Catalan’s constant : G=0.9159...

Edgar Valdebenito

abstract

In this note we give some formulas for Catalan’s constant:

$$G = \int_0^1 \frac{\tan^{-1} x}{x}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965594177\ \ldots$$

Keywords: Catalan constant ,formulas, series.

1. Introducción

Recordamos algunas fórmulas usuales para la constante de Catalan $(1814-1894)$:

$$G = -\int_0^1 \frac{\ln x}{1+x^2}\, dx \tag{1}$$

$$G = 1 - \sum_{n=1}^{\infty} \frac{n\,\zeta(2n+1)}{16^n} \tag{2}$$

$$G = \frac{1}{2}\int_0^{\infty} \frac{x}{\cosh x}\, dx \tag{3}$$

$$G = \int_0^{\pi/4} \frac{x}{\operatorname{sen} x \cos x}\, dx \tag{4}$$

$$G = \frac{1}{4}\int_0^{\pi/2} \ln\left(\frac{1+\cos x}{1-\cos x}\right) dx \tag{5}$$

$$G = \frac{1}{4}\int_0^{\pi/2} \ln\left(\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right) dx \tag{6}$$

$$G = -2\int_0^{\pi/4} \ln(2 \operatorname{sen} x)\, dx \tag{7}$$

$$G = 2\int_0^{\pi/4} \ln(2 \cos x)\, dx \tag{8}$$

$$G = \sum_{n=0}^{\infty} \frac{2^{2n-1}\,(n!)^2}{(2n)!\,(2n+1)^2} \tag{9}$$

$$G = \frac{\pi}{2}\ln 2 - \frac{\pi}{32}\sum_{n=0}^{\infty} \frac{((2n+1)!)^2}{2^{4n}\,(n!)^4\,(n+1)^3} \tag{10}$$

En esta nota mostramos una colección de fórmulas relacionadas con la constante de Catalan.

2. Algunas fórmulas

- **Entry 1.**

$$G = \int_C \frac{\ln z}{1+z^2}\, dz \;,\; C: \; z = \frac{1}{2} + \frac{1}{2} e^{i\theta} \;,\; 0 \le \theta \le \pi \tag{11}$$

$$\frac{G}{2} - \frac{\pi \ln 2}{8} = \int_C \frac{\ln z}{4+z^2}\, dz \;,\; C: \; z = 1 + e^{i\theta} \;,\; 0 \le \theta \le \pi \tag{12}$$

- **Entry 2.**

$$\frac{G}{2} - \frac{\pi \ln 2}{8} = \sum_{n=0}^{\infty} (-1)^n \, 5^{-n-1} \, c_n \, f_n \tag{13}$$

donde

$$f_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k \, 2^{k+1}}{k+1} \left(\frac{1}{k+1} - \ln 2 \right) \tag{14}$$

$$c_{n+2} = -2\, c_{n+1} - 5\, c_n \;,\; c_0 = 1 \;,\; c_1 = -2 \tag{15}$$

- **Entry 3.**

$$G = \int_0^{\infty} \int_0^1 \frac{e^{-y} \operatorname{sen}(x\, y)}{x\, y}\, dx\, dy \tag{16}$$

$$G = \int_0^{\infty} \frac{e^{-x}}{x} \operatorname{Si}(x)\, dx \tag{17}$$

donde

$$\operatorname{Si}(x) = \int_0^x \frac{\operatorname{sen} t}{t}\, dt \tag{18}$$

$$G = \sum_{n=0}^{\infty} e^{-n} \int_0^1 \frac{e^{-x}}{n+x} \operatorname{Si}(n+x)\, dx \tag{19}$$

- **Entry 4.**

$$G = \lim_{n\to\infty} n \int_0^1 \frac{1}{1+n\, x} \tan^{-1}\left(\frac{1+n\, x}{n+1} \right) dx \tag{20}$$

$$G = \lim_{n\to\infty} \int_0^1 \frac{1}{x} \tan^{-1}\left(\frac{n\, x}{n+1} \right) dx \tag{21}$$

$$G = \int_0^1 \frac{\tan^{-1}(1-x)}{1-x}\, dx \tag{22}$$

$$G = -\int_0^1 \frac{\ln(1-x)}{2 - 2\, x + x^2}\, dx \tag{23}$$

$$G = (n+1) \int_0^1 \int_0^1 \frac{x^n \tan^{-1}(x\, y)}{y}\, dx\, dy + \int_0^1 \int_0^1 \frac{x^{n+1}}{1 + x^2\, y^2}\, dx\, dy \;,\; n \ge 0 \tag{24}$$

$$G = \int_0^{1/3} \frac{\tan^{-1} x}{x\,(1-2\, x)}\, dx + \int_0^{1/2} \frac{\tan^{-1} x}{x\,(1-x)}\, dx \tag{25}$$

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n \, a^{2\,n+1}}{(2\,n+1)^2} + \int_0^1 \frac{1}{x} \tan^{-1}\left(\frac{(1-a)\, x}{1 + a\, x^2} \right) dx \;,\; 0 \le a \le 1 \tag{26}$$

$$G = \frac{1}{m} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)^2} + \int_0^1 \frac{1}{x} \tan^{-1}\left(\frac{x - a x^m}{1 + a x^{m+1}} \right) dx \tag{27}$$

donde $0 \le a \le 1$, $m \in \mathbb{N}$.

$$G = \frac{1}{p} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)^2 \binom{(2n+1)(p+q)}{(2n+1)p}} + \int_0^1 \frac{1}{x} \tan^{-1}\left(\frac{x - a x^p (1-x)^q}{1 + a x^{p+1} (1-x)^q} \right) dx \tag{28}$$

donde $0 \le a \le 1$, $p \in \mathbb{N}$, $q \in \mathbb{N} \bigcup \{0\}$.

$$G = \frac{n+1}{n} \int_0^1 \frac{1}{x} \tan^{-1}\left(\frac{x - x^{n+1}}{1 + x^{n+2}} \right) dx \ , \ n \in \mathbb{N} \tag{29}$$

$$G = 1 - \int_0^1 \frac{1}{x} \tan^{-1}\left(\frac{\tan x - x}{1 + x \tan x} \right) dx \tag{30}$$

$$G = \int_0^{\pi/2} \tan^{-1}(\operatorname{sen} x) \cot x \, dx \tag{31}$$

$$G = \int_0^{\pi/2} \tan^{-1}(\cos x) \tan x \, dx \tag{32}$$

$$G = 2 \int_0^{\infty} \frac{\tan^{-1}(\tanh x)}{\operatorname{senh}(2x)} dx \tag{33}$$

$$G = \int_0^{\infty} \tan^{-1}(e^{-x}) \, dx \tag{34}$$

$$G = \int_1^{e} \frac{\tan^{-1}(\ln x)}{x \ln x} dx \tag{35}$$

$$G = \sum_{n=0}^{\infty} \frac{2^{-2n}}{2n+1} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} \operatorname{Si}(2n - 2k + 1) + \int_0^1 \frac{1}{x} \tan^{-1}\left(\frac{x - \operatorname{sen} x}{1 + x \operatorname{sen} x} \right) dx \tag{36}$$

$$G = \frac{1}{2i} \int_0^1 \int_{1-iy}^{1+iy} \frac{1}{xy} \, dx \, dy \tag{37}$$

$$G = \int_1^{\phi} \frac{(2x-1) \tan^{-1}(x^2 - x)}{x(x-1)} dx \ , \ \ \phi = \frac{1 + \sqrt{5}}{2} \tag{38}$$

$$G = \int_1^{\alpha_m} \frac{(m x^{m-1} - 1) \tan^{-1}(x^m - x)}{x^m - x} dx \tag{39}$$

donde $m \in \mathbb{N} - \{1\}$, $\alpha_m = \sqrt[m]{1 + \sqrt[m]{1 + \sqrt[m]{1 + \ldots}}}$.

$$G = \int_0^1 \int_0^1 \frac{y}{(1 - xy)(2 - 2y + y^2)} dx \, dy \tag{40}$$

$$(a+b)\, G = a \left(\frac{\pi}{4} - \frac{\ln 2}{2} \right) + \int_0^1 \int_0^1 \frac{(axy + b)\, y}{(1 - xy)(2 - 2y + y^2)} dx \, dy \ , \ \ a, \, b \in \mathbb{R} \tag{41}$$

$$\frac{G}{1 + a^2} - \frac{a \pi^2}{48 (1 + a^2)} - \frac{a}{1 + a^2} \operatorname{Li}_2(-a) = \int_0^1 \int_0^1 \frac{1}{(1 + axy)(1 + x^2 y^2)} dx \, dy \tag{42}$$

donde $-1 < a < 1$, $\operatorname{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$, $|x| \le 1$.

■ Entry 5.

$$G = \sum_{n=0}^{\infty} a_n A_n \tag{43}$$

donde

$$a_{2n} = (-1)^n \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}, \quad n \in \mathbb{N} \bigcup \{0\} \tag{44}$$

$$a_{2n+1} = (-1)^n \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}, \quad n \in \mathbb{N} \bigcup \{0\} \tag{45}$$

$$A_n = \int_0^1 \int_0^1 e^{-xy} (xy)^n \, dx \, dy \ , \quad n \in \mathbb{N} \bigcup \{0\} \tag{46}$$

$$A_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k+1)^2} \ , \quad n \in \mathbb{N} \bigcup \{0\} \tag{47}$$

$$A_n = \frac{1}{(n+1)^2} F(n+1,\ n+1;\ n+2,\ n+2;\ -1) \ , \quad n \in \mathbb{N} \bigcup \{0\} \tag{48}$$

$$A_n = -\gamma(n,\ 1) + n\, A_{n-1} \ , \quad n \in \mathbb{N} \tag{49}$$

$$\gamma(n,\ 1) = (n-1)! \left(1 - e^{-1} \sum_{m=0}^{n-1} \frac{1}{m!} \right), \quad n \in \mathbb{N} \tag{50}$$

En (48) $F(a,\ b;\ c;\ d;\ x)$ es la función hipergeométrica.

■ Entry 6.

$$G = \sum_{n=0}^{\infty} b_n B_n \tag{51}$$

donde

$$b_{2n} = (-1)^n \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}, \quad n \in \mathbb{N} \bigcup \{0\} \tag{52}$$

$$b_{2n+1} = (-1)^{n+1} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}, \quad n \in \mathbb{N} \bigcup \{0\} \tag{53}$$

$$B_n = \int_0^1 \int_0^1 e^{xy} (xy)^n \, dx \, dy \ , \quad n \in \mathbb{N} \bigcup \{0\} \tag{54}$$

$$B_n = \sum_{k=0}^{\infty} \frac{1}{k!\,(n+k+1)^2} \ , \quad n \in \mathbb{N} \bigcup \{0\} \tag{55}$$

$$B_n = \frac{1}{(n+1)^2} F(n+1,\ n+1;\ n+2,\ n+2;\ 1) \ , \quad n \in \mathbb{N} \bigcup \{0\} \tag{56}$$

$$B_n = (-1)^n \gamma(n,\ -1) - n\, B_{n-1} \ , \quad n \in \mathbb{N} \tag{57}$$

$$\gamma(n,\ -1) = (n-1)! \left(1 - e \sum_{m=0}^{n-1} \frac{(-1)^m}{m!} \right), \quad n \in \mathbb{N} \tag{58}$$

En (56) $F(a,\ b;\ c;\ d;\ x)$ es la función hipergeométrica.

■ **Entry 7.**

$$G=\sum_{n=0}^{\infty}(-1)^n c_n C_n \tag{59}$$

donde

$$c_n=\sum_{k=0}^{n}\frac{1}{k!}\ ,\ \ n\in\mathbb{N}\bigcup\{0\} \tag{60}$$

$$C_n=\int_0^1\int_0^1 e^{(x\,y)^2}(x\,y)^{2\,n}\,dx\,dy\ ,\ \ n\in\mathbb{N}\bigcup\{0\} \tag{61}$$

$$C_n=\sum_{k=0}^{\infty}\frac{1}{k!\,(2\,n+2\,k+1)^2}\ ,\ \ n\in\mathbb{N}\bigcup\{0\} \tag{62}$$

$$C_n=\frac{1}{(2\,n+1)^2}\,F\left(n+\frac{1}{2},\ n+\frac{1}{2};\,n+\frac{3}{2},\ n+\frac{3}{2};\,1\right)\ ,\ \ n\in\mathbb{N}\bigcup\{0\} \tag{63}$$

$$C_n=\frac{1}{4}\,(-1)^{-(n-1/2)}\,\gamma\left(n-\frac{1}{2},\ -1\right)-\frac{2\,n-1}{2}\,C_{n-1}\ \ ,\ \ n\in\mathbb{N} \tag{64}$$

En (63) $F(a,\ b;\,c;\,d;\,x)$ es la función hipergeométrica.

■ **Entry 8.**

$$G=\sum_{n=0}^{\infty}(-1)^n d_n D_n \tag{65}$$

donde

$$d_n=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\ ,\ \ n\in\mathbb{N}\bigcup\{0\} \tag{66}$$

$$D_n=\int_0^1\int_0^1 e^{-(x\,y)^2}(x\,y)^{2\,n}\,dx\,dy\ ,\ \ n\in\mathbb{N}\bigcup\{0\} \tag{67}$$

$$D_n=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(2\,n+2\,k+1)^2}\ ,\ \ n\in\mathbb{N}\bigcup\{0\} \tag{68}$$

$$D_n=\frac{1}{(2\,n+1)^2}\,F\left(n+\frac{1}{2},\ n+\frac{1}{2};\,n+\frac{3}{2},\ n+\frac{3}{2};\,-1\right)\ ,\ \ n\in\mathbb{N}\bigcup\{0\} \tag{69}$$

$$D_n=-\frac{1}{4}\,\gamma\left(n-\frac{1}{2},\ 1\right)+\frac{2\,n-1}{2}\,D_{n-1}\ \ ,\ \ n\in\mathbb{N} \tag{70}$$

En las fórmulas (49), (57), (64), (70) , aparece la función gamma incompleta :

$$\gamma(a,\ x)=\int_0^x t^{a-1}\,e^{-t}\,dt \tag{71}$$

En (69) $F(a,\ b;\,c;\,d;\,x)$ es la función hipergeométrica.

■ **Entry 9.**

Sea $m\in\mathbb{N}$, se tiene :

$$G=\sum_{n=0}^{\infty}c(m,\ n)\,I(m,\ n) \tag{72}$$

donde

$$c(m,\ n) = \binom{n+m-1}{n} - c(m,\ n-2)\ ,\ \ n \in \mathbb{N} - \{1\} \tag{73}$$

$$c(m,\ 0) = 1\ ,\ \ c(m,\ 1) = m \tag{74}$$

$$c(m,\ 2\,n) = (-1)^n \sum_{k=0}^{n} \frac{(-1)^k\,(m)_{2\,k}}{(2\,k)\,!}\ ,\ \ n \in \mathbb{N} \bigcup \{0\} \tag{75}$$

$$c(m,\ 2\,n+1) = (-1)^n \sum_{k=0}^{n} \frac{(-1)^k\,(m)_{2\,k+1}}{(2\,k+1)\,!}\ ,\ \ n \in \mathbb{N} \bigcup \{0\} \tag{76}$$

$$c(m,\ 2\,n) = \frac{(-1)^n}{(m-1)\,!} \sum_{k=0}^{n} (-1)^k\,(2\,k+1)_{m-1}\ ,\ \ n \in \mathbb{N} \bigcup \{0\} \tag{77}$$

$$c(m,\ 2\,n+1) = \frac{(-1)^n}{(m-1)\,!} \sum_{k=0}^{n} (-1)^k\,(2\,k+2)_{m-1}\ ,\ \ n \in \mathbb{N} \bigcup \{0\} \tag{78}$$

$$I(m,\ n) = \int_0^1 \int_0^1 (1-x\,y)^m\,(x\,y)^n\,dx\,dy\ ,\ \ n \in \mathbb{N} \bigcup \{0\} \tag{79}$$

$$I(m,\ n) = \sum_{k=0}^{n} \binom{m}{k} \frac{(-1)^k}{(n+k+1)^2}\ ,\ \ n \in \mathbb{N} \bigcup \{0\} \tag{80}$$

$$I(m,\ n) = \int_0^1 \int_0^y \frac{(1-x)^m\,x^n}{y}\,dx\,dy\ ,\ \ n \in \mathbb{N} \bigcup \{0\} \tag{81}$$

$$I(m,\ n) = \int_0^1 \int_{1-y}^1 \frac{(1-x)^n\,x^m}{y}\,dx\,dy\ ,\ \ n \in \mathbb{N} \bigcup \{0\} \tag{82}$$

$$0 \le I(m+1,\ n+1) \le \frac{1}{4}\,I(m,\ n)\ ,\ \ m \in \mathbb{N},\ n \in \mathbb{N} \bigcup \{0\} \tag{83}$$

$$0 \le I(m,\ n) \le \frac{1}{(n+1)^2}\ ,\ \ m \in \mathbb{N},\ n \in \mathbb{N} \bigcup \{0\} \tag{84}$$

$$0 \le I(m,\ n+1) \le I(m,\ n)\ ,\ \ m \in \mathbb{N},\ n \in \mathbb{N} \bigcup \{0\} \tag{85}$$

$$0 \le I(m,\ n) \le \left(\frac{m}{n+m}\right)^m \left(\frac{n}{n+m}\right)^n \le 1\ ,\ \ m \in \mathbb{N},\ n \in \mathbb{N} \bigcup \{0\} \tag{86}$$

$$\sum_{n=0}^{\infty} I(1,\ n) = 1 \tag{87}$$

$$\sum_{n=0}^{\infty} I(m,\ n) = \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(-1)^k}{(k+1)^2}\ ,\ \ m \in \mathbb{N} \tag{88}$$

$$\sum_{n=0}^{\infty} (-1)^n\,I(m,\ n) = \int_0^1 \int_0^1 \frac{(1-x\,y)^m}{1+x\,y}\,dx\,dy\ ,\ \ m \in \mathbb{N} \tag{89}$$

$$\int_0^1 \int_0^1 \frac{(1-x\,y)^m}{1+x\,y}\,dx\,dy = \frac{2^m\,\pi^2}{12} + \sum_{k=1}^{m} (-1)^k \binom{m}{k} \frac{2^{m-k}}{k} \sum_{j=1}^{k} \binom{k}{j} \frac{1}{j}\ ,\ \ m \in \mathbb{N} \tag{90}$$

■ Entry 10.

$$G = \frac{\pi}{4} \ln\!\left(2+\sqrt{3}\right) - \frac{3}{2} \sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n f_n \tag{91}$$

donde

$$f_n = \int_0^{\pi/6} x\,(\operatorname{sen} x)^{2n-1}\, dx \ , \ \ n \in \mathbb{N} \tag{92}$$

$$f_n = \frac{(1/2)^{2n-2}}{(2n-1)^2}\left(\frac{1}{2} - (2n-1)\,\frac{\pi\sqrt{3}}{12}\right) + \frac{2n-2}{2n-1}\, f_{n-1} \ , \ \ n \in \mathbb{N} - \{1\} \tag{93}$$

$$f_1 = \frac{1}{2} - \frac{\pi\sqrt{3}}{12} \tag{94}$$

■ **Entry 11.**

$$G = \frac{\pi^2}{8} - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\int_0^{\pi/2}\cos\!\left(\frac{(2n+1)\,x}{\operatorname{sen} x}\right) dx \tag{95}$$

$$G = \frac{\pi}{8}\ln\!\left(2+\sqrt{3}\right) + \frac{\pi^2}{16} - \frac{3}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\int_0^{\pi/6}\cos\!\left(\frac{(2n+1)\,x}{\operatorname{sen} x}\right) dx \tag{96}$$

■ **Entry 12.**

$$G = \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{a_n\,(\pi/4)^{n+1}\,(1+(-1)^n)}{n+1} = \sqrt{2}\sum_{n=0}^{\infty}\frac{a_{2n}\,(\pi/4)^{2n+1}}{2n+1} \tag{97}$$

donde

$$a_0 = \frac{\pi}{4} \ , \ \ a_n = c_{n-1} + c_n\,\frac{\pi}{4} \tag{98}$$

$$c_0 = 1 \ , \ \ c_n = -\sum_{k=1}^{n}\frac{(-1)^{[k/2]}}{k!}\, c_{n-k} \tag{99}$$

$$G = \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{c_n\,(\pi/4)^{n+2}\,(2n+3+(-1)^n)}{(n+1)\,(n+2)} \tag{100}$$

■ **Entry 13.**

$$G = \frac{4}{5}\sum_{n=0}^{\infty} c_n\, 5^{-n}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{1}{2}\right)^{n-k}(k+1)^{-2} \tag{101}$$

donde

$$c_{n+2} = -4\,c_{n+1} - 20\,c_n \ , \ \ c_0 = 1 \ , \ \ c_1 = -4 \tag{102}$$

$$c_n = 2^{n-2}\,((2-i)\,(-1-2\,i)^n + (2+i)\,(-1+2\,i)^n) \ , \ \ n \in \mathbb{N} \cup \{0\} \tag{103}$$

■ **Entry 14.**

$$G = \frac{4}{5}\sum_{n=0}^{\infty}\left(-\frac{4}{5}\right)^n\sum_{k=0}^{n}\binom{n}{k}\sum_{m=0}^{n+k}\binom{n+k}{m}\left(-\frac{1}{2}\right)^{n+k-m}(m+1)^{-2} \tag{104}$$

■ **Entry 15.**

$$G = \int_0^{\pi/4}\int_0^{\pi/4}\frac{1}{(\cos x\cos y)^2 + (\operatorname{sen} x\operatorname{sen} y)^2}\, dx\, dy \tag{105}$$

$$G = \int_0^{\pi/4}\int_0^{\pi/4}\frac{1}{1 - (\operatorname{sen} x)^2 - (\operatorname{sen} y)^2 + 2\,(\operatorname{sen} x\operatorname{sen} y)^2}\, dx\, dy \tag{106}$$

$$G = \int_0^{\infty}\int_0^{\infty}\frac{1}{(\cosh x\cosh y)^2 + (\operatorname{senh} x\operatorname{senh} y)^2}\, dx\, dy \tag{107}$$

$$G = \int_0^\infty \int_0^\infty \frac{1}{1 + (\operatorname{senh} x)^2 + (\operatorname{senh} y)^2 + 2\,(\operatorname{senh} x \operatorname{senh} y)^2}\, dx\, dy \tag{108}$$

■ **Entry 16.**

$$G = \frac{\pi^2}{8} + \frac{\pi \ln 2}{4} - 4\left(\sqrt{2} - 1\right) \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(2\sqrt{2} - 2\right)^k \sum_{m=0}^{k} \binom{k}{m} f(k, m) \tag{109}$$

donde

$$f(k, m) = \int_0^{\pi/2} x\,(\operatorname{sen} x)^{k-m} (\cos x)^{m+1}\, dx \tag{110}$$

■ **Entry 17.**

$$G = -\frac{\pi^2}{8} + \frac{\pi \ln 2}{4} + 4\left(\sqrt{2} - 1\right) \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(2\sqrt{2} - 2\right)^k \sum_{m=0}^{k} \binom{k}{m} f(k, m) \tag{111}$$

donde

$$f(k, m) = \int_0^{\pi/2} x\,(\operatorname{sen} x)^{m+1} (\cos x)^{k-m}\, dx \tag{112}$$

■ **Entry 18.**

$$G = \frac{\pi}{2} \ln\left(\frac{2 + \sqrt{2}}{2}\right) - 2 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(4 - 2\sqrt{2}\right)^k f(k) \tag{113}$$

donde

$$f(k) = \int_0^{\pi/4} (\cos x)^k\, dx \;,\; k \in \mathbb{N} \cup \{0\} \tag{114}$$

$$f(k) = \frac{1}{k}\left(\frac{1}{\sqrt{2}}\right)^k + \frac{k-1}{k} f(k-2) \;,\; f(0) = \frac{\pi}{4} \;,\; f(1) = \frac{1}{\sqrt{2}} \tag{115}$$

■ **Entry 19.**

$$G = -\frac{\pi \ln 2}{4} + 2\int_0^\infty \operatorname{sen}^{-1}\left(\frac{e^{-x}}{2}\right) dx + \int_{-\ln 2}^{0} \operatorname{sen}^{-1}\left(\frac{e^{-x/2}}{2}\right) dx \tag{116}$$

■ **Entry 20.**

$$G = 1 + \int_0^\Omega \frac{\tan^{-1} x}{x}\, dx - \int_\Omega^1 \frac{1}{x} \tan^{-1}\left(\frac{\tan(1/\Omega) - x}{1 + x\tan(1/\Omega)}\right) dx \tag{117}$$

donde

$$\Omega = 0.56714329\ldots \;,\; \Omega = e^{-\Omega} \;,\; \Omega : \text{ omega constant} \tag{118}$$

■ **Entry 21.**

$$G = s_n + \int_0^1 \frac{1}{x} \tan^{-1}\left(\frac{x - \tan(s_n x)}{1 + x\tan(s_n x)}\right) dx \;,\; n \in \mathbb{N} \tag{119}$$

donde para s_n se tienen distintas alternativas :

$$s_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(2k-1)^2} \;,\; n \in \mathbb{N} \tag{120}$$

$$s_n = \left\{1, \frac{10}{11}, \frac{11}{12}, \frac{98}{107}, \frac{109}{119}, \frac{9690}{10\,579}, \frac{38\,869}{42\,435}, \ldots\right\} \tag{121}$$

en (121) s_n son los convergentes de la fracción continua para G.

$$s_n = \left\{\frac{\pi}{3}, \frac{2\pi}{7}, \frac{7\pi}{24}, \frac{114\pi}{391}, \frac{919\pi}{3152}, \ldots\right\} = \{c_n\,\pi\} \tag{122}$$

en (122) c_n son los convergentes de la fracción continua para G/π.

■ Entry 22.

$$G = f(a) + \theta \ln a + \int_{1/a}^{1} \frac{1}{x} \tan^{-1}\left(\frac{x - \tan\theta}{1 + x\tan\theta}\right) dx\ ,\ \ a \geq 1 \tag{123}$$

donde

$$f(a) = \int_0^{1/a} \frac{\tan^{-1} x}{x}\, dx \tag{124}$$

$$f(a) = \frac{1}{a}\, {}_3F_2\left(1/2,\ 1/2,\ 1/2\,;\, 3/2,\ 3/2\,;\, -a^{-2}\right) \tag{125}$$

$$f(a) = \frac{i}{2}\left(\mathrm{Li}_2\left(-\frac{i}{a}\right) - \mathrm{Li}_2\left(\frac{i}{a}\right)\right) \tag{126}$$

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}\ ,\ \ |z| \leq 1 \tag{127}$$

Ejemplos :

$$G = f(2) + \frac{\pi}{12}\ln 2 + \int_{1/2}^{1} \frac{1}{x}\tan^{-1}\left(\frac{x - 2 + \sqrt{3}}{1 + x\left(2 - \sqrt{3}\right)}\right) dx \tag{128}$$

$$G = f(2) + \frac{\pi}{8}\ln 2 + \int_{1/2}^{1} \frac{1}{x}\tan^{-1}\left(\frac{x - \sqrt{2} + 1}{1 + x\left(\sqrt{2} - 1\right)}\right) dx \tag{129}$$

$$G = f(2) + \frac{\pi}{6}\ln 2 + \int_{1/2}^{1} \frac{1}{x}\tan^{-1}\left(\frac{x\sqrt{3} - 1}{x + \sqrt{3}}\right) dx \tag{130}$$

$$G = f(2) + \frac{\pi}{4}\ln 2 + \int_{1/2}^{1} \frac{1}{x}\tan^{-1}\left(\frac{x - 1}{x + 1}\right) dx \tag{131}$$

$$G = f(2) + \frac{\pi}{3}\ln 2 + \int_{1/2}^{1} \frac{1}{x}\tan^{-1}\left(\frac{x - \sqrt{3}}{1 + x\sqrt{3}}\right) dx \tag{132}$$

En (125) ${}_3F_2\,(a,\ b,\ c\,;\,d,\ e\,;\,x)$ es la función hipergeométrica.

■ Entry 23.

$$G = \frac{\pi \ln 3}{4} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{3}\right)^{n-k} \left(\frac{2}{3}\right)^{k} f(k) \tag{133}$$

donde

$$f(k) = \int_0^{\pi} (\operatorname{sen} x)^k\, dx\ ,\ \ k \in \mathbb{N} \cup \{0\} \tag{134}$$

$$f(k) = \frac{k-1}{k}\, f(k-2)\ ,\ \ f(0) = \pi\ ,\ \ f(1) = 2 \tag{135}$$

■ **Entry 24.**

$$G = \frac{\pi \ln\left(2+\sqrt{3}\right)}{4} - \frac{3}{2}\int_{1/2}^{1} \frac{1}{x\sqrt{1-x^2}} \tan^{-1}\left(\frac{\sqrt{3-3x^2} - x}{\sqrt{1-x^2} + x\sqrt{3}}\right) dx \tag{136}$$

$$G = -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\infty}\left(\frac{\pi}{2} - \operatorname{sen}^{-1}(1-e^{-x})\right) dx \tag{137}$$

$$G = -\frac{\pi}{4}\ln 2 + \int_0^{\infty}\left(\frac{\pi}{2} - \tan^{-1}(e^{x}-1)\right) dx \tag{138}$$

$$G = -\frac{\pi}{4}\ln 2 + 2\int_0^{\infty}\left(\frac{\pi}{4} - \operatorname{sen}^{-1}\left(\frac{\sqrt{2-e^{-2x}} - e^{-x}}{2}\right)\right) dx \tag{139}$$

$$G = \frac{\pi}{2}\ln 2 - \int_0^{\ln 2} \operatorname{sen}^{-1}\left(\frac{e^{x/2} - \sqrt{2-e^{x}}}{2}\right) dx \tag{140}$$

$$G = \frac{\pi}{4}\ln 2 + \int_0^{\pi^2/16} \tan\left(\sqrt{x}\right) dx \tag{141}$$

■ **Entry 25.**

$$G = \frac{\pi}{4}(3\ln 2 - \ln 3) - \frac{2}{3}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{(-1)^k}{n+1}\sum_{m=0}^{k+1}\binom{k+1}{m}\left(-\frac{1}{3}\right)^{k-m+1}\left(\frac{2}{3}\right)^{m}(2m+1)^{-1} \tag{142}$$

■ **Entry 26.**

$$\frac{\pi}{4}\ln 2 + G = \int_0^{1}\frac{x\,e^{-x}}{1-2e^{-x}+2e^{-2x}}\,dx + \sum_{n=0}^{\infty}\frac{e^{-n-1}(n+2)\,c_n}{(n+1)^2} \tag{143}$$

donde

$$c_{n+2} = 2c_{n+1} - 2c_n\,,\;\; c_0 = 1,\; c_1 = 2 \tag{144}$$

$$c_n = \left(\frac{1+i}{2}\right)((1-i)^n - i(1+i)^n)\,,\;\; n\in\mathbb{N}\cup\{0\}\,,\;\; i=\sqrt{-1} \tag{145}$$

■ **Entry 27.**

$$G = \frac{\pi}{4}\ln 2 + 4\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^{\pi/4}\left(\frac{\cos x + \operatorname{sen} x - 1}{\cos x + \operatorname{sen} x + 1}\right)^{2n+1} dx \tag{146}$$

$$G = \frac{\pi}{2}\ln 2 - \sum_{n=0}^{\infty}\int_0^{3-2\sqrt{2}} x^{2n}\operatorname{sen}^{-1}\left(\frac{1+x-\sqrt{1-6x+x^2}}{2-2x}\right) dx \tag{147}$$

■ **Entry 28.**

$$G = \frac{\pi}{2}\ln 2 - 2\int_0^1\int_0^1\frac{x\,y^2}{(1+x^2y^2)(1+y^2)}\,dx\,dy \tag{148}$$

■ **Entry 29.**

$$G = \frac{3\pi}{8}\ln 2 + 2\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^1\left(\frac{x-x^2}{2+x+x^2}\right)^{2n+1}\frac{1}{1+x^2}\,dx \tag{149}$$

$$I_n = \int_0^1\left(\frac{x-x^2}{2+x+x^2}\right)^{2n+1}\frac{1}{1+x^2}\,dx\,,\;\; n\in\mathbb{N}\cup\{0\} \tag{150}$$

$$I_0 = \frac{\pi}{4} - \frac{4}{\sqrt{7}} \tan^{-1}\left(\frac{\sqrt{7}}{5}\right) \tag{151}$$

$$I_1 = \frac{\pi}{4} + \frac{2}{343}\left(7 - 110\sqrt{7}\tan^{-1}\left(\frac{\sqrt{7}}{5}\right)\right) \tag{152}$$

$$I_2 = \frac{\pi}{4} + \frac{1}{7203}\left(2177 - 6084\sqrt{7}\tan^{-1}\left(\frac{\sqrt{7}}{5}\right)\right) \tag{153}$$

■ Entry 30.

$$G = \frac{3\pi}{8}\ln 2 + \frac{4}{9}\sum_{n=0}^{\infty} 3^{-2n} \sum_{k=0}^{n} 3^k \sum_{m=0}^{n-k} \binom{2n-k-2m}{k} \frac{3^m f(n,k,m)}{2n-2k-2m+1} \tag{154}$$

donde

$$f(n,k,m) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} \sum_{s=0}^{r} \binom{r}{s} \sum_{t=0}^{m} (-1)^t 2^t \binom{m}{t} g(n,k,m,r,s,t) \tag{155}$$

$$g(n,k,m,r,s,t) = \frac{(2n-2k-2m+r+s+t+1)!\,(2n-2k-2m+1)!}{(4n-4k-4m+r+s+2t+3)!} \tag{156}$$

■ Entry 31.

$$G = -1 + \frac{\pi}{4}\ln 2 + \frac{\pi}{2} - \frac{\pi^2}{16} + 3\sum_{n=0}^{\infty}(n+1)\int_0^{\pi/4} x^2 (\operatorname{sen} x)^{2n+2}\,dx \tag{157}$$

$$I_n = \int_0^{\pi/4} x^2 (\operatorname{sen} x)^{2n+2}\,dx\ ,\ \ n \in \mathbb{N} \cup \{0\} \tag{158}$$

$$I_n = \frac{\pi}{2^{n+5}(n+1)^2}\left(2 - \frac{(n+1)\pi}{2}\right) + \frac{2n+1}{2n+2} I_{n-1} - \frac{J_n}{2(n+1)^2}\ ,\ \ n \in \mathbb{N} \tag{159}$$

$$J_n = -\frac{2^{-n-2}}{n+1} + \frac{2n+1}{2n+2} J_{n-1}\ ,\ \ n \in \mathbb{N} \tag{160}$$

$$I_0 = \frac{48 - 6\pi^2 + \pi^3}{384}\ ,\ \ J_0 = \frac{\pi - 2}{8} \tag{161}$$

■ Entry 32.

$$G = \frac{\pi}{2}\ln 2 - 2\left(2 + \sqrt{2}\right)\sum_{n=0}^{\infty} \frac{\left(\sqrt{2} - 1\right)^{4n+3} F\left(1, 1; 2n + (5/2); \left(2 - \sqrt{2}\right)/4\right)}{(2n+1)(4n+3)} \tag{162}$$

donde $F(a, b; c; x)$ es la función hipergeométrica.

■ Entry 33.

$$G = \frac{\pi}{2}\ln 2 - 8\sum_{n=0}^{\infty} \frac{c_n}{2n+1} \tag{163}$$

donde

$$c_n = -\frac{\pi}{8} + \sqrt{2} - 1 - \sum_{k=1}^{n}\left(\frac{\left(\sqrt{2}-1\right)^{4k-1}}{4k-1} - \frac{\left(\sqrt{2}-1\right)^{4k+1}}{4k+1}\right) \tag{164}$$

$$c_n = p_n \sqrt{2} - q_n - \frac{\pi}{8} \tag{165}$$

$$p_{n+1} = \left(\frac{3}{4n+5} - \frac{1}{4n+3}\right) a_n + \frac{2}{4n+5} b_n + p_n \tag{166}$$

$$q_{n+1} = \frac{4}{4n+5} a_n + \left(\frac{3}{4n+5} - \frac{1}{4n+3}\right) b_n + q_n \tag{167}$$

$$a_{n+1} = 17 a_n + 12 b_n \tag{168}$$

$$b_{n+1} = 24 a_n + 17 b_n \tag{169}$$

$$p_0 = 1,\ q_0 = 1,\ a_0 = 5,\ b_0 = 7 \tag{170}$$

■ **Entry 34.**

$$G = \frac{\pi}{2} \ln 2 - 8 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n+k} \left(\sqrt{2} - 1\right)^{2n+2k+3}}{(2k+1)(2n+2k+3)} \tag{171}$$

$$G = \frac{\pi}{2} \ln 2 - 8 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\sqrt{2} - 1\right)^{2n+3}}{2n+3} \sum_{k=0}^{[n/2]} \frac{1}{2k+1} \tag{172}$$

■ **Entry 35.**

$$G = \frac{\pi}{4} \ln 2 + 4\left(2 - \sqrt{2}\right) \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(4 - 2\sqrt{2}\right)^k f(k) \tag{173}$$

donde

$$f(k) = \int_0^{\pi/4} x \operatorname{sen} x (\cos x)^k \, dx \ , \ \ k \in \mathbb{N} \cup \{0\} \tag{174}$$

■ **Entry 36.**

$$G = \pi \left(\frac{1}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \left(-1 + n \ln\left(\frac{2n+1}{2n-1}\right)\right)\right) \tag{175}$$

$$G = \frac{\pi \ln\left(2 + \sqrt{3}\right)}{8} + \frac{3}{4} \pi \left(\frac{1}{6} + \sum_{n=1}^{\infty} (-1)^{n-1} \left(-\frac{1}{3} + n \ln\left(\frac{6n+1}{6n-1}\right)\right)\right) \tag{176}$$

■ **Entry 37.**

$$G = \frac{1}{2} \int_0^{\pi/2} \sec\left(\frac{x}{2}\right) \sec\left(\frac{x}{4}\right) \sec\left(\frac{x}{8}\right) \sec\left(\frac{x}{16}\right) \ldots \, dx \tag{177}$$

$$g_n = \frac{1}{2} \int_0^{\pi/2} \prod_{k=1}^{n} \sec\left(\frac{x}{2^k}\right) dx \ , \ \ g_n \to G \tag{178}$$

■ **Entry 38.**

$$G = \frac{3\pi}{8} \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k F(n+1, (n+k+1)/2; (n+k+3)/2; -1)}{n+k+1} \tag{179}$$

$$G = \frac{3\pi}{8} \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{-n-1}}{n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k F(n+1, 1; (n+k+3)/2; 1/2)}{n+k+1} \tag{180}$$

$$G = \frac{3\pi}{8} \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k 2^{-(n+k+1)/2} f(n, k)}{n+k+1} \tag{181}$$

$$f(n,k)=F((n+k+1)/2,\ -(n-k-1)/2;\ (n+k+3)/2;\ 1/2) \tag{182}$$

$$G=\frac{3\pi}{8}\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,2^{-n}}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\,F(-(n-k-1)/2,\ 1;\ (n+k+3)/2;\ -1)}{n+k+1} \tag{183}$$

donde $F(a,\ b\ ;\ c\ ;\ x)$ es la función hipergeométrica.

■ Entry 39.

$$G=\int_0^1\int_0^1\frac{2x}{(1+x^2)(1-x^2y^2)}\,dx\,dy \tag{184}$$

$$G+\frac{\pi}{4}\ln 2=\int_0^1\int_0^1\frac{1+2xy+x^2}{(1+x^2)(1+xy)(x+y)}\,dx\,dy \tag{185}$$

$$G=\frac{3\pi}{8}\ln 2+\int_0^1\int_0^1\frac{x-x^2}{(1+x^2)(1+xy)(1+x^2y)}\,dx\,dy \tag{186}$$

■ Entry 40.

$$G=\frac{3\pi}{8}\ln 2+\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1\left(\frac{x-x^2}{1+x}\right)^n\frac{1}{1+x^2}\,dx \tag{187}$$

$$G=\frac{3\pi}{8}\ln 2+\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1\frac{x^n-x^{2n}}{(1+x+x^2)^n}\,\frac{1}{1+x^2}\,dx \tag{188}$$

■ Entry 41.

$$G=\frac{3\pi}{8}\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\binom{2n}{n}^{-1}\frac{{}_3F_2\,(n+1,\ (n+1)/2,\ (n+2)/2;\ n+1,\ (2n+3)/2;\ -1)}{2n+1} \tag{189}$$

donde ${}_3F_2\,(a,\ b,\ c\ ;\ d,\ e\ ;\ x)$ es la función hipergeométrica.

■ Entry 42.

$$G=\frac{3\pi}{8}\ln 2+\mathrm{s1}+\mathrm{s2} \tag{190}$$

donde

$$\mathrm{s1}=\sum_{n=0}^{\infty}2^{-n-2}\,a_n \tag{191}$$

$$\mathrm{s2}=\sum_{n=0}^{\infty}2^{-2n-3}\,(-1)^n\,b_n \tag{192}$$

$$a_n=\sum_{k=0}^{n}\frac{1}{n-k+1}\left(\sum_{m=0}^{n+1}\binom{n+1}{m}\frac{(-1)^m\,2^{-2m-1}}{n+m+2}-\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r\,2^{-n+k-2r-2}}{n+r+2}\right) \tag{193}$$

$$b_n=\frac{1}{n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}-\frac{1}{2n+3}\sum_{k=0}^{n}\frac{2+(-1)^k}{2k+2} \tag{194}$$

■ Entry 43.

$$G=\frac{\pi}{12}+\int_0^{1/\sqrt{3}}\frac{\tan^{-1}x}{x}\,dx+\int_a^b\frac{1}{1-x}\tan^{-1}\!\left(\frac{1}{1-x}\tan^{-1}\!\left(\frac{1}{1-x}\tan^{-1}\!\left(\frac{1}{1-x}\ ...\right)\right)\right)dx \tag{195}$$

donde

$$a = 1 - \frac{\pi\sqrt{3}}{6}, \quad b = 1 - \frac{\pi}{4} \tag{196}$$

■ **Entry 44.**

$$G = \frac{\pi}{2} - 1 + \frac{\pi}{4}\ln 2 - 2\int_0^1\int_0^1 \frac{x^4}{(1+x^2)(1+x^2 y^2)}\,dx\,dy \tag{197}$$

■ **Entry 45.**

$$G + \frac{\pi}{4}\ln 2 = \sum_{n=0}^{\infty} 2^{-3n} \sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k}(2k+1)^{-2} \tag{198}$$

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