A class of position-dependent mass Liénard differential equations via a general nonlocal transformation

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Abstract

The objective, in this paper, consists of mapping the damped linear harmonic oscillator equation onto a class of Liénard nonlinear differential equations that incorporates the well known position dependent mass Mathews-Lakshmanan oscillator equations as specific examples through a general nonlocal transformation.

1. **General theory**

Suppose, to formulate the general class of exactly solvable quadratically dissipative Liénard type differential equations, that the damped linear harmonic oscillator equation is of the form

$$y'' + \mu y' + \omega^2 y = 0 \tag{1}$$

where prime means differentiation with respect to τ , μ and ω are arbitrary parameters. Consider, now, the general nonlocal transformation

$$y = \int g(x)^l dx\,,\ d\tau = \exp(\gamma\varphi(x))dt \tag{2}$$

where the exponent l and γ are arbitrary parameters, $g(x) \neq 0$, and $\varphi(x)$ are arbitrary functions of x . So, the damped linear harmonic oscillator equation, by application of the general nonlocal transformation (2), may be mapped onto the generalized mixed Liénard-type nonlinear oscillator equation

$$\ddot{x} + (l\frac{g'(x)}{g(x)} - \gamma\varphi'(x))\dot{x}^2 + \mu\dot{x}\exp(\gamma\varphi(x)) + \frac{\omega^2 \exp(2\gamma\varphi(x))\int g(x)^l dx}{g(x)^l} = 0 \tag{3}$$

which represents the desired general class of exactly integrable mixed Liénard type differential nonlinear equations. To obtain the expected general class of position-dependent mass Liénard type equations, let $\varphi(x) = \ln(f(x))$, where $\ln$ designates the natural logarithm. Then the equation (3) reduces to

$$\ddot{x} + (l\frac{g'(x)}{g(x)} - \gamma\frac{f'(x)}{f(x)})\dot{x}^2 + \mu\dot{x}f(x)^\gamma + \frac{\omega^2 f(x)^{2\gamma}\int g(x)^l dx}{g(x)^l} = 0 \tag{4}$$

The parametric choice $l = \frac{1}{2}$, $\gamma = 1$, and $\mu = 0$, transforms the equation (4) to the generalized quadratic Liénard type differential equation

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$$\ddot{x}+\frac{1}{2}(\frac{g'(x)}{g(x)}-2\frac{f'(x)}{f(x)})\dot{x}^2+\frac{\omega^2 f(x)^2 g(x)^{1/2}\int g(x)^{1/2}dx}{g(x)}=0 \qquad (5)$$

which represents the desired general class of position-dependent mass Liénard type differential equations.

2. **Application**

The equation (5) is consistent with the equation derived in [2] from a generalized position dependent mass nonlocal point transformation. It suffices, as proof, to substitute [2]

$f(x)=\frac{1}{1\pm\lambda x^2}$, and $g(x)=f(x)^3=\frac{1}{(1\pm\lambda x^2)^3}$, into the equation (5) to get the famous position-dependent mass Mathews-Lakshmanan nonlinear oscillator equations [1, 2]

$$\ddot{x}\mp\frac{\lambda x}{1\pm\lambda x^2}\dot{x}^2+\frac{\omega^2 x}{1\pm\lambda x^2}=0 \qquad (6)$$

which, first, were derived from a non-standard Lagrangian using the Euler-Lagrange equation [1]. It is worth to note that the equation (4) may give the damped Mathews-Lakshmanan oscillator equations with the above functions and parametric choice for $\mu\neq 0$.

References

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