Proof.** Assertions 2) and 3) follow from the definition of the operation  $C()$  and Lemma 1.

We prove 1). If  $X_1$  is a finite set then

$$C(X_1) = X_1 \subseteq X_1 \cup X_2 \subseteq C(X_1 \cup X_2).$$

Assume that  $X_1$  is an infinite set. Let  $D_1$  be an arbitrary non-principal ultrafilter on  $X_1$ . By Lemma 1 there exists a non-principal ultrafilter  $D$  on  $X_1 \cup X_2$  that  $X_1 \in D, D|_{X_1} = D_1$  and  $\lim_D(X_1 \cup X_2) = \lim_{D_1}(X_1)$ . We see that  $C(X_1) \subseteq C(X_1 \cup X_2)$ . Similarly  $C(X_2) \subseteq C(X_1 \cup X_2)$ , i.e.

$$C(X_1) \cup C(X_2) \subseteq C(X_1 \cup X_2).$$

Let us prove the item 1) in the opposite direction. Let  $D$  be an arbitrary non-principal ultrafilter on  $X_1 \cup X_2$ . It is obvious that  $X_1 \in D$  or  $X_2 \setminus X_1 \in D$ . If  $X_2 \setminus X_1 \in D$  then  $X_2 \in D$ . Hence it is  $X_1 \in D$  or  $X_2 \in D$ . Let  $X_1 \in D$  for definiteness. From Lemma 1 we have  $\lim_D(X_1 \cup X_2) = \lim_{D_1}(X_1)$ , where  $D_1 = D|_{X_1}$ . From this it follows that

$$C(X_1 \cup X_2) \subseteq C(X_1) \cup C(X_2).$$

The item 1) of the Lemma is proved.

Let us prove the item 4). If  $X$  is finite then  $C(X) = X \subseteq Y \subseteq C(Y)$ . Suppose that  $X$  is infinite. Let  $D$  be an arbitrary non-principal ultrafilter on  $X$ . From Lemma 1 we have that there is a non-principal ultrafilter  $G$  on  $Y$  that  $X \in G$ ,  $G|_X = D$ ,  $\lim_D(X) = \lim_G(Y)$ , i.e.  $C(X) \subseteq C(Y)$ . The item 4) of the the Lemma is proved.

**Lemma 3.** Let  $\chi = (M, \leq)$  be a complete upper semilattice. The set of all sets such that  $X \subseteq M$ ,  $C(X) = X$  (we assume that  $X$  is closed) is a topology.

**Proof.** Let  $R = \{X \subseteq M | C(X) = X\}$ . Obviously  $\emptyset, X \in R$ . It is sufficient to prove:

- 1) there is  $\cup P \in R$  for every finite  $P \subseteq R$ ;
- 2) there is  $\cap P \in R$  for each  $P \subseteq R$ .

The associativity of the union of a sets implies that property 1) suffices to prove for two sets in  $R$ . Let  $X_1, X_2 \in R$ . From Lemma 2 it follows that  $C(X_1 \cup X_2) = C(X_1) \cup C(X_2) = X_1 \cup X_2$ . That is  $X_1 \cup X_2 \in R$ . The property 1) is proved.

We prove the property 2). Let  $P \subseteq R$ . We consider an arbitrary set  $Y \in P$ . It is obviously that  $\cap P \subseteq Y$ . We obtain by Lemma 2 that  $C(\cap P) \subseteq C(Y) = Y$ , i.e.  $C(\cap P) \subseteq Y$ . As  $Y \in P$  is arbitrary, we have  $C(\cap P) \subseteq \cap P$ . Since  $\cap P \subseteq C(\cap P)$ , we have  $C(\cap P) = \cap P$ , i.e.  $\cap P \in R$ .

**Definition 4.** The topological space is defined by the Lemma 3 will be denoted by  $atop(\chi)$ .

**Lemma 4.** Let  $\chi = (M, \leq)$  is a complete upper semilattice,  $\kappa$  is an infinite cardinal and  $(X_\lambda)_{\lambda < \kappa}$  is a not increasing sequence of closed in the topology  $atop(\chi)$  sets, i.e.  $X_\lambda \supseteq X_{\lambda+1}$ ,  $C(X_\lambda) = X_\lambda$  for all ordinals  $\lambda < \kappa$ . If  $X_\lambda \neq \emptyset$  for all  $\lambda < \kappa$  then this sequence has a non-empty intersection.

**Proof.** If  $\kappa$  is't a regular cardinal then we can choose some subsequence of the size of the regular cardinal  $cf(\kappa)$  (cofinality  $\kappa$ ) that for any ordinal  $\lambda_0 < \kappa$  there is an ordinal  $\lambda > \lambda_0$  corresponding to the element of the selected subsequence. It is clear that the intersection of the original sequence and the selected subsequence are the some. Therefore, we can assume without loss of generality that  $\kappa$  is a regular cardinal.

Let  $h : \kappa \rightarrow M$  is an arbitrary mapping for which  $h(\lambda) \in X_\lambda$  for all  $\lambda < \kappa$ .

If  $|Rang(h)| < \kappa$  then (in accordance with the regularity of  $\kappa$ ) there is an element  $a \in Rang(h)$  that  $|h^{-1}(a)| = \kappa$ . For any ordinal  $\lambda_0 < \kappa$  there is an ordinal  $\lambda$  that  $\lambda_0 < \lambda < \kappa$  and  $h(\lambda) = a$ . This means that  $a \in X_\lambda$  for all  $\lambda < \kappa$ , i.e. the sequence  $(X_\lambda)_{\lambda < \kappa}$  has a non empty intersection.

Let us assume that  $|Rang(h)| = \kappa$ . Let  $D$  be a non-principal ultrafilter on the set  $Rang(h)$  that if  $Z \in D$  then  $|Z| = \kappa$ . We consider an arbitrary ordinal  $\lambda_0 < \kappa$ . Let  $E = \{h(\lambda) | \lambda < \lambda_0\}$ . It is obvious that the  $Rang(h) \setminus E \in D$ . According to Lemma 1, we have

$$\lim_D(Rang(h)) = \lim_F(Rang(h) \setminus E),$$

where  $F = D|_{Rang(h) \setminus E}$ . We note that  $Rang(h) \setminus E \subseteq X_{\lambda_0}$ . By lemma 1 there is a non-principal ultrafilter  $G$  on the set  $X_{\lambda_0}$  that

$$\lim_G(X_{\lambda_0}) = \lim_F(Rang(h) \setminus E).$$

Since  $X_{\lambda_0}$  is a close set,  $C(X_{\lambda_0}) = X_{\lambda_0}$  and

$$\lim_D(Rang(h)) = \lim_F(Rang(h) \setminus E) = \lim_G(X_{\lambda_0}) \in X_{\lambda_0}.$$

Since  $\lambda_0 < \kappa$  is an arbitrary ordinal,

$$\lim_D(Rang(h)) \in \bigcap_{\lambda < \kappa} X_\lambda$$

Lemma is proved.

**Theorem 1.** Let  $\chi = (M, \leq)$  be a complete upper semilattice then the topological space  $atop(\chi)$  is compact.

**Proof.** We prove this theorem in two ways.

1) By Lemma 2, every point of  $atop(\chi)$  is a close set, i.e.  $atop(\chi)$  is a  $T_1$  space. Lemma 4 implies that every well ordered sequence of non-empty closed decreasing sets is non-empty intersection. The theorem (Alexandrov P. S. Uryson P.S., [1, p.26] ) for  $T_1$  topological spaces implies that if every well ordered sequence of non-empty closed decreasing sets is non-empty intersection then the topology is a compact. Thus,  $atop(\chi)$  is a compact topology.

2) The second proof uses the methods of the proof of the existence of a finite subcovering of a countable cover of a countably compact (with the modern interpretation) topological space (F. Hausdorff [3, p.141]). In this proof we construct a finite subcovering for a arbitrary covering. This proof is longer, but it is useful for future analysis.

We will show that each open covers of  $atop(\chi)$  has a subcovering of a smaller cardinality. Let  $\kappa$  be an infinite cardinal and let  $(G_\lambda)_{\lambda < \kappa}$  be an

open covering of the cardinality  $\kappa$ . Hence  $(F_\lambda)_{\lambda < \kappa}$ , where  $F_\lambda = G_0 \cup \dots \cup G_\lambda$ , is a non-decreasing sequence of open sets, which is covered. We consider the corresponding sequence of the close sets  $M \setminus F_0 \supseteq \dots \supseteq M \setminus F_\lambda \supseteq \dots$ , where  $\lambda < \kappa$ . This sequence has empty intersection. By Lemma 4 there exists an ordinal  $\lambda_0 < \kappa$  that  $M \setminus F_\lambda = \emptyset$  for all  $\lambda < \kappa$ , which  $\lambda > \lambda_0$ . We see that the set of open sets  $G_0, \dots, G_{\lambda_0}$  is covered and the cardinality of the cover is  $|\lambda_0| < \kappa$ . If  $|\lambda_0|$  is an infinite cardinal then we can repeat the same procedure and we can get an open covering of a cardinality less than  $|\lambda_0|$  and so on. Thus for a finite number of steps we can get a finite covering.

**Theorem 2.** Let  $\chi = (M, \leq)$  be a complete upper semilattice,  $\Delta$  be an approximation base,  $|\Delta| = \omega_0$ ,  $X \subseteq M$  be an infinite,  $D$  be a non-principal ultrafilter on  $X$ . There is a countable subset  $X_0 \subseteq X$  that

$$\lim_D(X) = \sup(\Delta_0) = \lim_{D_0}(X_0),$$

where

$$\Delta_0 = \{\alpha \in \Delta \mid \{x \in X_0 \mid \alpha \not\leq x\} \text{ is finite}\},$$

$$\Delta_1 = \Delta \setminus \Delta_0 = \{\alpha \in \Delta \mid \{x \in X_0 \mid \alpha \leq x\} \text{ is finite}\},$$

$D_0$  is an arbitrary non-principal ultrafilter on  $X_0$ .

**Proof.**

For anyone  $\alpha \in \Delta$  we denote

$$X_\alpha = \{x \in X \mid \alpha \leq x\},$$

$$\bar{X}_\alpha = \{x \in X \mid \alpha \not\leq x\},$$

$$\Delta_0 = \{\alpha \in \Delta \mid X_\alpha \in D\},$$

$$\Delta_1 = \{\alpha \in \Delta \mid \bar{X}_\alpha \in D\}.$$

It is obvious that

$$\lim_D(X) = \lim_1(\Delta, D, X) = \sup(\Delta_0),$$

$$\Delta_0 \cap \Delta_1 = \emptyset, \Delta_0 \cup \Delta_1 = \Delta.$$

Let the sequence  $\alpha_0, \alpha_1, \alpha_2, \dots$  be a list of all elements of  $\Delta$ . We define the sequence  $X^{(0)}, X^{(1)}, X^{(2)}, \dots$  by the induction.

1) If  $\alpha_0 \in \Delta_0$  then  $X^{(0)} = X_{\alpha_0}$ , if  $\alpha_0 \in \Delta_1$  then  $X^{(0)} = \bar{X}_{\alpha_0}$ .

2) If  $X^{(i)}$  is determined then

if  $\alpha_{i+1} \in \Delta_0$  then

$$X^{(i+1)} = X^{(i)} \cap X_{\alpha_{i+1}},$$

if  $\alpha_{i+1} \in \Delta_1$  then

$$X^{(i+1)} = X^{(i)} \cap \bar{X}_{\alpha_{i+1}}.$$



We note that for all  $i \in N$  we have

$$X^{(i+1)} \subseteq X^{(i)} \subseteq X, X^{(i)} \in D.$$

If  $\alpha_i \in \Delta_0$  then  $\alpha_i \leq x$  for all  $x \in X^{(i)}$ . If  $\alpha_i \in \Delta_1$  then  $\alpha_i \not\leq x$  for all  $x \in X^{(i)}$ .

Since  $X^{(i)}$  is infinite set, we can construct a sequence  $x_0, x_1, x_2, \dots$  of  $X$  for which  $x_i \in X^{(i)}$ ,  $x_i \neq x_j$  for all  $i \neq j$ . Denote  $X_0$  the set of all elements of the sequence. Obviously  $|X_0| = \omega_0$ .

Let  $n \in N$ . If  $\alpha_n \in \Delta_0$  then  $\alpha_n \leq x_m$  for all  $m \geq n$ . If  $\alpha_n \in \Delta_1$  then  $\alpha_n \not\leq x_m$  for all  $m \geq n$ . We see that for any  $\alpha \in \Delta_0$  the set  $\{x \in X_0 | \alpha \not\leq x\}$  is finite, for any  $\alpha \in \Delta_1$  the set  $\{x \in X_0 | \alpha \leq x\}$  is finite.

Since  $\Delta_0 \cap \Delta_1 = \emptyset, \Delta_0 \cup \Delta_1 = \Delta$ , we have

$$\Delta_0 = \{\alpha \in \Delta | \{x \in X_0 | \alpha \not\leq x\} \text{ is finite}\},$$

$$\Delta_1 = \{\alpha \in \Delta | \{x \in X_0 | \alpha \leq x\} \text{ is finite}\}.$$

Let  $D_0$  be an arbitrary non-principal ultrafilter on  $X_0$ . Obviously if  $\alpha \in \Delta_0$  then

$$\{x \in X_0 | \alpha \leq x\} \in D_0,$$

if  $\alpha \in \Delta_1$  then

$$\{x \in X_0 | \alpha \not\leq x\} \in D_0,$$

i.e

$$\Delta_0 = \{\alpha \in \Delta | \{x \in X_0 | \alpha \leq x\} \in D_0\},$$

$$\Delta_1 = \{\alpha \in \Delta | \{x \in X_0 | \alpha \not\leq x\} \in D_0\}.$$

Thus we have

$$\begin{aligned} \lim_D(X) &= \sup(\Delta_0) = \sup\{\alpha \in \Delta | \{x \in X_0 | \alpha \not\leq x\} \text{ is finite}\} = \\ &= \sup\{\alpha \in \Delta | \{x \in X_0 | \alpha \leq x\} \in D_0\} = \lim_{D_0}(X_0). \end{aligned}$$

The theorem is proved.

### § 3. Examples

**Example 1.** Let  $\chi = ([0, 1], \leq)$  be a lattice on the interval  $[0, 1]$  with standard interpretation of the relation " $\leq$ ". Obviously  $\chi$  is a complete lattice.

Let  $X \subseteq [0, 1]$ ,  $D$  is an arbitrary non-principal ultrafilter on  $X$ . We will show that the point

$$a = \lim_D(X)$$

is a limit point in the usual topology, i.e. any open interval of the point  $a$  contains points of the set  $X \setminus \{a\}$ . We assume that  $a \neq 0, a \neq 1$ . The cases  $a = 0, a = 1$  are analyzed in the similar way. We suppose the opposite, i.e. there is an open interval  $(b, c) \subset [0, 1]$  that  $a \in (b, c)$  and  $(b, c) \cap X = \{a\}$ . We will obtain a contradiction.

Since  $D$  is a non-principal ultrafilter, then  $X \setminus \{a\} \in D$ . Let  $X_0 = X \setminus \{a\}$ . By Lemma 2

$$\lim_D(X) = \lim_{D_0}(X_0),$$

where  $D_0 = D|_{X_0}$ . For every  $Y \in D_0$  we have  $Y \cap (b, c) = \emptyset$ , i.e. either  $\lim_2(D_0, X_0) \leq b$  or  $\lim_2(D_0, X_0) \geq c$ . By Proposition 2 we have either  $\lim_{D_0}(X_0) \leq b$  or  $\lim_{D_0}(X_0) \geq c$ . Thus it is  $a \neq \lim_D(X)$ . This is a contradiction with the assumption that the point  $a$  is't a limit point in the usual topology.

Now let the point  $a$  is a limit point of the set  $X$  in the usual topology, i.e. any open interval containing the point  $a$  intersects with  $X \setminus \{a\}$ . We will show that there is a non-principal ultrafilter  $D$  on  $X$  that  $\lim_D(X) = a$ . We assume that  $a \neq 0, a \neq 1$ . The cases  $a = 0, a = 1$  are analyzed in the similar way.

Since any open interval containing the point  $a$  has a non-empty intersection with  $X$ , then this intersection contains an infinite number of elements. Otherwise it would be possible to pick up an open interval containing the point  $a$  and has no intersection with  $X$ . Let

$$R = \{X \cap (b, c) | a \in (b, c), (b, c) \subset [0, 1]\}.$$

In view of the above remarks any set of  $R$  is infinite, the intersection of two sets of  $R$  also belongs to  $R$ . There exists a non-principal ultrafilter  $D$  on  $X$  that  $R \subseteq D$ . By construction  $D$ , we have

$$\lim_D(X) = \lim_2(D, X) = a.$$

Thus the topology  $atop(\chi)$  coincides with the usual topology on  $[0, 1]$ , which is a compact topology.

**Example 2.** Let  $M = \omega \cup \{\omega\}$ ,  $\chi = (M, \leq)$ . Obviously  $\chi$  is a complete lattice. Close sets in the topology  $atop(\chi)$  are finite sets and sets containing  $\omega$ . Consequently open sets are sets that do not contain  $\omega$  and sets with a finite supplement. Topology  $atop(\chi)$  is a compact. Any open covering must to cover the point  $\omega$ . The covering must include an open set containing  $\omega$  and having finite supplement that is covered by a finite number of open sets

of the selected covering.

**Example 3** Let  $\chi = (P(\mathbb{N}), \subseteq)$  be the lattice of the subsets of the natural numbers  $\mathbb{N}$  by inclusion. Obviously  $\chi$  is a complete lattice. We consider an approximation base

$$\Delta = \{\{n\} | n \in \mathbb{N}\} \cup \emptyset.$$

Since  $|\Delta| = \omega_0$ , according to Theorem 2 the closure of any set  $X \subseteq P(\mathbb{N})$  of  $atop(\chi)$  can be reduced to the closure of all countable subsets of  $X$ .

By Theorem 1 the topology  $atop(\chi)$  is a compact.

**Example 4** Let  $a, b \in \mathbb{R}$  and  $a < b$ . Let  $M$  be the set of all real functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  that  $a \leq f(x) \leq b$ . Let  $\chi = (M, \leq)$ , where " $\leq$ " is a pointwise comparison of functions. Obviously  $\chi$  is a complete lattice. By Theorem 1 the topology  $atop(\chi)$  is a compact.

**Example 5.** We consider the propositional logic  $L = L(A, \Omega, Z, I)$ , where  $A = \{p_1, p_2, \dots\}$  are propositional variables,  $\Omega = \{\neg, \wedge, \vee, \rightarrow\}$  are logical connectives,  $Z$  is a set of inference rules (the rule of inference is modus ponens),  $I$  is a set of Hilbert axioms.

Let  $\Phi$  be a set of all formulas of  $L$ . Let  $\Psi_1, \Psi_2 \subseteq \Phi$ . We assume  $\Psi_1 \prec \Psi_2 \Leftrightarrow$  for any  $\phi \in \Psi_1$  we have  $I, \Psi_2 \vdash \phi$ .

We denote

$$[\Psi] = \{X \subseteq \Phi | \Psi \prec X \text{ and } X \prec \Psi\}.$$

$[\Psi]$  is a class of equivalence of  $\Psi$ .

Let  $\chi = (M, \leq)$ , where  $M = \{[X] | X \subseteq \Phi\}$ , if  $X_1, X_2 \subseteq \Phi$  then we assume  $[X_1] \leq [X_2] \Leftrightarrow X_1 \prec X_2$ .

It is obvious that if  $S \subseteq P(M)$  then  $sup(S) = [\cup S]$ .

Thus,  $\chi$  is a complete upper semilattice.

It is obvious that

- 1)  $[I] = [\emptyset] = [T]$ , where  $T$  is a set of all formulas  $\phi \in \Phi$  that  $I \vdash \phi$ ;
- 2)  $[I] \leq [X]$  for any  $X \subseteq \Phi$ , i.e. for any non-principal ultrafilter  $D$  on  $M$  we have  $[I] \leq \lim_D(X)$ ;
- 3)  $sup(\Phi) = [\Phi] = [\{p_1, \neg p_1\}]$ ;
- 4) for any non-principal ultrafilter  $D$  on  $M$  we have  $\lim_D(X) \neq [\{p_1, \neg p_1\}]$ ;
- 5) if  $X \subseteq \Phi$  and  $|X| < \omega_0$  then there exist a non-principal ultrafilter  $D$  on  $M$  that  $\lim_D(X) = [X]$ .
- 6) if  $X \subseteq \Phi$  and  $X$  is a full set of formulas (for any  $\phi \in \Phi$  we have  $X \vdash \phi$  or  $X \vdash \neg\phi$ ) then  $[X]$  is't a limit point.

By Theorem 2, the topology  $atop(\chi)$  is a compact.

**Example 6 (Semilattice of facts).** We consider some set  $\Phi$  of real facts. We consider relation " $\prec$ " on  $\Phi$ . We assume  $s_1 \prec s_2 \Leftrightarrow$  when the fact

" $s_2$  implies  $s_1$ " belong to  $\Phi$ . We define classes of equivalence  $M$  on subsets of  $\Phi$  similar with Example 6.

Let  $\chi = (M, \leq)$ , where  $M = \{[X] | X \subseteq \Phi\}$ , if  $X_1, X_2 \subseteq \Phi$  then we assume  $[X_1] \leq [X_2] \Leftrightarrow X_1 \prec X_2$ .

It is obvious that if  $S \subseteq P(M)$  then  $\sup(S) = [\cup S]$ .

Thus,  $\chi$  is a complete upper semilattice. By Theorem 2, the topology  $\text{atop}(\chi)$  is a compact.

## REFERENCES

[1] Alexandrov P. S. Uryson P.S., Memoir on compact topological spaces, FIZMATLIT, Moscow, 2009.

[2] Ershov Yu. L., Palyutin E. A., Mathematical logic, Nauka, Moscow, 1987.

[3] F. Hausdorff, Set Theory, Moscow, 1937.

[4] R. C. Moore, G. S. Mrowka, Topologies determined by countable objects, Notices Amer. Math. Soc. 11, 1964.

[5] Kolmogorov A. N., Fomin S. V. Elements of the theory of functions and functional analysis, Nauka, Moscow, 1989.

[6] D. Scott Continuous lattice, Lecture Notes, 274, 1972.