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A Generalized Pignistic Transformation

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Abstract: *This chapter introduces a generalized pignistic transformation (GPT) developed in the DSmT framework as a tool for decision-making at the pignistic level. The GPT allows to construct quite easily a subjective probability measure from any generalized basic belief assignment provided by any corpus of evidence. We focus our presentation on the 3D case and we provide the full result obtained by the proposed GPT and its validation drawn from the probability theory.*

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7.1 A short introduction to the DSm cardinality

One important notion involved in the definition of the Generalized Pignistic Transformation (GPT) is the *DSm cardinality* introduced in chapter 3 (section 3.2.2) and in [1]. The *DSm cardinality* of any element A of hyper-power set D^Θ, denoted $\mathcal{C}_\mathcal{M}(A)$, corresponds to the number of parts of A in the corresponding fuzzy/vague Venn diagram of the problem (model $\mathcal{M}$) taking into account the set of integrity constraints (if any), i.e. all the possible intersections due to the nature of the elements θ_i. This *intrinsic cardinality* depends on the model $\mathcal{M}$ (free, hybrid or Shafer's model). $\mathcal{M}$ is the model that contains A, which depends both on the dimension $n = |\Theta|$ and on the number of non-empty intersections present in its associated Venn diagram. The DSm cardinality depends on the cardinal of $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ and on the model of D^Θ (i.e., the number of intersections and between what elements of Θ - in a word the structure) at the same time; it is not necessarily that every singleton, say θ_i, has the same DSm cardinal, because each singleton has a different structure; if its structure is the simplest (no intersection of this elements with other elements) then $\mathcal{C}_\mathcal{M}(\theta_i) = 1$, if the structure is more complicated (many intersections) then $\mathcal{C}_\mathcal{M}(\theta_i) > 1$; let's consider a singleton θ_i: if it has 1 intersection only then $\mathcal{C}_\mathcal{M}(\theta_i) = 2$, for 2 intersections only $\mathcal{C}_\mathcal{M}(\theta_i)$ is 3 or 4 depending on the model $\mathcal{M}$, for m intersections it is between $m+1$ and 2^m depending on the model; the maximum DSm cardinality is 2^{n-1} and occurs for $\theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ in the free model $\mathcal{M}^f$; similarly for any set from D^Θ: the more complicated structure it has, the bigger is the DSm cardinal; thus the DSm cardinality measures the complexity of en element from D^Θ, which is a nice characterization in our opinion; we may say that for the singleton θ_i not even $|\Theta|$ counts, but only its structure (= how many other singletons intersect θ_i). Simple illustrative examples have already been presented in chapter 3. One has $1 \leq \mathcal{C}_\mathcal{M}(A) \leq 2^n - 1$. $\mathcal{C}_\mathcal{M}(A)$ must not be confused with the classical cardinality $|A|$ of a given set A (i.e. the number of its distinct elements) - that's why a new notation is necessary here.

It has been shown in [1], that $\mathcal{C}_\mathcal{M}(A)$, is exactly equal to the sum of the elements of the row of $\mathbf{D}_n$ corresponding to proposition A in the $\mathbf{u}_n$ basis (see chapter 2). Actually $\mathcal{C}_\mathcal{M}(A)$ is very easy to compute by programming from the algorithm of generation of D^Θ given in chapter 2 and in [2].

If one imposes a constraint that a set B from D^Θ is empty (i.e. we choose a hybrid DSm model), then one suppresses the columns corresponding to the parts which compose B in the matrix $\mathbf{D}_n$ and the row of B and the rows of all elements of D^Θ which are subsets of B, getting a new matrix $\mathbf{D'}_n$ which represents a new hybrid DSm model $\mathcal{M}'$. In the $\mathbf{u}_n$ basis, one similarly suppresses the parts that form B, and now this basis has the dimension $2^n - 1 - \mathcal{C}_\mathcal{M}(B)$.

7.2 The Classical Pignistic Transformation (CPT)

We follow here Smets' point of view [8] about the assumption that beliefs manifest themselves at two mental levels: the *credal* level where beliefs are entertained and the *pignistic* level where belief functions are used to make decisions. Pignistic terminology has been coined by Philippe Smets and comes from *pignus*, a bet in Latin. The probability functions, usually used to quantify the beliefs at both levels, are actually used here only to quantify the uncertainty when a decision is really necessary, otherwise we argue as Philippe Smets does, that beliefs are represented by belief functions. To take a rational decision, we propose to transform generalized beliefs into pignistic probability functions through the Generalized Pignistic Transformation (the GPT) which will be presented in the following. We first recall the Classical Pignistic Transformation (the CPT) based on Dempster-Shafer Theory (DST) and then we generalize it within the Dezert-Smarandache Theory (DSmT) framework.

When a decision must be taken, we use the expected utility theory which requires to construct a probability function $P\{.\}$ from basic belief function $m(.)$ [8]. This is achieved by the so-called classical Pignistic Transformation. In the Transferable Belief Model (the TBM) context [7] with open-world assumption, Philippe Smets derives the pignistic probabilities from any non normalized basic belief assignment $m(.)$ (i.e. for which $m(\emptyset) \geq 0$) by the following formula [8]:

$$P\{A\} = \sum_{X \subseteq \Theta} \frac{|X \cap A|}{|X|} \frac{m(X)}{1 - m(\emptyset)} \tag{7.1}$$

where $|A|$ denotes the number of worlds in the set A (with convention $|\emptyset|/|\emptyset| = 1$, to define $P\{\emptyset\}$). $P\{A\}$ corresponds to $BetP(A)$ in Smets' notation [8]. Decisions are achieved by computing the expected utilities of the acts using the subjective/pignistic $P\{.\}$ as the probability function needed to compute expectations. Usually, one uses the maximum of the pignistic probability as decision criterion. The max. of $P\{.\}$ is often considered as a prudent betting decision criterion between the two other alternatives (max of plausibility or max. of credibility which appears to be respectively too optimistic or too pessimistic). It is easy to show that $P\{.\}$ is indeed a probability function (see [7]).

It is important to note that if the belief mass $m(.)$ results from the combination of two independent sources of evidence (i.e. $m(.) = [m_1 \oplus m_2](.)$) then, at the pignistic level, the classical pignistic probability measure $P(.)$ remains the same when using Dempster's rule or when using Smets' rule in his TBM open-world approach working with $m(\emptyset) > 0$. Thus the problem arising with the combination of highly conflicting sources when using Dempster's rule (see chapter 5), and apparently circumvented with the TBM at the credal level, still fundamentally remains at the pignistic level. The problem is only transferred from credal level to pignistic level when using TBM. TBM does not help to improve the reliability of the decision-making with respect to Dempster's rule of combination because the pignistic probabilities are strictly and mathematically equivalent. In other words, if the result of the combination is wrong or at least very questionable or counter-intuitive when the degree of the conflict $m(\emptyset)$ becomes high, then the decision based on pignistic probabilities will become inevitably wrong or very questionable too.

Taking into account the previous remark, we rather prefer to adopt from now on the classical Shafer's definition for basic belief assignment $m(.) : 2^\Theta \to [0,1]$ which imposes to take $m(\emptyset) = 0$ and $\sum_{X\in 2^\Theta} m(X) = 1$. We adopt therefore the following definition for the Classical Pignistic Transformation (CPT):

$$P\{A\} = \sum_{X\in 2^\Theta} \frac{|X\cap A|}{|X|} m(X) \tag{7.2}$$

7.3 A Generalized Pignistic Transformation (GPT)

7.3.1 Definition

To take a rational decision within the DSmT framework, it is necessary to generalize the Classical Pignistic Transformation in order to construct a pignistic probability function from any generalized basic belief assignment $m(.)$ drawn from the DSm rules of combination (the classic or the hybrid ones - see chapter 1). We propose here the simplest and direct extension of the CPT to define a Generalized Pignistic Transformation as follows:

$$\forall A \in D^\Theta, \qquad P\{A\} = \sum_{X\in D^\Theta} \frac{\mathcal{C}_\mathcal{M}(X\cap A)}{\mathcal{C}_\mathcal{M}(X)} m(X) \tag{7.3}$$

where $\mathcal{C}_\mathcal{M}(X)$ denotes the DSm cardinal of proposition X for the DSm model $\mathcal{M}$ of the problem under consideration.

The decision about the solution of the problem is usually taken by the maximum of pignistic probability function $P\{.\}$. Let's remark the close ressemblance of the two pignistic transformations (7.2) and (7.3). It can be shown that (7.3) reduces to (7.2) when the hyper-power set D^Θ reduces to classical power set 2^Θ if we adopt Shafer's model. But (7.3) is a generalization of (7.2) since it can be used for computing pignistic probabilities for any models (including Shafer's model).

7.3.2 $P\{.\}$ is a probability measure

It is important to prove that $P\{.\}$ built from GPT is indeed a (subjective/pignistic) probability measure satisfying the following axioms of probability theory [4, 5]:

- **Axiom 1** (nonnegativity): The (generalized pignistic) probability of any event A is bounded by 0 and 1

$$0 \leq P\{A\} \leq 1$$

- **Axiom 2** (unity): Any sure event (the sample space) has unity (generalized pignistic) probability

$$P\{\mathcal{S}\} = 1$$

- **Axiom 3** (additivity over mutually exclusive events): If A, B are disjoint (i.e. $A \cap B = \emptyset$) then $P(A \cup B) = P(A) + P(B)$

The axiom 1 is satisfied because, by the definition of the generalized basic belief assignment $m(.)$, one has $\forall \alpha_i \in D^\Theta$, $0 \leq m(\alpha_i) \leq 1$ with $\sum_{\alpha_i \in D^\Theta} m(\alpha_i) = 1$ and since all coefficients involved within GPT are bounded by 0 and 1, it follows directly that pignistic probabilities are also bounded by 0 and 1.

The axiom 2 is satisfied because all the coefficients involved in the sure event $\mathcal{S} \triangleq \theta_1 \cup \theta_2 \cup ... \cup \theta_n$ are equal to one because $\mathcal{C}_\mathcal{M}(X \cap \mathcal{S})/\mathcal{C}_\mathcal{M}(X) = \mathcal{C}_\mathcal{M}(X)/\mathcal{C}_\mathcal{M}(X) = 1$, so that $P\{\mathcal{S}\} \equiv \sum_{\alpha_i \in D^\Theta} m(\alpha_i) = 1$.

The axiom 3 is satisfied. Indeed, from the definition of GPT, one has

$$P\{A \cup B\} = \sum_{X \in D^\Theta} \frac{\mathcal{C}_\mathcal{M}(X \cap (A \cup B))}{\mathcal{C}_\mathcal{M}(X)} m(X) \tag{7.4}$$

But if we consider A and B exclusive (i.e. $A \cap B = \emptyset$), then it follows:

$$\mathcal{C}_\mathcal{M}(X \cap (A \cup B)) = \mathcal{C}_\mathcal{M}((X \cap A) \cup (X \cap B)) = \mathcal{C}_\mathcal{M}(X \cap A) + \mathcal{C}_\mathcal{M}(X \cap B)$$

By substituting $\mathcal{C}_\mathcal{M}(X \cap (A \cup B))$ by $\mathcal{C}_\mathcal{M}(X \cap A) + \mathcal{C}_\mathcal{M}(X \cap B)$ into (7.4), it comes:

$$\begin{aligned} P\{A \cup B\} &= \sum_{X \in D^\Theta} \frac{\mathcal{C}_\mathcal{M}(X \cap A) + \mathcal{C}_\mathcal{M}(X \cap B)}{\mathcal{C}_\mathcal{M}(X)} m(X) \\ &= \sum_{X \in D^\Theta} \frac{\mathcal{C}_\mathcal{M}(X \cap A)}{\mathcal{C}_\mathcal{M}(X)} m(X) + \sum_{X \in D^\Theta} \frac{\mathcal{C}_\mathcal{M}(X \cap B)}{\mathcal{C}_\mathcal{M}(X)} m(X) \\ &= P\{A\} + P\{B\} \end{aligned}$$

which completes the proof. From the coefficients $\frac{\mathcal{C}_\mathcal{M}(X \cap A)}{\mathcal{C}_\mathcal{M}(X)}$ involved in (7.3), it can also be easily checked that $A \subset B \Rightarrow P\{A\} \leq P\{B\}$. One can also easily prove the Poincaré' equality: $P\{A \cup B\} = P\{A\} + P\{B\} - P\{A \cap B\}$ because $\mathcal{C}_\mathcal{M}(X \cap (A \cup B) = \mathcal{C}_\mathcal{M}((X \cap A) \cup (X \cap B)) = \mathcal{C}_\mathcal{M}(X \cap A) + \mathcal{C}_\mathcal{M}(X \cap B) - \mathcal{C}_\mathcal{M}(X \cap (A \cap B))$ (one has substracted $\mathcal{C}_\mathcal{M}(X \cap (A \cap B))$, i.e. the number of parts of $X \cap (A \cap B)$ in the Venn diagram, due to the fact that these parts were added twice: once in $\mathcal{C}_\mathcal{M}(X \cap A)$ and second time in $\mathcal{C}_\mathcal{M}(X \cap B)$.

7.4 Some examples for the GPT

7.4.1 Example for the 2D case

- **With the free DSm model**:

Let's consider $\Theta = \{\theta_1, \theta_2\}$ and the generalized basic belief function $m(.)$ over the hyper-power set $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$. It is easy to construct the pignistic probability $P\{.\}$. According to the definition of the GPT given in (7.3), one gets:

$$
\begin{aligned}
P\{\emptyset\} &= 0 \\
P\{\theta_1\} &= m(\theta_1) + \frac{1}{2}m(\theta_2) + m(\theta_1 \cap \theta_2) + \frac{2}{3}m(\theta_1 \cup \theta_2) \\
P\{\theta_2\} &= m(\theta_2) + \frac{1}{2}m(\theta_1) + m(\theta_1 \cap \theta_2) + \frac{2}{3}m(\theta_1 \cup \theta_2) \\
P\{\theta_1 \cap \theta_2\} &= \frac{1}{2}m(\theta_2) + \frac{1}{2}m(\theta_1) + m(\theta_1 \cap \theta_2) + \frac{1}{3}m(\theta_1 \cup \theta_2) \\
P\{\theta_1 \cup \theta_2\} &= P\{\Theta\} = m(\theta_1) + m(\theta_2) + m(\theta_1 \cap \theta_2) + m(\theta_1 \cup \theta_2) = 1
\end{aligned}
$$

It is easy to prove that $0 \leq P\{.\} \leq 1$ and $P\{\theta_1 \cup \theta_2\} = P\{\theta_1\} + P\{\theta_2\} - P\{\theta_1 \cap \theta_2\}$

- **With Shafer's model**:

If one adopts Shafer's model (we assume $\theta_1 \cap \theta_2 \overset{\mathcal{M}^0}{\equiv} \emptyset$), then after applying the hybrid DSm rule of combination, one gets a basic belief function with non null masses only on θ_1, θ_2 and $\theta_1 \cup \theta_2$. By applying the GPT, one gets:

$$P\{\emptyset\} = 0$$

$$P\{\theta_1 \cap \theta_2\} = 0$$

$$P\{\theta_1\} = m(\theta_1) + \frac{1}{2}m(\theta_1 \cup \theta_2)$$

$$P\{\theta_2\} = m(\theta_2) + \frac{1}{2}m(\theta_1 \cup \theta_2)$$

$$P\{\theta_1 \cup \theta_2\} = m(\theta_1) + m(\theta_2) + m(\theta_1 \cup \theta_2) = 1$$

which naturally corresponds in this case to the pignistic probability built with the classical pignistic transformation (7.2).

7.4.2 Example for the 3D case

- **With the free DSm model**:

X	$\frac{\mathcal{C}_\mathcal{M}(X\cap\alpha_6)}{\mathcal{C}_\mathcal{M}(X)}$	$\leq$	$\frac{\mathcal{C}_\mathcal{M}(X\cap\alpha_{10})}{\mathcal{C}_\mathcal{M}(X)}$	X	$\frac{\mathcal{C}_\mathcal{M}(X\cap\alpha_6)}{\mathcal{C}_\mathcal{M}(X)}$	$\leq$	$\frac{\mathcal{C}_\mathcal{M}(X\cap\alpha_{10})}{\mathcal{C}_\mathcal{M}(X)}$
α_1	1	$\leq$	1	α_{10}	(3/4)	$\leq$	1
α_2	1	$\leq$	1	α_{11}	(2/4)	$\leq$	(2/4)
α_3	(1/2)	$\leq$	(1/2)	α_{12}	(3/5)	$\leq$	(3/5)
α_4	1	$\leq$	1	α_{13}	(3/5)	$\leq$	(4/5)
α_5	(2/3)	$\leq$	(2/3)	α_{14}	(3/5)	$\leq$	(3/5)
α_6	1	$\leq$	1	α_{15}	(3/6)	$\leq$	(4/6)
α_7	(2/3)	$\leq$	(2/3)	α_{16}	(3/6)	$\leq$	(3/6)
α_8	(3/4)	$\leq$	(3/4)	α_{17}	(3/6)	$\leq$	(4/6)
α_9	(2/4)	$\leq$	(2/4)	α_{18}	(3/7)	$\leq$	(4/7)

Table 7.1: Coefficients $\frac{\mathcal{C}_\mathcal{M}(X\cap\alpha_6)}{\mathcal{C}_\mathcal{M}(X)}$ and $\frac{\mathcal{C}_\mathcal{M}(X\cap\alpha_{10})}{\mathcal{C}_\mathcal{M}(X)}$

Let's consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$, its hyper-power set $D^\Theta = \{\alpha_0, \ldots, \alpha_{18}\}$ (with α_i, $i = 0, \ldots, 18$ corresponding to propositions shown in table 3.1 of chapter 3, and the generalized basic belief assignment $m(.)$ over the hyper-power set D^Θ. The six tables presented in the appendix show the full derivations of all generalized pignistic probabilities $P\{\alpha_i\}$ for $i = 1, \ldots, 18$ ($P\{\emptyset\} = 0$ by definition) according to the GPT formula (7.3).

Note that $P\{\alpha_{18}\} = 1$ because $(\theta_1 \cup \theta_2 \cup \theta_3)$ corresponds to the sure event in our subjective probability space and $\sum_{\alpha_i \in D^\Theta} m(\alpha_i) = 1$ by the definition of any generalized basic belief assignment $m(.)$ defined on D^Θ.

It can be verified (as expected) on this example, although being a quite tedious task, that Poincaré' s equality holds:

$$P\{A_1 \cup \ldots \cup A_n\} = \sum_{\substack{I \subset \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} P\{\bigcap_{i \in I} A_i\} \tag{7.5}$$

It is also easy to verify that $\forall A \subset B \Rightarrow P\{A\} \leq P\{B\}$ holds. By example, for $(\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2) \subset \alpha_{10} \triangleq \theta_2)$ and from the expressions of $P\{\alpha_6\}$ and $P\{\alpha_{10}\}$ given in appendix, we directly conclude that $P\{\alpha_6\} \leq P\{\alpha_{10}\}$ because

$$\forall X \in D^\Theta, \qquad \frac{\mathcal{C}_\mathcal{M}(X \cap \alpha_6)}{\mathcal{C}_\mathcal{M}(X)} \leq \frac{\mathcal{C}_\mathcal{M}(X \cap \alpha_{10})}{\mathcal{C}_\mathcal{M}(X)} \tag{7.6}$$

as shown in the table above.

- **Example with a given hybrid DSm model**:

Consider now the hybrid DSm model $\mathcal{M} \neq \mathcal{M}^f$ in which we force all possible conjunctions to be empty, but $\theta_1 \cap \theta_2$ according to the second Venn diagram presented in Chapter 3 and shown in Figure 3.2. In this case the hyper-power set D^Θ reduces to 9 elements $\{\alpha_0, \ldots, \alpha_8\}$ shown in table 3.2 of Chapter 3. The following tables present the full derivations of the pignistic probabilities $P\{\alpha_i\}$ for $i = 1, \ldots, 8$ from the GPT formula (7.3) applied to this hybrid DSm model.

$P\{\alpha_1\} =$	$P\{\alpha_2\} =$	$P\{\alpha_3\} =$	$P\{\alpha_4\} =$
$(1/1)m(\alpha_1)$	$(0/1)m(\alpha_1)$	$(1/1)m(\alpha_1)$	$(1/1)m(\alpha_1)$
$+(0/1)m(\alpha_2)$	$+(1/1)m(\alpha_2)$	$+(0/2)m(\alpha_2)$	$+(0/1)m(\alpha_2)$
$+(1/2)m(\alpha_3)$	$+(0/2)m(\alpha_3)$	$+(2/2)m(\alpha_3)$	$+(1/2)m(\alpha_3)$
$+(1/2)m(\alpha_4)$	$+(0/2)m(\alpha_4)$	$+(1/2)m(\alpha_4)$	$+(2/2)m(\alpha_4)$
$+(1/3)m(\alpha_5)$	$+(0/3)m(\alpha_5)$	$+(2/3)m(\alpha_5)$	$+(2/3)m(\alpha_5)$
$+(1/3)m(\alpha_6)$	$+(1/3)m(\alpha_6)$	$+(2/3)m(\alpha_6)$	$+(1/3)m(\alpha_6)$
$+(1/3)m(\alpha_7)$	$+(1/3)m(\alpha_7)$	$+(1/3)m(\alpha_7)$	$+(2/3)m(\alpha_7)$
$+(1/4)m(\alpha_8)$	$+(1/4)m(\alpha_8)$	$+(2/4)m(\alpha_8)$	$+(2/4)m(\alpha_8)$

Table 7.2: Derivation of $P\{\alpha_1 \triangleq \theta_1 \cap \theta_2\}$, $P\{\alpha_2 \triangleq \theta_3\}$, $P\{\alpha_3 \triangleq \theta_1\}$ and $P\{\alpha_4 \triangleq \theta_2\}$

$P\{\alpha_5\} =$	$P\{\alpha_6\} =$	$P\{\alpha_7\} =$	$P\{\alpha_8\} =$
$(1/1)m(\alpha_1)$	$(1/1)m(\alpha_1)$	$(1/1)m(\alpha_1)$	$(1/1)m(\alpha_1)$
$+(0/1)m(\alpha_2)$	$+(1/1)m(\alpha_2)$	$+(2/2)m(\alpha_2)$	$+(2/2)m(\alpha_2)$
$+(2/2)m(\alpha_3)$	$+(2/2)m(\alpha_3)$	$+(1/2)m(\alpha_3)$	$+(2/2)m(\alpha_3)$
$+(2/2)m(\alpha_4)$	$+(1/2)m(\alpha_4)$	$+(2/2)m(\alpha_4)$	$+(2/2)m(\alpha_4)$
$+(3/3)m(\alpha_5)$	$+(2/3)m(\alpha_5)$	$+(2/3)m(\alpha_5)$	$+(3/3)m(\alpha_5)$
$+(2/3)m(\alpha_6)$	$+(3/3)m(\alpha_6)$	$+(2/3)m(\alpha_6)$	$+(3/3)m(\alpha_6)$
$+(2/3)m(\alpha_7)$	$+(2/3)m(\alpha_7)$	$+(3/3)m(\alpha_7)$	$+(3/3)m(\alpha_7)$
$+(3/4)m(\alpha_8)$	$+(3/4)m(\alpha_8)$	$+(3/4)m(\alpha_8)$	$+(4/4)m(\alpha_8)$

Table 7.3: Derivation of $P\{\alpha_5 \triangleq \theta_1 \cup \theta_2\}$, $P\{\alpha_6 \triangleq \theta_1 \cup \theta_3\}$, $P\{\alpha_7 \triangleq \theta_2 \cup \theta_3\}$ and $P\{\alpha_8 \triangleq \theta_1 \cup \theta_2 \cup \theta_3\}$

- **Example with Shafer's model**:

Consider now Shafer's model $\mathcal{M}^0 \neq \mathcal{M}^f$ in which we force all possible conjunctions to be empty according to the third Venn diagram presented in Chapter 3. In this case the hyper-power set

D^Θ reduces to the classical power set 2^Θ with 8 elements $\{\alpha_0, \ldots, \alpha_7\}$ explicated in table 3.3 of Chapter 3. Applying, the GPT formula (7.3), one gets the following pignistic probabilities $P\{\alpha_i\}$ for $i = 1, \ldots, 7$ which naturally coincide, in this particular case, with the values obtained directly by the classical pignistic transformation (7.2):

$P\{\alpha_1\} =$	$P\{\alpha_2\} =$	$P\{\alpha_3\} =$
$(1/1)m(\alpha_1)$	$(0/1)m(\alpha_1)$	$(0/1)m(\alpha_1)$
$+(0/1)m(\alpha_2)$	$+(1/1)m(\alpha_2)$	$+(0/1)m(\alpha_2)$
$+(0/1)m(\alpha_3)$	$+(0/1)m(\alpha_3)$	$+(1/1)m(\alpha_3)$
$+(1/2)m(\alpha_4)$	$+(1/2)m(\alpha_4)$	$+(0/2)m(\alpha_4)$
$+(1/2)m(\alpha_5)$	$+(0/2)m(\alpha_5)$	$+(1/2)m(\alpha_5)$
$+(0/2)m(\alpha_6)$	$+(1/2)m(\alpha_6)$	$+(1/2)m(\alpha_6)$
$+(1/3)m(\alpha_7)$	$+(1/3)m(\alpha_7)$	$+(1/3)m(\alpha_7)$

Table 7.4: Derivation of $P\{\alpha_1 \triangleq \theta_1\}$, $P\{\alpha_2 \triangleq \theta_2\}$ and $P\{\alpha_3 \triangleq \theta_3\}$

$P\{\alpha_4\} =$	$P\{\alpha_5\} =$	$P\{\alpha_6\} =$	$P\{\alpha_7\} =$
$(1/1)m(\alpha_1)$	$(1/1)m(\alpha_1)$	$(0/1)m(\alpha_1)$	$(1/1)m(\alpha_1)$
$+(1/1)m(\alpha_2)$	$+(0/1)m(\alpha_2)$	$+(1/1)m(\alpha_2)$	$+(1/1)m(\alpha_2)$
$+(0/1)m(\alpha_3)$	$+(1/1)m(\alpha_3)$	$+(1/1)m(\alpha_3)$	$+(1/1)m(\alpha_3)$
$+(2/2)m(\alpha_4)$	$+(1/2)m(\alpha_4)$	$+(1/2)m(\alpha_4)$	$+(2/2)m(\alpha_4)$
$+(1/2)m(\alpha_5)$	$+(2/2)m(\alpha_5)$	$+(1/2)m(\alpha_5)$	$+(2/2)m(\alpha_5)$
$+(1/2)m(\alpha_6)$	$+(1/2)m(\alpha_6)$	$+(2/2)m(\alpha_6)$	$+(2/2)m(\alpha_6)$
$+(2/3)m(\alpha_7)$	$+(2/3)m(\alpha_7)$	$+(2/3)m(\alpha_7)$	$+(3/3)m(\alpha_7)$

Table 7.5: Derivation of $P\{\alpha_4 \triangleq \theta_1 \cup \theta_2\}$, $P\{\alpha_5 \triangleq \theta_1 \cup \theta_3\}$, $P\{\alpha_6 \triangleq \theta_2 \cup \theta_3\}$ and $P\{\alpha_7 \triangleq \theta_1 \cup \theta_2 \cup \theta_3\} = 1$

7.5 Conclusion

A generalization of the classical pignistic transformation developed originally within the DST framework has been proposed in this chapter. This generalization is based on the new theory of plausible and paradoxical reasoning (DSmT) and provides a new mathematical issue to help the decision-making under uncertainty and paradoxical (i.e. highly conflicting) sources of information. The generalized pignistic transformation (GPT) proposed here allows to build a subjective/pignistic probability measure over the hyper-power set of the frame of the problem under consideration for all kinds of models (free, hybrid or Shafer's model). The GPT coincides naturally with the classical pignistic transformation whenever Shafer's model is adopted. It corresponds with the assumptions of classical pignistic probability generalized to the free DSm model. A relation of GPT on general hybrid DSm models to assumptions of classical PT is still in the process of investigation. Several examples for the 2D and 3D cases for different kinds of models have been presented to illustrate the validity of the GPT.

7.6 References

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Appendix: Derivation of the GPT for the 3D free DSm model

$P\{\alpha_1\} =$	$P\{\alpha_2\} =$	$P\{\alpha_3\} =$
$m(\alpha_1)$	$m(\alpha_1)$	$m(\alpha_1)$
$+(1/2)m(\alpha_2)$	$+m(\alpha_2)$	$+(1/2)m(\alpha_2)$
$+(1/2)m(\alpha_3)$	$+(1/2)m(\alpha_3)$	$+m(\alpha_3)$
$+(1/2)m(\alpha_4)$	$+(1/2)m(\alpha_4)$	$+(1/2)m(\alpha_4)$
$+(1/3)m(\alpha_5)$	$+(1/3)m(\alpha_5)$	$+(2/3)m(\alpha_5)$
$+(1/3)m(\alpha_6)$	$+(2/3)m(\alpha_6)$	$+(1/3)m(\alpha_6)$
$+(1/3)m(\alpha_7)$	$+(2/3)m(\alpha_7)$	$+(2/3)m(\alpha_7)$
$+(1/4)m(\alpha_8)$	$+(2/4)m(\alpha_8)$	$+(2/4)m(\alpha_8)$
$+(1/4)m(\alpha_9)$	$+(2/4)m(\alpha_9)$	$+(2/4)m(\alpha_9)$
$+(1/4)m(\alpha_{10})$	$+(2/4)m(\alpha_{10})$	$+(1/4)m(\alpha_{10})$
$+(1/4)m(\alpha_{11})$	$+(1/4)m(\alpha_{11})$	$+(2/4)m(\alpha_{11})$
$+(1/5)m(\alpha_{12})$	$+(2/5)m(\alpha_{12})$	$+(2/5)m(\alpha_{12})$
$+(1/5)m(\alpha_{13})$	$+(2/5)m(\alpha_{13})$	$+(2/5)m(\alpha_{13})$
$+(1/5)m(\alpha_{14})$	$+(2/5)m(\alpha_{14})$	$+(2/5)m(\alpha_{14})$
$+(1/6)m(\alpha_{15})$	$+(2/6)m(\alpha_{15})$	$+(2/6)m(\alpha_{15})$
$+(1/6)m(\alpha_{16})$	$+(2/6)m(\alpha_{16})$	$+(2/6)m(\alpha_{16})$
$+(1/6)m(\alpha_{17})$	$+(2/6)m(\alpha_{17})$	$+(2/6)m(\alpha_{17})$
$+(1/7)m(\alpha_{18})$	$+(2/7)m(\alpha_{18})$	$+(2/7)m(\alpha_{18})$

Derivation of $P\{\alpha_1\}$, $P\{\alpha_2\}$ and $P\{\alpha_3\}$

$P\{\alpha_{10}\} =$	$P\{\alpha_{11}\} =$	$P\{\alpha_{12}\} =$
$m(\alpha_1)$	$m(\alpha_1)$	$m(\alpha_1)$
$+m(\alpha_2)$	$+(1/2)m(\alpha_2)$	$+m(\alpha_2)$
$+(1/2)m(\alpha_3)$	$+m(\alpha_3)$	$+m(\alpha_3)$
$+m(\alpha_4)$	$+m(\alpha_4)$	$+m(\alpha_4)$
$+(2/3)m(\alpha_5)$	$+m(\alpha_5)$	$+m(\alpha_5)$
$+m(\alpha_6)$	$+(2/3)m(\alpha_6)$	$+m(\alpha_6)$
$+(2/3)m(\alpha_7)$	$+(2/3)m(\alpha_7)$	$+m(\alpha_7)$
$+(3/4)m(\alpha_8)$	$+(3/4)m(\alpha_8)$	$+m(\alpha_8)$
$+(2/4)m(\alpha_9)$	$+(2/4)m(\alpha_9)$	$+(3/4)m(\alpha_9)$
$+m(\alpha_{10})$	$+(2/4)m(\alpha_{10})$	$+(3/4)m(\alpha_{10})$
$+(2/4)m(\alpha_{11})$	$+m(\alpha_{11})$	$+m(\alpha_{11})$
$+(3/5)m(\alpha_{12})$	$+(4/5)m(\alpha_{12})$	$+m(\alpha_{12})$
$+(4/5)m(\alpha_{13})$	$+(3/5)m(\alpha_{13})$	$+(4/5)m(\alpha_{13})$
$+(3/5)m(\alpha_{14})$	$+(3/5)m(\alpha_{14})$	$+(4/5)m(\alpha_{14})$
$+(4/6)m(\alpha_{15})$	$+(3/6)m(\alpha_{15})$	$+(4/6)m(\alpha_{15})$
$+(3/6)m(\alpha_{16})$	$+(4/6)m(\alpha_{16})$	$+(4/6)m(\alpha_{16})$
$+(4/6)m(\alpha_{17})$	$+(4/6)m(\alpha_{17})$	$+(5/6)m(\alpha_{17})$
$+(4/7)m(\alpha_{18})$	$+(4/7)m(\alpha_{18})$	$+(5/7)m(\alpha_{18})$

Derivation of $P\{\alpha_{10}\}$, $P\{\alpha_{11}\}$ and $P\{\alpha_{12}\}$

$P\{\alpha_4\} =$	$P\{\alpha_5\} =$	$P\{\alpha_6\} =$
$m(\alpha_1)$	$m(\alpha_1)$	$m(\alpha_1)$
$+(1/2)m(\alpha_2)$	$+(1/2)m(\alpha_2)$	$+m(\alpha_2)$
$+(1/2)m(\alpha_3)$	$+m(\alpha_3)$	$+(1/2)m(\alpha_3)$
$+m(\alpha_4)$	$+m(\alpha_4)$	$+m(\alpha_4)$
$+(2/3)m(\alpha_5)$	$+m(\alpha_5)$	$+(2/3)m(\alpha_5)$
$+(2/3)m(\alpha_6)$	$+(2/3)m(\alpha_6)$	$+m(\alpha_6)$
$+(1/3)m(\alpha_7)$	$+(2/3)m(\alpha_7)$	$+(2/3)m(\alpha_7)$
$+(2/4)m(\alpha_8)$	$+(3/4)m(\alpha_8)$	$+(3/4)m(\alpha_8)$
$+(1/4)m(\alpha_9)$	$+(2/4)m(\alpha_9)$	$+(2/4)m(\alpha_9)$
$+(2/4)m(\alpha_{10})$	$+(2/4)m(\alpha_{10})$	$+(3/4)m(\alpha_{10})$
$+(2/4)m(\alpha_{11})$	$+(3/4)m(\alpha_{11})$	$+(2/4)m(\alpha_{11})$
$+(2/5)m(\alpha_{12})$	$+(3/5)m(\alpha_{12})$	$+(3/5)m(\alpha_{12})$
$+(2/5)m(\alpha_{13})$	$+(3/5)m(\alpha_{13})$	$+(3/5)m(\alpha_{13})$
$+(2/5)m(\alpha_{14})$	$+(3/5)m(\alpha_{14})$	$+(3/5)m(\alpha_{14})$
$+(2/6)m(\alpha_{15})$	$+(3/6)m(\alpha_{15})$	$+(3/6)m(\alpha_{15})$
$+(2/6)m(\alpha_{16})$	$+(3/6)m(\alpha_{16})$	$+(3/6)m(\alpha_{16})$
$+(2/6)m(\alpha_{17})$	$+(3/6)m(\alpha_{17})$	$+(3/6)m(\alpha_{17})$
$+(2/7)m(\alpha_{18})$	$+(3/7)m(\alpha_{18})$	$+(3/7)m(\alpha_{18})$

Derivation of $P\{\alpha_4\}$, $P\{\alpha_5\}$ and $P\{\alpha_6\}$

$P\{\alpha_{10}\} =$	$P\{\alpha_{11}\} =$	$P\{\alpha_{12}\} =$
$m(\alpha_1)$	$m(\alpha_1)$	$m(\alpha_1)$
$+m(\alpha_2)$	$+(1/2)m(\alpha_2)$	$+m(\alpha_2)$
$+(1/2)m(\alpha_3)$	$+m(\alpha_3)$	$+m(\alpha_3)$
$+m(\alpha_4)$	$+m(\alpha_4)$	$+m(\alpha_4)$
$+(2/3)m(\alpha_5)$	$+m(\alpha_5)$	$+m(\alpha_5)$
$+m(\alpha_6)$	$+(2/3)m(\alpha_6)$	$+m(\alpha_6)$
$+(2/3)m(\alpha_7)$	$+(2/3)m(\alpha_7)$	$+m(\alpha_7)$
$+(3/4)m(\alpha_8)$	$+(3/4)m(\alpha_8)$	$+m(\alpha_8)$
$+(2/4)m(\alpha_9)$	$+(2/4)m(\alpha_9)$	$+(3/4)m(\alpha_9)$
$+m(\alpha_{10})$	$+(2/4)m(\alpha_{10})$	$+(3/4)m(\alpha_{10})$
$+(2/4)m(\alpha_{11})$	$+m(\alpha_{11})$	$+m(\alpha_{11})$
$+(3/5)m(\alpha_{12})$	$+(4/5)m(\alpha_{12})$	$+m(\alpha_{12})$
$+(4/5)m(\alpha_{13})$	$+(3/5)m(\alpha_{13})$	$+(4/5)m(\alpha_{13})$
$+(3/5)m(\alpha_{14})$	$+(3/5)m(\alpha_{14})$	$+(4/5)m(\alpha_{14})$
$+(4/6)m(\alpha_{15})$	$+(3/6)m(\alpha_{15})$	$+(4/6)m(\alpha_{15})$
$+(3/6)m(\alpha_{16})$	$+(4/6)m(\alpha_{16})$	$+(4/6)m(\alpha_{16})$
$+(4/6)m(\alpha_{17})$	$+(4/6)m(\alpha_{17})$	$+(5/6)m(\alpha_{17})$
$+(4/7)m(\alpha_{18})$	$+(4/7)m(\alpha_{18})$	$+(5/7)m(\alpha_{18})$

Derivation of $P\{\alpha_{13}\}$, $P\{\alpha_{14}\}$ and $P\{\alpha_{15}\}$

$P\{\alpha_7\} =$	$P\{\alpha_8\} =$	$P\{\alpha_9\} =$
$m(\alpha_1)$	$m(\alpha_1)$	$m(\alpha_1)$
$+m(\alpha_2)$	$+m(\alpha_2)$	$+m(\alpha_2)$
$+m(\alpha_3)$	$+m(\alpha_3)$	$+m(\alpha_3)$
$+(1/2)m(\alpha_4)$	$+m(\alpha_4)$	$+(1/2)m(\alpha_4)$
$+(2/3)m(\alpha_5)$	$+m(\alpha_5)$	$+(2/3)m(\alpha_5)$
$+(2/3)m(\alpha_6)$	$+m(\alpha_6)$	$+(2/3)m(\alpha_6)$
$+m(\alpha_7)$	$+m(\alpha_7)$	$+m(\alpha_7)$
$+(3/4)m(\alpha_8)$	$+m(\alpha_8)$	$+(3/4)m(\alpha_8)$
$+(3/4)m(\alpha_9)$	$+(3/4)m(\alpha_9)$	$+m(\alpha_9)$
$+(2/4)m(\alpha_{10})$	$+(3/4)m(\alpha_{10})$	$+(2/4)m(\alpha_{10})$
$+(2/4)m(\alpha_{11})$	$+(3/4)m(\alpha_{11})$	$+(2/4)m(\alpha_{11})$
$+(3/5)m(\alpha_{12})$	$+(4/5)m(\alpha_{12})$	$+(3/5)m(\alpha_{12})$
$+(3/5)m(\alpha_{13})$	$+(4/5)m(\alpha_{13})$	$+(3/5)m(\alpha_{13})$
$+(3/5)m(\alpha_{14})$	$+(4/5)m(\alpha_{14})$	$+(4/5)m(\alpha_{14})$
$+(3/6)m(\alpha_{15})$	$+(4/6)m(\alpha_{15})$	$+(4/6)m(\alpha_{15})$
$+(3/6)m(\alpha_{16})$	$+(4/6)m(\alpha_{16})$	$+(4/6)m(\alpha_{16})$
$+(3/6)m(\alpha_{17})$	$+(4/6)m(\alpha_{17})$	$+(3/6)m(\alpha_{17})$
$+(3/7)m(\alpha_{18})$	$+(4/7)m(\alpha_{18})$	$+(4/7)m(\alpha_{18})$

Derivation of $P\{\alpha_7\}$, $P\{\alpha_8\}$ and $P\{\alpha_9\}$

$P\{\alpha_{16}\} =$	$P\{\alpha_{17}\} =$	$P\{\alpha_{18}\} =$
$m(\alpha_1)$	$m(\alpha_1)$	$m(\alpha_1)$
$+m(\alpha_2)$	$+m(\alpha_2)$	$+m(\alpha_2)$
$+m(\alpha_3)$	$+m(\alpha_3)$	$+m(\alpha_3)$
$+m(\alpha_4)$	$+m(\alpha_4)$	$+m(\alpha_4)$
$+m(\alpha_5)$	$+m(\alpha_5)$	$+m(\alpha_5)$
$+m(\alpha_6)$	$+m(\alpha_6)$	$+m(\alpha_6)$
$+m(\alpha_7)$	$+m(\alpha_7)$	$+m(\alpha_7)$
$+m(\alpha_8)$	$+m(\alpha_8)$	$+m(\alpha_8)$
$+m(\alpha_9)$	$+(3/4)m(\alpha_9)$	$+m(\alpha_9)$
$+(3/4)m(\alpha_{10})$	$+m(\alpha_{10})$	$+m(\alpha_{10})$
$+m(\alpha_{11})$	$+m(\alpha_{11})$	$+m(\alpha_{11})$
$+m(\alpha_{12})$	$+m(\alpha_{12})$	$+m(\alpha_{12})$
$+(4/5)m(\alpha_{13})$	$+m(\alpha_{13})$	$+m(\alpha_{13})$
$+m(\alpha_{14})$	$+(4/5)m(\alpha_{14})$	$+m(\alpha_{14})$
$+(5/6)m(\alpha_{15})$	$+(5/6)m(\alpha_{15})$	$+m(\alpha_{15})$
$+m(\alpha_{16})$	$+(5/6)m(\alpha_{16})$	$+m(\alpha_{16})$
$+(5/6)m(\alpha_{17})$	$+m(\alpha_{17})$	$+m(\alpha_{17})$
$+(6/7)m(\alpha_{18})$	$+(6/7)m(\alpha_{18})$	$+m(\alpha_{18})$

Derivation of $P\{\alpha_{16}\}$, $P\{\alpha_{17}\}$ and $P\{\alpha_{18}\}$