Theory of Gravity: A classical field approach in curved space–time

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A new approach to the theory of gravity is proposed. A second-rank tensor field is chosen to be the potential of the gravitational field. The gravitational field is related to the metric tensor of space–time, and all phenomena occur in curved space–time. A variational principle is established, and the gravitational field equations are derived. The energy–momentum density tensor of the gravitational field and its conservation law are obtained. The source of the gravitational field is the energy–momentum density of all kinds of matter, including the gravitational field itself. A Lagrangian of the gravitational field is proposed that correctly describes local observable gravitational phenomena in the second-order approximation. The energy density of the gravitational field is positive. Estimates are obtained for the gravitational energy defect, the difference between the inertial and gravitational masses of a body, and the effect of the external gravitational field on the mass of a body. The new approach to the description of the gravitational field and its energy provides additional incentives search possibilities of experimental verification of the phenomena of gravitation in strong fields.

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I. INTRODUCTION

The idea that matter curves space allowed Einstein to develop general relativity theory (GR) [1], which generalizes Newton's theory of gravitation [2–6] and is a generally accepted theory of gravitation and of the space-time structure. The predictions of local gravitational phenomena made by GR agree with the results of experiments.

GR is based on Einstein's equation relating the space–time curvature tensor to the energy–momentum density tensor of matter:

$$R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi kT_{ij}, \tag{1}$$

where $R_{ij} = R^k_{ikj}$ are the components of the Ricci tensor, $R = R_{ij}g^{ij}$ is the scalar curvature of the space, T_{ij} are the components of the energy–momentum density tensor of matter, and $k = 6.673 \times 10^{-8} cm^3 g^{-1} s^{-2}$ is the gravitation constant. Hilbert [7] established that Einstein's equation can be derived from the variational principle if the Lagrangian of the gravitational field (GF) is given by the scalar curvature of space–time and the gravitational potentials are given by the components of the metric tensor. In this approach, which can be called the Einstein–Hilbert approach, the curvature of space–time is determined by the energy–momentum density tensor of all kinds of matter, except for the GF. The GF itself has no energy–momentum density. The reason is the covariant constancy of the metric tensor. According to Einstein, the GF can be defined without introducing the field strength and energy density [8]; therefore, this field may essentially differ from other fields, in particular, from electromagnetic field.

A variety of gravity theories have been developed and proposed (see, for example, the surveys in [9–22]). As pointed out in [20], most variants of gravity theory are based on a modification or extension of the Einstein–Hilbert approach, and the Lagrangian of the GF is based on the scalar curvature of space–time. This may apply to both earlier and modern studies. In particular, in scalar–tensor theories of gravity [17], in addition to the field of the metric tensor components, scalar fields are used as the gravitational potentials.

A somewhat different approach is implemented in the two-metric formalism (see, for example, [9, 11, 23–26]). In space–time, one simultaneously considers two metric tensors. Gravitational phenomena occur in planar geometry, while all the other phenomena of the material world occur in pseudo-Riemannian geometry described by a metric tensor depending on the GF. In contrast to GR, there is a well-defined energy–momentum density tensor of the GF in two-metric theories. However, the question remains open of whether it is necessary that the two geometries—a plane (background) and a locally curved pseudo-Riemannian—should exist simultaneously.

The description of gravitational phenomena by a tensor field in a flat space (see, for example, [16]) is not as consistent as GR.

In this study, we support Einstein's point of view: all phenomena occur in the same curved pseudo-Riemannian space–time with the metric tensor $g_{\boldsymbol{\mu\nu}}$. We assume that at each point in space-time, in addition to the metric tensor $g_{\boldsymbol{\mu\nu}}$, are given two more tensors — the tensor field of gravity $G_{\boldsymbol{\mu\nu}}$, and a tensor $\eta_{\boldsymbol{\mu\nu}}$, which will be called the background tensor. The tensors $g_{\boldsymbol{\mu\nu}}$, $\eta_{\boldsymbol{\mu\nu}}$, $G_{\boldsymbol{\mu\nu}}$ associated asymptotic ratio $g_{\boldsymbol{\mu\nu}} \to \eta_{\boldsymbol{\mu\nu}}$ with $G_{\boldsymbol{\mu\nu}} \to 0$. The tensor $\eta_{\boldsymbol{\mu\nu}}$ is a given value and describes the metric properties of space - time that could take place in the absence of a gravitational field. Not used to describe the geometry of any space. This distinguishes the approach proposed from two-metric formalism. The only metric tensor $g_{\boldsymbol{\mu\nu}}$ is an algebraic function of the tensor $\eta_{\boldsymbol{\mu\nu}}$ and $G_{\boldsymbol{\mu\nu}}$. The covariant derivative of GF yields its strength; accordingly, the GF has energy–momentum density tensor. The question is if such a consistent description of all gravitational phenomena is possible? The goal of this study is to show that such an approach is possible. To this end, we define a relation between the gravitational potentials and the metric tensor, establish a variational principle, and derive field equations and conservation laws of the energy–momentum of matter and the GF. As the Lagrangian of the GF, we take the simplest scalar invariant composed of GF strengths. Consider an example of a specific Lagrangian of the GF that correctly describes all observable local gravitational phenomena in the second-order approximation in the same manner as in GR. This is a nontrivial result, because, on the one hand, the GF in the theory proposed is an ordinary physical field in the sense of Faraday–Maxwell that has positive energy density, and, on the other hand, all phenomena occur, just as in GR, in curved space–time.

Conventionally, tensor and spinor fields in GR are considered as a set of components that are defined at each point of the space in a given system of coordinates and are appropriately transformed when passing to another coordinate system. A variation of the fields in the space is described by covariant derivatives [10, 27, 28]. There also exists another approach, in which all fields are assumed to be abstract geometric objects [10, 29]. Below, we will apply a method of abstract tensor and spinor indices proposed by Penrose [29, 30]. This method allows us to operate with tensors irrespective of the coordinate system and coordinate bases and retains all the advantages of the component approach.

Denote by Greek letters $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, ...$ abstract vector indices and by Latin letters $i, j, k, ...$ taking values of 0, 1, 2, 3, vector components in a given basis. We assume that the space–time is a 4-dimensional manifold whose points are uniquely parameterized, at least in one of the coordinate charts, by a coordinate system $r^i = (r^0, r^1, r^2, r^3)$, where $r^0 = ct$, t is time, r^1, r^2, r^3 are spatial coordinates, and c is the velocity of light. The coordinate system is assigned a natural vector basis $e_i^{\boldsymbol{\mu}}$. By a small variation of the coordinates dr^i, we can obtain an abstract small-displacement vector

$$dr^{\boldsymbol{\mu}} = dr^i \boldsymbol{e}_i^{\boldsymbol{\mu}}. \tag{2}$$

Here the standard convention of summing over repeated upper and lower indices is used. Similarly, any abstract vector $A^{\boldsymbol{\mu}}$ can be represented as $A^i e_i^{\boldsymbol{\mu}}$, and any abstract multicomponent tensor, as $S^{\boldsymbol{\mu}}_{\boldsymbol{\nu\rho}} = S^i_{jk} e_i^{\boldsymbol{\mu}} e^j_{\boldsymbol{\nu}} e^k_{\boldsymbol{\rho}}$, where A^i and S^i_{jk} are the components of the vector and tensor, respectively.

Let us define a metric structure on the manifold considered. To this end, at each point we define 16 numbers g_{ij}, called covariant metric coefficients, which define the scalar product of two vectors $e_i^{\boldsymbol{\mu}}$ and $e_j^{\boldsymbol{\nu}}$ of the coordinate basis,

$$g_{ij} = g_{\boldsymbol{\mu\nu}} e_i^{\boldsymbol{\mu}} e_j^{\boldsymbol{\nu}} = e_{\boldsymbol{\mu} i} e_j^{\boldsymbol{\mu}}, \tag{3}$$

where $g_{\boldsymbol{\mu\nu}}$ is a symbol of convolution over two abstract indices. Identical upper and lower indices also imply convolution. From the relations

$$g_{ik} g^{jk} = \delta_i^j, \quad e^i_{\boldsymbol{\mu}} e_j^{\boldsymbol{\mu}} = \delta^i_j, \tag{4}$$

where δ_i^j is the Levi-Civita symbol ($\delta_i^j = 1$ for $i = j$ and $\delta_i^j = 0$ for $i \neq j$), we obtain the contravariant metric coefficients g^{ik} and a basis $e^i_{\boldsymbol{\mu}}$ dual to the coordinate basis $e_j^{\boldsymbol{\mu}}$.

The representation

$$g_{\boldsymbol{\mu\nu}} = g_{ij} e^i_{\boldsymbol{\mu}} e^j_{\boldsymbol{\nu}} \tag{5}$$

gives a convenient expression for the metric coefficients, which are the components of the abstract metric tensor in the coordinate basis. For abstract indices, we have

$$g_{\boldsymbol{\mu\sigma}} g^{\boldsymbol{\nu\sigma}} = \delta^{\boldsymbol{\nu}}_{\boldsymbol{\mu}}, \quad \delta^{\boldsymbol{\mu}}_{\boldsymbol{\mu}} = 4, \tag{6}$$

where $\delta^{\boldsymbol{\nu}}_{\boldsymbol{\mu}}$ is an abstract analog of the Levi-Civita coordinate symbol. The square of the invariant interval

$$ds^2 = g_{\boldsymbol{\mu\nu}} dr^{\boldsymbol{\mu}} dr^{\boldsymbol{\nu}} = g_{ij} dr^i dr^j \tag{7}$$

can be expressed in either the abstract or the coordinate form. By definition, $g_{ij} = g_{ji}$ and g_{ij} have 10 independent components.

Let $\partial_k = \frac{\partial}{\partial r^k}$ be the coordinate derivative. When passing from one point of the space to another, the vectors of the coordinate basis are changed by the quantity

$$\delta e_j^{\boldsymbol{\mu}} = \partial_i e_j^{\boldsymbol{\mu}} dr^i = \Gamma_{ij}^n e_n^{\boldsymbol{\mu}} dr^i, \qquad (8)$$

where Γ_{ij}^n—the Christoffel symbols—are connection coefficients of the coordinate basis. For example, the derivative of the second-rank tensor field

$$\partial_k S^{\boldsymbol{\mu\nu}} = (\partial_k S^{ij} + \Gamma_{kn}^i S^{nj} + \Gamma_{kn}^j S^{in}) e_i^{\boldsymbol{\mu}} e_j^{\boldsymbol{\nu}}$$

has components that coincide with the components of the covariant derivative. Let us restrict our consideration to torsion-free spaces; then the connection coefficients are symmetric:

$$\Gamma_{ij}^n = \Gamma_{ji}^n. \qquad (9)$$

The metric space is a Riemannian space if

$$\partial_k (g_{\boldsymbol{\mu\nu}} e_i^{\boldsymbol{\mu}} e_j^{\boldsymbol{\nu}}) = g_{\boldsymbol{\mu\nu}} \partial_k (e_i^{\boldsymbol{\mu}} e_j^{\boldsymbol{\nu}}). \qquad (10)$$

Conditions (9) and (10) lead to the following relationship between the Christoffel symbols and the metric coefficients:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kn} (\partial_i g_{nj} + \partial_j g_{in} - \partial_n g_{ij}). \qquad (11)$$

When we use abstract indices, the covariant derivative is replaced by the differential operator $\partial_{\boldsymbol{\mu}}$ [29, 30], which we define as

$$\partial_{\boldsymbol{\mu}} = e_{\boldsymbol{\mu}}^j \partial_j. \qquad (12)$$

Condition (10) can be rewritten as

$$\partial_{\boldsymbol{\sigma}} g_{\boldsymbol{\mu\nu}} = 0, \qquad (13)$$

which corresponds to covariant constancy of the metric tensor. Let us define the commutator of derivatives in a similar way:

$$\Delta_{\boldsymbol{\mu\nu}} = \partial_{\boldsymbol{\mu}} \partial_{\boldsymbol{\nu}} - \partial_{\boldsymbol{\nu}} \partial_{\boldsymbol{\mu}}. \qquad (14)$$

For a tensor field, we have [29, 30]

$$\Delta_{\boldsymbol{\mu\nu}} G_{\boldsymbol{\chi}}^{\boldsymbol{\pi}} = R_{\boldsymbol{\mu\nu\sigma}}^{\boldsymbol{\pi}} G_{\boldsymbol{\chi}}^{\boldsymbol{\sigma}} - R_{\boldsymbol{\mu\nu\chi}}^{\boldsymbol{\sigma}} G_{\boldsymbol{\sigma}}^{\boldsymbol{\pi}},$$

where $R_{\boldsymbol{\mu\nu\sigma}}^{\boldsymbol{\pi}}$ is the curvature tensor of space–time with components $R_{ijk}^n = \partial_i \Gamma_{jk}^n - \partial_j \Gamma_{ik}^n + \Gamma_{jk}^m \Gamma_{im}^n - \Gamma_{ik}^m \Gamma_{jm}^n$.

Let $g = \det(g_{ij})$ be a determinant composed of metric coefficients. We have

$$\partial_{\boldsymbol{\mu}} \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{ij} \partial_{\boldsymbol{\mu}} g_{ij}. \qquad (15)$$

The following relation holds for an arbitrary vector field $X^{\boldsymbol{\mu}}$:

$$\sqrt{-g} \partial_{\boldsymbol{\sigma}} X^{\boldsymbol{\sigma}} = \partial_k (X^k \sqrt{-g}), \qquad (16)$$

which shows that $\sqrt{-g} \partial_{\boldsymbol{\sigma}} X^{\boldsymbol{\sigma}}$ is a total derivative.

II. GRAVITATIONAL FIELD AND ITS RELATION TO THE METRIC STRUCTURE OF THE SPACE

The world in which we live is a space with distributed matter. The space is three-dimensional. The forth dimension—time—is a parameter whose variation changes the distribution of matter in the space. Matter consists of various fields distributed in the space. According to Einstein's idea, all phenomena occur in curved pseudo-Riemannian space–time, and the matter distributed in the space affects its structure. In particular, in GR Einstein represented this effect as a direct relationship between the curvature of space–time and the energy–momentum density of matter. A variation in the density of energy-momentum of matter changes the curvature of space–time, and accordingly, the metric structure of space-time.

In the present study, we consider another possibility. Matter is a source of the GF. The GF affects the metric structure of space–time. A variation in the metric structure of space–time leads to a change in the curvature of space–time.

As a gravitational potential, we take a symmetric tensor $G_{\boldsymbol{\mu\nu}}$, which is the sole dynamical field. The main assumption of this paper is as follows: the GF tensor and the metric tensor are related by the formula

$$g_{\boldsymbol{\mu\nu}} = \eta_{\boldsymbol{\mu\nu}} + G_{\boldsymbol{\mu\nu}}, \qquad (17)$$

where $\eta_{\boldsymbol{\mu\nu}}$ is the background tensor. When $G_{\boldsymbol{\mu\nu}} = 0$, there is no GF, $g_{\boldsymbol{\mu\nu}} = \eta_{\boldsymbol{\mu\nu}}$. When considering the local properties of space–time, the background tensor can be called the Minkowski tensor. The equation

$$\eta_{\boldsymbol{\mu\alpha}} \zeta^{\boldsymbol{\alpha\nu}} = \delta_{\boldsymbol{\mu}}^{\boldsymbol{\nu}} \qquad (18)$$

defines the second background tensor $\zeta^{\boldsymbol{\alpha\nu}}$, which is the inverse of the tensor $\eta_{\boldsymbol{\mu\alpha}}$. The tensors thus defined possess the following properties:

$$\begin{aligned} \partial_{\boldsymbol{\sigma}} g_{\boldsymbol{\mu\nu}} &= 0, \\ \partial_{\boldsymbol{\sigma}} \eta_{\boldsymbol{\mu\nu}} &= -\partial_{\boldsymbol{\sigma}} G_{\boldsymbol{\mu\nu}}, \\ \partial_{\boldsymbol{\sigma}} \zeta^{\boldsymbol{\mu\nu}} &= \zeta^{\boldsymbol{\mu\alpha}} \zeta^{\boldsymbol{\nu\beta}} \partial_{\boldsymbol{\sigma}} G_{\boldsymbol{\alpha\beta}}. \end{aligned} \qquad (19)$$

The derivative of the tensor $G_{\boldsymbol{\mu\nu}}$, unlike the derivative of the metric tensor, is not identically zero.

The assumptions made allow us to formulate a theory of GF within the standard classical field theory and the principle of stationary action.

III. FIELD EQUATIONS

Consider a system consisting of the GF $G_{\boldsymbol{\mu\nu}}$ and the fields of matter, which we denote by a generalized symbol Ψ. The field equations are derived from the extremality of action, which we write as

$$S_s = \int L \sqrt{-g} d^4 r, \qquad (20)$$

where L is the Lagrangian density of the system and $\sqrt{-g}d^4r$ is an invariant volume. Assume that the matter field and the GF satisfy the Euler–Lagrange equations of at most second order. Then, L generally depends on the metric tensor, the background tensor, the matter and gravitational fields, and the first-order derivatives of these fields:

$$L = L(g^{\mu\nu}, \zeta^{\mu\nu}, \Psi, \partial_\sigma \Psi, G_{\mu\nu}, \partial_\sigma G_{\mu\nu}). \quad (21)$$

Let us subject the fields to infinitesimal variations:

$$\Psi \to \Psi + \delta\Psi, \quad (22)$$
$$G_{\mu\nu} \to G_{\mu\nu} + \delta G_{\mu\nu},$$

where $\delta\Psi$ is a variation of the matter fields and $\delta G_{\mu\nu}$ is a variation of the GF. The background tensors $\eta_{\mu\nu}$ and $\zeta^{\mu\nu}$ are given quantities, and their variations vanish. Taking into account the relation

$$\delta g^{\sigma\pi} = -g^{\sigma\alpha} g^{\beta\pi} \delta g_{\alpha\beta}, \quad (23)$$

and the relation (17), we obtain the following variations related to the metric tensor:

$$\delta g_{\sigma\pi} = \delta G_{\sigma\pi}, \quad (24)$$
$$\delta g^{\sigma\pi} = -g^{\sigma\mu} g^{\pi\nu} \delta G_{\mu\nu}, \quad (25)$$
$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{\mu\nu} \delta G_{\mu\nu}. \quad (26)$$

The total variation of the action is expressed as

$$\delta S_s = \int \delta L \sqrt{-g} d^4 r, \quad (27)$$

where

$$\begin{aligned} \delta L &= \frac{1}{\sqrt{-g}} \delta(L\sqrt{-g}) \\ &= \left[\frac{\partial L}{\partial \Psi} - \partial_\sigma \frac{\partial L}{\partial \partial_\sigma \Psi}\right] \delta\Psi \\ &+ \left[\frac{\partial L}{\partial G_{\mu\nu}} - \partial_\sigma \frac{\partial L}{\partial \partial_\sigma G_{\mu\nu}} - \frac{1}{2} T_{\sigma\pi} g^{\sigma\mu} g^{\pi\nu}\right] \delta G_{\mu\nu} \quad (28) \\ &+ \partial_\sigma \left[\frac{\partial L}{\partial \partial_\sigma \Psi} \delta\Psi + \frac{\partial L}{\partial \partial_\sigma G_{\mu\nu}} \delta G_{\mu\nu}\right]. \end{aligned}$$

The tensor

$$T_{\sigma\pi} = \frac{2}{\sqrt{-g}} \frac{\partial(L\sqrt{-g})}{\partial g^{\sigma\pi}} \quad (29)$$

is the metric tensor of energy–momentum density of the system. Setting the coefficients of the variations $\delta\Psi$ and $\delta G_{\mu\nu}$ to zero, we obtain the Euler–Lagrange equations.

For the matter field, we have

$$\frac{\partial L}{\partial \Psi} - \partial_\sigma \frac{\partial L}{\partial \partial_\sigma \Psi} = 0, \quad (30)$$

while, for the gravitational field, we have

$$\frac{\partial L}{\partial G_{\mu\nu}} - \partial_\sigma \frac{\partial L}{\partial \partial_\sigma G_{\mu\nu}} = \frac{1}{2} T^{\mu\nu}. \quad (31)$$

IV. ENERGY–MOMENTUM DENSITY TENSOR IN CURVED SPACE-TIME

To obtain the energy–momentum density tensor and its conservation law, we apply a method similar to that of [27, 31]. Since the Lagrangian L of the system of the matter field and the GF does not explicitly depend on coordinates, applying the rule of differentiation of complex functions, we can write the identity

$$\begin{aligned} \partial_\pi L &= \frac{\partial L}{\partial \Psi} \partial_\pi \Psi + \frac{\partial L}{\partial \partial_\sigma \Psi} \partial_\pi \partial_\sigma \Psi \quad (32) \\ &+ \frac{\partial L}{\partial \zeta^{\mu\nu}} \partial_\pi \zeta^{\mu\nu} + \frac{\partial L}{\partial G_{\mu\nu}} \partial_\pi G_{\mu\nu} + \frac{\partial L}{\partial \partial_\sigma G_{\mu\nu}} \partial_\pi \partial_\sigma G_{\mu\nu}. \end{aligned}$$

Taking into account the noncommutativity of the derivatives, we rewrite it as

$$\begin{aligned} \partial_\pi L &= \left(\frac{\partial L}{\partial \Psi} - \partial_\sigma \frac{\partial L}{\partial \partial_\sigma \Psi}\right) \partial_\pi \Psi \\ &+ \left(\frac{\partial L}{\partial \zeta^{\gamma\delta}} \zeta^{\gamma\mu} \zeta^{\delta\nu} + \frac{\partial L}{\partial G_{\mu\nu}} - \partial_\sigma \frac{\partial L}{\partial \partial_\sigma G_{\mu\nu}}\right) \partial_\pi G_{\mu\nu} \quad (33) \\ &+ \partial_\sigma \left(\frac{\partial L}{\partial \partial_\sigma \Psi} \partial_\pi \Psi + \frac{\partial L}{\partial \partial_\sigma G_{\mu\nu}} \partial_\pi G_{\mu\nu}\right) \\ &+ \frac{\partial L}{\partial \partial_\sigma \Psi} \Delta_{\pi\sigma} \Psi + \frac{\partial L}{\partial \partial_\sigma G_{\mu\nu}} \Delta_{\pi\sigma} G_{\mu\nu}. \end{aligned}$$

Assuming that the Euler–Lagrange equations (30), (31) hold, we find that the tensor

$$t_\pi^\sigma = -L\delta_\pi^\sigma + \frac{\partial L}{\partial \partial_\sigma \Psi} \partial_\pi \Psi + \frac{\partial L}{\partial \partial_\sigma G_{\mu\nu}} \partial_\pi G_{\mu\nu}, \quad (34)$$

which is the canonical tensor of energy–momentum density of the system, satisfies the following conservation law:

$$\partial_\sigma t_\pi^\sigma = f_\pi, \quad (35)$$

where

$$\begin{aligned} f_\pi &= -\left(\frac{1}{2} T^{\mu\nu} + \frac{\partial L}{\partial \zeta^{\gamma\delta}} \zeta^{\gamma\mu} \zeta^{\delta\nu}\right) \partial_\pi G_{\mu\nu} \\ &- \frac{\partial L}{\partial \partial_\sigma \Psi} \Delta_{\pi\sigma} \Psi - \frac{\partial L}{\partial \partial_\sigma G_{\mu\nu}} \Delta_{\pi\sigma} G_{\mu\nu} \quad (36) \end{aligned}$$

is the four-vector of the density of forces caused by the curvature of the space and by the effect of the GF strength $\partial_\pi G_{\mu\nu}$ on the density of the metric tensor of energy–momentum of the system. The force f_π is an analog of the density of the Lorentz force $F_{\mu\nu} j^\nu$ for electromagnetic interaction.

In the case of a real GF, the Lagrangian density of the system can be represented as

$$L = L_m + L_g, \quad (37)$$

where L_m is the density of the matter Lagrangian and L_g is the density of the GF Lagrangian. We will consider the case of minimal coupling, when the Lagrangian density of gravitation depends only on the GF and the Lagrangian density of matter depends only on the matter fields and the metric tensor. In this case, the general conservation law (35) can be represented as two separate conservation laws. Indeed, writing, for L_g and L_m, identities similar to (32) and applying the field equations (30) and (31), we obtain the canonical energy–momentum density tensors. For the GF, we have

$$t_{\boldsymbol{\pi}}^{\boldsymbol{\sigma}(g)} = -L_g \delta_{\boldsymbol{\pi}}^{\boldsymbol{\sigma}} + \frac{\partial L_g}{\partial \partial_{\boldsymbol{\sigma}} G_{\boldsymbol{\mu\nu}}} \partial_{\boldsymbol{\pi}} G_{\boldsymbol{\mu\nu}}, \tag{38}$$

with the conservation law

$$\partial_{\boldsymbol{\sigma}} t_{\boldsymbol{\pi}}^{\boldsymbol{\sigma}(g)} = f_{\boldsymbol{\pi}}^{(g)}, \tag{39}$$

where

$$f_{\boldsymbol{\pi}}^{(g)} = -\left(\frac{1}{2} T^{\boldsymbol{\mu\nu}} + \frac{\partial L}{\partial \zeta^{\boldsymbol{\gamma\delta}}} \zeta^{\boldsymbol{\gamma\mu}} \zeta^{\boldsymbol{\delta\nu}}\right) \partial_{\boldsymbol{\pi}} G_{\boldsymbol{\mu\nu}} - \frac{\partial L}{\partial \partial_{\boldsymbol{\sigma}} G_{\boldsymbol{\mu\nu}}} \Delta_{\boldsymbol{\pi\sigma}} G_{\boldsymbol{\mu\nu}}. \tag{40}$$

For the matter fields, we have

$$t_{\boldsymbol{\mu}}^{\boldsymbol{\sigma}(m)} = -L_m \delta_{\boldsymbol{\mu}}^{\boldsymbol{\sigma}} + \frac{\partial L_m}{\partial \partial_{\boldsymbol{\sigma}} \Psi} \partial_{\boldsymbol{\mu}} \Psi, \tag{41}$$

with the conservation law

$$\partial_{\sigma} t_{\boldsymbol{\pi}}^{\boldsymbol{\sigma}(m)} = f_{\boldsymbol{\pi}}^{(m)} = -\frac{\partial L_m}{\partial \partial_{\boldsymbol{\sigma}} \Psi} \Delta_{\boldsymbol{\pi\sigma}} \Psi, \tag{42}$$

where $f_{\boldsymbol{\pi}}^{(m)}$ is the force due to the curvature of space acting on the matter field. This force vanishes for scalar fields, as well as for spinor fields of spin-$\frac{1}{2}$ particles (leptons, quarks) due to the algebraic structure of the Lagrangian density of such fields. For an ideal fluid, which will be used as a model of matter, we should set

$$\partial_{\boldsymbol{\sigma}} t_{\boldsymbol{\pi}}^{\boldsymbol{\sigma}(m)} = 0, \tag{43}$$

just as in GR. For the canonical energy–momentum density tensor (34) we obtain

$$t_{\boldsymbol{\pi}}^{\boldsymbol{\sigma}} = t_{\boldsymbol{\pi}}^{\boldsymbol{\sigma}(m)} + t_{\boldsymbol{\pi}}^{\boldsymbol{\sigma}(g)}. \tag{44}$$

A similar decomposition can be obtained for the metric tensor (29):

$$T_{\boldsymbol{\sigma\pi}} = T_{\boldsymbol{\sigma\pi}}^{(m)} + T_{\boldsymbol{\sigma\pi}}^{(g)}, \tag{45}$$

where

$$T_{\boldsymbol{\sigma\pi}}^{(m)} = \frac{2}{\sqrt{-g}} \frac{\partial (L_m \sqrt{-g})}{\partial g^{\boldsymbol{\sigma\pi}}}, \quad T_{\boldsymbol{\sigma\pi}}^{(g)} = \frac{2}{\sqrt{-g}} \frac{\partial (L_g \sqrt{-g})}{\partial g^{\boldsymbol{\sigma\pi}}}. \tag{46}$$

The metric energy–momentum density tensor may differ from the canonical tensor for both matter anthe GF. We choose Lagrangians L_m, L_g so that the tensors are consistent:

$$T_{\boldsymbol{\sigma\pi}} = g_{(\boldsymbol{\sigma\alpha}} t_{\boldsymbol{\pi})}^{\boldsymbol{\alpha}} + \partial_{(\boldsymbol{\sigma}} Z_{\boldsymbol{\pi})}, \tag{47}$$

where $Z_{\boldsymbol{\pi}}$ is a vector field.

In the general case, $f_{\boldsymbol{\pi}}$ (36) is different from zero; this leads to nonconservation of the energy of the system, whereas, for an isolated static system consisting of matter and GFs, the energy must be conserved. The conservation of energy can be achieved by using the nonuniqueness of the Lagrangian density L. Let us introduce an additional quantity

$$L_c = \frac{c^4}{64\pi k} \partial_{\boldsymbol{\sigma}} (U^{\boldsymbol{\sigma}} + g^{\boldsymbol{\sigma\pi}} D_{\boldsymbol{\pi}}) \tag{48}$$

to the Lagrangian, which is the 4-divergence of the vector fields $U^{\boldsymbol{\sigma}}$ and $D_{\boldsymbol{\pi}}$. The quantity L_c does not lead to additional field equations and does not affect the form of equations (30),(31); however, it makes an additional contribution to the metric tensor of energy–momentum density of the form

$$T_{\boldsymbol{\sigma\pi}}^{(c)} = \frac{c^4}{64\pi k} (\partial_{\boldsymbol{\sigma}} D_{\boldsymbol{\pi}} + \partial_{\boldsymbol{\pi}} D_{\boldsymbol{\sigma}}) - g_{\boldsymbol{\sigma\pi}} L_c \tag{49}$$

and thus affects the solutions of the GF equations. We choose the vector fields $U^{\boldsymbol{\sigma}}$ and $D_{\boldsymbol{\pi}}$ so that the solutions of the field equations for the total energy–momentum density tensor of the system satisfy the local conservation law

$$\partial^{\boldsymbol{\sigma}} T_{\boldsymbol{\sigma\pi}} = \partial^{\boldsymbol{\sigma}} (T_{\boldsymbol{\sigma\pi}}^{(m)} + T_{\boldsymbol{\sigma\pi}}^{(g)} + T_{\boldsymbol{\sigma\pi}}^{(c)}) = 0. \tag{50}$$

We will call $U^{\boldsymbol{\sigma}}$ and $D_{\boldsymbol{\pi}}$ correcting fields. In what follows, by the energy–momentum density tensor of the GF we will mean the tensor

$$T_{\boldsymbol{\sigma\pi}}^{(g)} + T_{\boldsymbol{\sigma\pi}}^{(c)}, \tag{51}$$

and by the Lagrangian of the GF we will mean the scalar $L_g + L_c$.

For a static GF, the vector of the local coordinate basis $e_0^{\boldsymbol{\mu}}$ is a Killing vector. This fact allows us from the local conservation law (50) to derive an integral conserved quantity—the rest energy of the system,

$$E = \int T_0^0 \sqrt{-g} d^3 r. \tag{52}$$

The definition of the rest energy of the system yields a definition of the inertial mass $m = E/c^2$.

The application of the variational principle of the classical field theory allows us to obtain non-linear equation (31), which is significantly different from Einstein's equation. A natural consequence of the new approach is the fact that the GF, just as all the other matter fields, is characterized by energy–momentum density tensor (51), which is obtained from the Lagrangian $L_g + L_c$ of the GF. The source of the GF is the energy–momentum density tensor of all kinds of matter, including the GF itself. A conserved integral characteristic of the source of the GF is its total rest energy (52) and the corresponding mass m of the source.

V. LAGRANGIAN OF THE GRAVITATIONAL FIELD

In Einstein's GR, the GF Lagrangian is given by the scalar curvature of space–time. The potentials of the GF are given by the components of the metric tensor. In the theory of gravity proposed, the gravitational potential is given by a second-rank symmetric tensor $G_{\mu\nu}$. Due to the relationship between the GF and the metric tensor of space-time (17), all phenomena occur, just as in GR, in curved space–time. This renders the gravitational interaction universal. In the presence of matter, the space–time has a nonzero curvature tensor; however, unlike GR, we do not use this tensor in our theory. By analogy with the electromagnetic field, we take the GF Lagrangian in the form of a simple quadratic combination of the first-order derivatives of the GF tensor, which correctly describes gravitational phenomena in the first- and second-order approximations,

$$L_g = \frac{c^4}{64\pi k} g^{\sigma\pi}(\zeta^{\alpha\gamma}\zeta^{\beta\delta} - \frac{1}{2}\zeta^{\alpha\beta}\zeta^{\gamma\delta})\partial_\sigma G_{\alpha\beta}\partial_\pi G_{\gamma\delta}. \quad (53)$$

Lagrangian (53) corresponds to the GF equation

$$\partial_\sigma \left((\zeta^{\alpha\gamma}\zeta^{\beta\delta} - \frac{1}{2}\zeta^{\alpha\beta}\zeta^{\gamma\delta})\partial^\sigma G_{\alpha\beta} \right) = -\frac{16\pi k}{c^4} T^{\gamma\delta}, \quad (54)$$

where $T^{\gamma\delta} = g^{\gamma\sigma}g^{\delta\pi}T_{\sigma\pi}$, $T_{\sigma\pi} = T^{(m)}_{\sigma\pi} + T^{(g)}_{\sigma\pi} + T^{(c)}_{\sigma\pi}$ is the total energy–momentum density tensor of the system consisting of matter, GF, and correcting fields. The energy–momentum density tensor of the GF is

$$T^{(g)}_{\sigma\pi} = \frac{c^4}{32\pi k}(\zeta^{\alpha\gamma}\zeta^{\beta\delta} - \frac{1}{2}\zeta^{\alpha\beta}\zeta^{\gamma\delta})\partial_\sigma G_{\alpha\beta}\partial_\pi G_{\gamma\delta} - g_{\sigma\pi}L_g \quad (55)$$

For Lagrangian (53), the density of the canonical energy–momentum density tensor (34) is consistent with the metric tensor

$$T^{(g)}_{\sigma\pi} = g_{\sigma\gamma}t^{\gamma(g)}_{\pi}. \quad (56)$$

It is the consistency condition that is responsible for the fact that the convolution of the tensors $\partial_\sigma G_{\alpha\beta}$ and $\partial_\pi G_{\gamma\delta}$ over the indices α, β and γ, δ is performed with the use of the background, rather than the metric, tensor. We also assume that a similar relation holds for the energy–momentum density of matter.

In the first-order approximation, the GF equation turns into the equation of the linearized theory of gravity [10]. As a consequence, the metric structure of space–time in this approximation coincides with the space–time structure of GR.

VI. SOLUTION OF GRAVITATIONAL FIELD EQUATIONS

The GF is determined by the source. Consider the simplest case: a spherically symmetric body of radius R_b consisting of an ideal fluid with energy–momentum density tensor

$$T^{(m)}_{\mu\nu} = (\varepsilon + P)u_\mu u_\nu - P g_{\mu\nu}, \quad (57)$$

where ε is the density of rest energy, P is pressure, and u_μ is the velocity 4-vector of an element of the body.

Let us obtain a solution to the equations for a fixed body in which there is no internal motion of matter. In this case, the velocity 4-vector of an element of the body is

$$u_\mu = \sqrt{g_{00}}e^0_\mu. \quad (58)$$

In the system of coordinates $r^i = (ct, r, \theta, \varphi)$, where r, θ, φ are spherical coordinates, we have

$$\eta_{\alpha\beta} = e^0_\alpha e^0_\beta - (e^1_\alpha e^1_\beta + r^2 e^2_\alpha e^2_\beta + r^2 sin^2(\theta)e^3_\alpha e^3_\beta), \quad (59)$$

$$\zeta^{\alpha\beta} = e^\alpha_0 e^\beta_0 - (e^\alpha_1 e^\beta_1 + \frac{1}{r^2}e^\alpha_2 e^\beta_2 + \frac{1}{r^2 sin^2(\theta)}e^\alpha_3 e^\beta_3), \quad (60)$$

$$\eta = det(\eta_{ik}) = -r^2 sin(\theta). \quad (61)$$

The spherical symmetry of the distribution of energy density and the pressure of matter (57) imposes the following constraint on the GF tensor, the correcting fields, and the metric tensor:

$$G_{\mu\nu} = F(r)e^0_\mu e^0_\nu + A(r)e^1_\mu e^1_\nu + [A(r) + B(r)]r^2(e^2_\mu e^2_\nu + sin^2(\vartheta)e^3_\mu e^3_\nu), \quad (62)$$

$$U^\sigma = U(r)e^\sigma_1, \quad D_\pi = D(r)e^1_\pi, \quad (63)$$

$$P = P(r), \quad (64)$$

$$g_{\mu\nu} = [1 + F(r)]e^0_\mu e^0_\nu - [1 - A(r)]e^1_\mu e^1_\nu - [1 - A(r) - B(r)]r^2(e^2_\mu e^2_\nu + sin^2(\vartheta)e^3_\mu e^3_\nu), \quad (65)$$

$$-g = [1 + F(r)][1 - A(r)][(1 - A(r) - B(r)]^2\eta, \quad (66)$$

where $F(r), A(r), B(r), U(r), D(r)$, and $P(r)$ are functions of r that are to be determined. The nonzero $B(r)$ leads to the anisotropy of the metric tensor.

Let ε_0, E_0, and m_0 be the energy density, energy, and the mass of a stationary body defined in the absence of GF, and ε, E, and m be the energy density, energy, and the mass of the body in the presence of the GF. In the general case, the GF acts on both the matter and the spatial volume. Let us neglect the effect of the GF on the matter of the body. The variation of the differential volume is

$$d^3V = \kappa d^3V_0, \quad (67)$$

where d^3V_0 is the differential volume in the absence of GF and

$$\kappa = \sqrt{(1-A(r))(1-A(r)-B(r))^2}. \tag{68}$$

In the presence of GF, the density of the rest energy is

$$\varepsilon = \varepsilon_0 \kappa^{-1}. \tag{69}$$

For the total energy of the body we obtain

$$E = \int (\varepsilon + T_0^{0(g)})\sqrt{-g}d^3r = \int \varepsilon_0 \sqrt{g_{00}}\sqrt{\eta}d^3r + E_{gf}, \tag{70}$$

where $T_0^{0(g)}$ and E_{gf} are the energy density and the total energy of the GF of the body.

The GF is determined by three equations: the GF equation (54), the equation of conservation of the energy–momentum density tensor of the system (50), and the equation of conservation of the energy–momentum tensor of matter (43). In the component representation, we obtain five independent equations involving six independent unknown functions. One of these functions should be chosen as a parameter. The solution of the equations by the method of series expansion shows that one should take $B(r)$ as the parameter, because all the equations of the system can be consistently resolved only in this case. Equations (54) and (50) have a relatively simple tensor form. In the component representation, these equations constitute a complicated nonlinear system of coupled second-order differential equations even in the simplest case of spherical symmetry (see the Appendix, where we present the equations under the condition $B(r) = 0$). To obtain these equations, we used a computer algebra package TTC (Tools of Tensor Calculus) [32] in a Mathematica system [33] developed by A.Balfagón, P.Castellví, and X.Jaén.

Next, we consider the solution of the GF equations for a constant energy density of the matter of the body ε_0. In this case,

$$\varepsilon_0 = \frac{E_0}{\frac{4}{3}\pi R_b^3}.$$

In what follows, it will be convenient to introduce a dimensionless quantity

$$p = \frac{kE_0}{c^4 R_b} = \frac{km_0}{c^2 R_b} \tag{71}$$

which is the ratio of the gravitational radius of the body kE_0/c^4 to the geometric radius R_b. We will call p the compactness parameter of the body. The second interpretation of p is the absolute value of the Newton potential on the surface of the body. Usually, $p << 1$, which corresponds to the case of weak fields. In particular, $p \simeq 6.96 \times 10^{-10}$ for the Earth and $p \simeq 2.1 \times 10^{-6}$ for the Sun. The GF is weak at any distance from the center of the body. The GF may be strong near neutron stars, where p is greater than 0.1. The notations introduced allow us to write

$$16\pi k T_{\boldsymbol{\mu\nu}}^{(m)} = \frac{12p}{R_b^2}\left[(\kappa^{-1}+\Pi)u_{\boldsymbol{\mu}}u_{\boldsymbol{\nu}} - \Pi g_{\boldsymbol{\mu\nu}}\right], \tag{72}$$

where $\Pi = P/\varepsilon_0$.

Equation (43) leads to a single equation—hydrostatic equilibrium equation

$$\frac{F'(r)}{1+F(r)} + \frac{2\Pi'(r)}{\kappa^{-1}+\Pi(r)} = 0. \tag{73}$$

For weak fields, the external and internal solutions of the field equations can be obtained by a series expansion. The solution exhibits qualitatively different dependence on r inside the body, for $r \leq R_b$, and in the external domain, where there is no matter. On the boundary of the body, for $r = R_b$, $U(r)$ and $D(r)$ must be continuous, while $F(r)$, $A(r)$, and $B(r)$ must be continuous together with their first derivatives. Let us take the radius of the body R_b as the unit of length. Expand the external solution as

$$F(r) = F_1\frac{p}{r} + F_2\frac{p^2}{r^2} + F_3\frac{p^3}{r^3} + O(p^4), \tag{74}$$

$$D(r) = D_2\frac{p^2}{r^3} + D_3\frac{p^3}{r^4} + O(p^4), \tag{75}$$

and the internal solution as

$$F(r) = f_0 + f_1 pr^2 + f_2 p^2 r^4 + f_3 p^3 r^6 + O(p^4), \tag{76}$$

$$D(r) = d_0 p^2 r + d_2 p^2 r^3 + d_3 p^3 r^5 + O(p^4), \tag{77}$$

$$\Pi(r) = \Pi_0 p + \Pi_1 pr^2 + \Pi_2 p^2 r^4 + \Pi_3 p^3 r^6 + O(p^4), \tag{78}$$

where $F_i, f_i, D_i, d_i, \Pi_i$ are constants that depend on p. Similarly for $A(r), B(r)$, and $U(r)$. Let us substitute the expansions into equations (50), (54), and (73) and set the coefficients of appropriate powers of r to zero. We obtain a solution depending on the constants $F_1, A_1, f_0, a_0, d_0, u_0$, and Π_0. Expressing these constants as expansions in the compactness parameter p, we determine them from the condition of continuity on the boundary of the body. Performing the above-listed procedures, we obtain the following expressions in the third-order approximation.

For the external solution:

$$F(r) = -\frac{2p}{r}\left(1 - \frac{3p}{5} - \frac{3699p^2}{1400}\right) + \frac{1}{r^2}\left(\frac{p^2}{4} - \frac{3p^3}{10}\right) - \frac{22p^3}{15r^3}, \tag{79}$$

$$A(r) = -\frac{2p}{r}\left(1 - \frac{3p}{5} - \frac{507p^2}{1400}\right) - \frac{1}{r^2}\left(\frac{15p^2}{4} - \frac{9p^3}{2}\right) - \frac{p^3}{r^3}\left(\frac{104}{15} + \frac{2}{3}B_3\right), \tag{80}$$

$$B(r) = B_3 \frac{p^3}{r^3}, \tag{81}$$

$$U(r) = -\frac{1}{r^3}\left(2p^2 - \frac{12p^3}{5}\right) + \frac{18p^3}{5r^4}, \tag{82}$$

$$D(r) = \frac{1}{r^3}\left(2p^2 - \frac{12p^3}{5}\right) + \frac{16p^3}{5r^4}, \tag{83}$$

$$\begin{aligned}\sqrt{g_{00}} = 1 &- \frac{p}{r}\left(1 - \frac{3p}{5} - \frac{3699p^2}{1400}\right)\\ &- \frac{1}{r^2}\left(\frac{3p^2}{8} - \frac{9p^3}{20}\right) - \frac{133p^3}{120r^3},\end{aligned} \tag{84}$$

$$\begin{aligned}\sqrt{\frac{g}{\eta}} &= 1 + \frac{p}{r}\left(2 - \frac{6p}{5} + \frac{1089p^2}{700}\right)\\ &+ \frac{1}{r^2}\left(\frac{15p^2}{4} - \frac{9p^3}{2}\right) + \frac{p^3}{r^3}\left(\frac{37}{6} + B_3\right).\end{aligned} \tag{85}$$

For the internal solution:

$$\begin{aligned}F(r) = -3p &+ \frac{39p^2}{20} + \left(\frac{2897}{700} - \frac{21}{16}B_3\right)p^3\\ &+ r^2\left(p - \frac{3p^2}{20} - \frac{186p^3}{175}\right)\\ &- r^4\left(\frac{7p^2}{20} - \frac{81p^3}{200} - \frac{63p^3B_3}{16}\right)\\ &+ r^6p^3\left(\frac{31}{840} - \frac{21B_3}{8}\right),\end{aligned} \tag{86}$$

$$\begin{aligned}A(r) = -3p &- \frac{129p^2}{20} - \left(\frac{7937}{700} - \frac{35B_3}{16}\right)p^3\\ &+ r^2\left(p + \frac{93p^2}{20} + \frac{5097p^3}{350}\right)\\ &- r^4\left(\frac{3p^2}{4} + \frac{231p^3}{40} + \frac{153p^3B_3}{16}\right)\\ &+ r^6p^3\left(\frac{101}{120} + \frac{161}{24}B_3\right),\end{aligned} \tag{87}$$

$$B(r) = p^3B_3\left(\frac{9}{2}r^4 - \frac{7}{2}r^6\right), \tag{88}$$

$$\begin{aligned}U(r) &= p^2r\left(-\frac{6}{5} + \frac{3342p}{175}\right)\\ &- p^2r^3\left(\frac{4}{5} + \frac{396p}{25} - 63pB_3\right)\\ &+ p^3r^5\left(\frac{96}{35} - 63B_3\right),\end{aligned} \tag{89}$$

$$\begin{aligned}D(r) &= p^2r\left(6 + \frac{54p}{5}\right)\\ &- p^2r^3\left(4 + \frac{84p}{5} + 63pB_3\right)\\ &+ p^3r^5\left(\frac{34}{5} + 63B_3\right),\end{aligned} \tag{90}$$

$$\begin{aligned}\Pi(r) = \frac{p}{2} &- \frac{3p^2}{4} + \frac{1111p^3}{4200} + \frac{21B_3p^3}{32}\\ &- r^2\left(\frac{p}{2} - \frac{23p^2}{40} + \frac{411p^3}{1400}\right)\\ &+ r^4\left(\frac{7p^2}{40} + \frac{13p^3}{50} - \frac{63B_3p^3}{32}\right)\\ &+ r^6p^3\left(\frac{21}{16}B_3 - \frac{97}{420}\right),\end{aligned} \tag{91}$$

$$\begin{aligned}\sqrt{g_{00}} = 1 &- \frac{3p}{2} - \frac{3p^2}{20} + p^3\left(\frac{1291}{700} - \frac{21B_3}{32}\right)\\ &+ r^2\left(\frac{p}{2} + \frac{27p^2}{40} + \frac{1557p^3}{2800}\right)\\ &- r^4\left(\frac{3p^2}{10} + \frac{117p^3}{200} - \frac{63p^3B_3}{32}\right)\\ &+ r^6p^3\left(\frac{283}{1680} - \frac{21B_3}{16}\right),\end{aligned} \tag{92}$$

$$\begin{aligned}\sqrt{\frac{g}{\eta}} = 1 &+ 3p + \frac{123p^2}{20} + p^3\left(\frac{7999}{700} - \frac{63B_3}{16}\right)\\ &- r^2\left(p + \frac{81p^2}{20} + \frac{7893p^3}{700}\right)\\ &+ r^4\left(\frac{9p^2}{20} + \frac{693p^3}{200} + \frac{261p^3B_3}{16}\right)\\ &- p^3r^6\left(\frac{331}{840} + \frac{91}{8}B_3)\right).\end{aligned} \tag{93}$$

The solution obtained is expressed in terms of the compactness parameter p, which, in turn, is expressed in terms of the mass m_0 of the original matter of the body in the absence of GF. Introduce one more mass,

$$m_g = m_0(1 - \frac{3}{5}p - \frac{3699}{1400}p^2 + O(p^3)), \tag{94}$$

the gravitational mass. It is this value which is the ratio of the mass in r^{-1} in the component of the gravitational potential $F(r)$ (79). The corresponding compactness parameter $p_g = \frac{km_g}{c^2R_b}$ is related to p by the condition:

$$p_g = p - \frac{3}{5}p^2 - \frac{3699}{1400}p^3 + O(p^4), \tag{95}$$

or

$$p = p_g + \frac{3}{5}p_g^2 + \frac{4707}{1400}p_g^3 + O(p_g^4). \tag{96}$$

When expressed in terms of p_g, the external solution has the following form in the third-order approximation:

$$F(r) = -\frac{2p_g}{r} + \frac{p_g^2}{4r^2} - \frac{22p_g^3}{15r^3}, \tag{97}$$

$$A(r) = -\frac{2p_g}{r}\left(1 + \frac{57}{25}p_g^2\right) - \frac{15p_g^2}{4r^2} - \left(\frac{104}{15} + \frac{2}{3}B_3\right)\frac{p_g^3}{r^3}, \tag{98}$$

$$B(r) = B_3\frac{p_g^3}{r^3}, \tag{99}$$

$$U(r) = -\frac{2p_g^2}{r^3} + \frac{18p_g^3}{5r^4}, \tag{100}$$

$$D(r) = \frac{2p_g^2}{r^3} + \frac{16p_g^3}{5r^4}, \tag{101}$$

$$\sqrt{g_{00}} = 1 - \frac{p_g}{r} - \frac{3p_g^2}{8r^2} - \frac{133p_g^3}{120r^3}, \tag{102}$$

$$\sqrt{\frac{g}{\eta}} = 1 + \frac{p_g}{r}\left(2 + \frac{171}{25}p_g^2\right) + \frac{15p_g^2}{4r^2} + \left(\frac{37}{6} + B_3\right)\frac{p_g^3}{r^3}. \tag{103}$$

In the third-order approximation, the solution is determined by two parameters: the compactness parameter p and an arbitrary constant B_3. The indefinite quantity $B(r)$ arises because the number of unknown functions is greater than the number of equations that define these functions. This fact manifests itself only in the third and higher order expansions in the compactness parameter p. In the second-order approximation, $B(r) = 0$. The GF and the metric tensor are defined uniquely. Take $B(r) = 0$ in all orders. This condition establishes a correspondence between the number of unknown functions and the number of equations and allows us to write the GF tensor

$$G_{\mu\nu} = F(r)e^0_\mu e^0_\nu + A(r)(e^0_\mu e^0_\nu - \eta_{\mu\nu}) \tag{104}$$

in an isotropic form, where there is no chosen spatial direction. If necessary, we can obtain higher order expansion terms.

In the second-order approximation, for the external solution we have

$$g_{\mu\nu} = \left(1 - \frac{2km_g}{c^2r} + \frac{(km_g)^2}{4c^4r^2}\right)e^0_\mu e^0_\nu - \left(1 + \frac{2km_g}{c^2r} + \frac{15(km_g)^2}{4c^4r^2}\right)(e^0_\mu e^0_\nu - \eta_{\mu\nu}). \tag{105}$$

For comparison, in GR, the Schwarzschild solution in isotropic coordinates is as follows:

$$g_{\mu\nu} = \left(1 - \frac{2km_g}{c^2r} + \frac{2(km_g)^2}{c^4r^2}\right)e^0_\mu e^0_\nu - \left(1 + \frac{2km_g}{c^2r} + \frac{3(km_g)^2}{2c^4r^2}\right)(e^0_\mu e^0_\nu - \eta_{\mu\nu}). \tag{106}$$

In the parameterized post-Newtonian (PPN) formalism [21], the following parameters correspond to the metric tensor (105):

$$\gamma = 1, \beta = \frac{1}{8}, \xi = 0, \alpha_i = 0, \varsigma_k = 0. \tag{107}$$

The parameter $\gamma = 1$ is due to the coincidence of the GF Lagrangian with the Lagrangian of the linearized theory of gravity in the first-order approximation. The value of the parameter $\beta = \frac{1}{8}$ differs from the value $\beta = 1$ in GR. In the second-order approximation, the theories differ substantially.

In Newton's theory of gravity, the GF of a body is described by the potential

$$\varphi(r) = -\frac{km_g}{r}. \tag{108}$$

It is convenient to represent the external solution in terms of the Newton potential. In the second-order approximation, solution (79)–(85) is rewritten as

$$F(r) = \frac{2}{c^2}\varphi(r) + \frac{1}{4c^4}\varphi^2(r), \tag{109}$$

$$A(r) = \frac{2}{c^2}\varphi(r) - \frac{15}{4c^4}\varphi^2(r), \tag{110}$$

$$U(r) = -D(r) = -\frac{2}{c^4}\varphi^2\frac{R_b}{r}. \tag{111}$$

$$\sqrt{g_{00}} = 1 + \frac{1}{c^2}\varphi(r) - \frac{3}{8c^4}\varphi^2(r), \tag{112}$$

$$\sqrt{\frac{g}{\eta}} = 1 - \frac{2}{c^2}\varphi(r) + \frac{15}{4c^4}\varphi^2(r), \tag{113}$$

Solution (104) corresponds to a fixed source. For practical applications, we should also take into account the effect of motion of the source on its GF. To this end, we pass from the spherical spatial system of coordinates (r, θ, φ) to a Cartesian system with coordinates (x^1, x^2, x^3). We use the following notation: (x^1_0, x^2_0, x^3_0) are the coordinates of the center of the source of GF, $\mathbf{r} = (x^1 - x^1_0, x^2 - x^2_0, x^3 - x^3_0)$, $r = \sqrt{(\mathbf{rr})} = \sqrt{(r^1)^2 + (r^2)^2 + (r^3)^2}$, $\mathbf{v} = (\frac{dx^1_0}{dt}, \frac{dx^2_0}{dt}, \frac{dx^3_0}{dt})$, is the velocity vector of the source, $\frac{d\mathbf{r}}{dt} = -\mathbf{v}$, $(\mathbf{vr}) = v^1r^1 + v^2r^2 + v^3r^3$, $(\mathbf{vn}) = \frac{(\mathbf{vr})}{r}$, $\partial_n = \frac{\partial}{\partial_n}$, $\boldsymbol{\nabla} = (\partial_1, \partial_2, \partial_3)$ and

$\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$. Solution (104) in a Cartesian spatial system of coordinates is expressed as

$$\eta_{\boldsymbol{\alpha\beta}} = i^0_{\boldsymbol{\alpha}} i^0_{\boldsymbol{\beta}} - (i^1_{\boldsymbol{\alpha}} i^1_{\boldsymbol{\beta}} + i^2_{\boldsymbol{\alpha}} i^2_{\boldsymbol{\beta}} + i^3_{\boldsymbol{\alpha}} i^3_{\boldsymbol{\beta}}),$$

$$G_{\boldsymbol{\mu\nu}} = [F(r) + A(r)] i^0_{\boldsymbol{\mu}} i^0_{\boldsymbol{\nu}} - A(r)\eta_{\boldsymbol{\mu\nu}}, \tag{114}$$

and

$$g_{\boldsymbol{\mu\nu}} = [F(r) + A(r)] i^0_{\boldsymbol{\mu}} i^0_{\boldsymbol{\nu}} + [1 - A(r)]\eta_{\boldsymbol{\mu\nu}}, \tag{115}$$

where $i^k_{\boldsymbol{\mu}}$ are vectors dual to the vectors of the local coordinate basis $i^{\boldsymbol{\mu}}_k$ of the Cartesian system of coordinates. In the second-order approximation under the assumption that the velocities are small compared with the velocity of light, the motion of the source can be taken into account by an appropriate Lorentz transformation of the part of the solution corresponding to the first-order approximation. The sum of the transformed part of the solution and a part of second-order solution gives the solution of the problem posed. The above-mentioned Lorentz transformation is equivalent to the transition from solution (114), (115) to the solution in the form

$$G_{\boldsymbol{\mu\nu}} = [F(r) + A(r)] v_{\boldsymbol{\mu}} v_{\boldsymbol{\nu}} - A(r)\eta_{\boldsymbol{\mu\nu}}, \tag{116}$$

$$g_{\boldsymbol{\mu\nu}} = [F(r) + A(r)] v_{\boldsymbol{\mu}} v_{\boldsymbol{\nu}} + [1 - A(r)]\eta_{\boldsymbol{\mu\nu}},$$

where $v_{\boldsymbol{\mu}} = \frac{1}{\sqrt{1-v^2}}(i^0_{\boldsymbol{\mu}} + v^1 i^1_{\boldsymbol{\mu}} + v^2 i^2_{\boldsymbol{\mu}} + v^3 i^3_{\boldsymbol{\mu}})$. The tensor $\eta_{\boldsymbol{\mu\nu}}$ is invariant under Lorentz transformations.

In the case of a moving body, $\partial_0 r = -\frac{(\mathbf{vr})}{r} \neq 0$, and one should take into account the retardation in the Newtonian potential [27] by replacing (108) with

$$\begin{aligned} \varphi(r) &= -\left(\frac{km_g}{r} + \frac{km_g}{2}\partial_0^2 r\right) \\ &= -\frac{km_g}{r}\left(1 + \frac{1}{2c^2}\left[(\mathbf{vv}) - (\mathbf{vn})^2\right]\right). \end{aligned} \tag{117}$$

Consider also the GF of two bodies. In the second-order approximation,

$$G_{\boldsymbol{\mu\nu}} = G^{(1)}_{\boldsymbol{\mu\nu}} + G^{(2)}_{\boldsymbol{\mu\nu}} + \Phi_{\boldsymbol{\mu\nu}}, \tag{118}$$

where $G^{(1)}_{\boldsymbol{\mu\nu}}$ and $G^{(2)}_{\boldsymbol{\mu\nu}}$ are the GFs of bodies 1 and 2 in accordance with (116) and $\Phi_{\boldsymbol{\mu\nu}}$ is a second-order quantity that depends on the distances to body 1 and body 2. From the GF equation, we obtain

$$\Phi_{00} = 0, \Phi_{0n} = 0; \tag{119}$$

the components Φ_{nm} with $n, m = 1, 2, 3$ are defined as a solution to the equations

$$\begin{aligned} \Delta\Phi_{nm} &= -8(2\boldsymbol{\nabla}\varphi_1\boldsymbol{\nabla}\varphi_2 - \partial_n\varphi_1\partial_m\varphi_2), \quad n = m, \\ \Delta\Phi_{nm} &= 4(\partial_n\varphi_1\partial_m\varphi_2 + \partial_m\varphi_1\partial_n\varphi_2), \quad n \neq m, \end{aligned} \tag{120}$$

where φ_1 and φ_2 are the Newtonian potentials of the bodies.

VII. GRAVITATIONAL FIELD OF A ROTATING BODY

Consider a variation in the GF due to the rotation of the body with angular velocity N_φ about the axis $\theta = 0$. The volume elements move at velocity v_φ along the coordinate φ with the basis vector $e^{\boldsymbol{\mu}}_3$. Let us restrict ourselves to the approximation of $v_\varphi << 1, r_g << R_b$ and to a spherically symmetric body. Then $v_\varphi = N_\varphi r \sin\theta$. In the energy–momentum density tensor of matter, the following term appears due to rotation:

$$T^{(m)}_{\boldsymbol{\mu\nu}} = T^{(m)}_{\boldsymbol{\mu\nu}(0)} - \varepsilon N_\varphi r^2 \sin^2(\theta)(e^3_{\boldsymbol{\mu}} e^0_{\boldsymbol{\nu}} + e^0_{\boldsymbol{\mu}} e^3_{\boldsymbol{\nu}}), \tag{121}$$

where $T^{(m)}_{\boldsymbol{\mu\nu}(0)}$ is the energy–momentum density tensor without rotation. In the approximation linear with respect to N_φ, the source corresponds to the field

$$G_{\boldsymbol{\mu\nu}} = G^{(0)}_{\boldsymbol{\mu\nu}} - Qcr^2 \sin^2\theta(e^3_{\boldsymbol{\mu}} e^0_{\boldsymbol{\nu}} + e^0_{\boldsymbol{\mu}} e^3_{\boldsymbol{\nu}}), \tag{122}$$

where $G^{(0)}_{\boldsymbol{\mu\nu}}$ is the GF (62) of the source without rotation. From the field equations we obtain

$$Q = \left(5 - \frac{3r^2}{R_b^2}\right)\frac{kL_\varphi}{c^3 R_b^3} \tag{123}$$

for the internal solution and

$$Q = \frac{2kL_\varphi}{c^3 r^3} \tag{124}$$

for the external solution, where $L_\varphi = \frac{4}{5} M N_\varphi R^2$ is the angular momentum of a uniform sphere. The GF corresponds to the following metric tensor:

$$g_{\boldsymbol{\mu\nu}} = g^{(0)}_{\boldsymbol{\mu\nu}} - Qcr^2 \sin^2\theta(e^3_{\boldsymbol{\mu}} e^0_{\boldsymbol{\nu}} + e^0_{\boldsymbol{\mu}} e^3_{\boldsymbol{\nu}}). \tag{125}$$

Here $g^{(0)}_{\boldsymbol{\mu\nu}}$ is the metric tensor corresponding to a source without rotation.

VIII. RADIATION OF GRAVITATIONAL WAVES

Consider the gravitational radiation field produced by a compact system of bodies that move with velocities small compared with the velocity of light. At a sufficiently large distance from the bodies, the GF can be assumed weak and the space, flat. To describe the GF, it suffices to apply the first nonvanishing approximation that leads to the equation of the linearized theory of gravity

$$\partial_{\boldsymbol{\sigma}}\partial^{\boldsymbol{\sigma}}\left(G^{\boldsymbol{\mu\nu}} - \frac{1}{2}\zeta^{\boldsymbol{\mu\nu}} G^{\boldsymbol{\sigma}}_{\boldsymbol{\sigma}}\right) = -16\pi k T^{\boldsymbol{\mu\nu}}, \tag{126}$$

which is the wave equation and has the following solution:

$$G^{\boldsymbol{\mu\nu}} - \frac{1}{2}\zeta^{\boldsymbol{\mu\nu}} G^{\boldsymbol{\sigma}}_{\boldsymbol{\sigma}} = -4k\int \frac{T^{\boldsymbol{\mu\nu}}(\boldsymbol{r}', t - |\boldsymbol{r} - \boldsymbol{r}'|)}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3 r'. \tag{127}$$

At large distances from the source, from this solution we can single out a part that represents a divergent wave, which can be expressed (in the $T-T$ gauge) as [10]

$$h^{\boldsymbol{\mu\nu}}=-\frac{2k}{r}\left(P^{\boldsymbol{\mu}}_{\boldsymbol{\alpha}}P^{\boldsymbol{\nu}}_{\boldsymbol{\beta}}-\frac{1}{2}P^{\boldsymbol{\mu\nu}}P_{\boldsymbol{\alpha\beta}}\right)\ddot{D}^{\boldsymbol{\alpha\beta}}, \tag{128}$$

where $P_{\boldsymbol{\mu\alpha}}$ is the operator of projection onto a plane orthogonal to the radiation propagation direction, which possesses the properties $P_{\boldsymbol{\mu\alpha}}P^{\boldsymbol{\alpha}}_{\boldsymbol{\nu}}=P_{\boldsymbol{\mu\nu}}, P_{\boldsymbol{\mu\alpha}}P^{\boldsymbol{\mu\alpha}}=2$, $n_{\boldsymbol{\mu}}$ is a four-vector in the direction of propagation, $n_{\boldsymbol{\mu}}n^{\boldsymbol{\mu}}=-1$, and $D^{\boldsymbol{\alpha\beta}}=e^{\boldsymbol{\alpha}}_ae^{\boldsymbol{\beta}}_bD^{ab}$ is the reduced quadrupole moment tensor; the two dots over $D^{\boldsymbol{\alpha\beta}}$ denote the second-order time derivative; and

$$D^{ab}=\int\frac{\varepsilon}{c^2}\left(r^ar^b-\frac{1}{3}r^2\delta^{ab}\right)d^3r, \tag{129}$$

where $a,b=1,2,3$. Substituting solution (128) into the energy–momentum density tensor of the GF (55), we obtain the following expression for the energy flux density in the direction of propagation n_μ:

$$\begin{aligned}T_{\boldsymbol{\mu\nu}}e^{\boldsymbol{\mu}}_0n^{\boldsymbol{\nu}}&= \\ \frac{k}{8\pi r^2}\left(\dddot{D}_{\boldsymbol{\alpha\beta}}\dddot{D}^{\boldsymbol{\alpha\beta}}-2\dddot{D}_{\boldsymbol{\sigma\alpha}}\dddot{D}^{\boldsymbol{\sigma}}_{\boldsymbol{\beta}}n^{\boldsymbol{\alpha}}n^{\boldsymbol{\beta}}+\frac{1}{2}(\dddot{D}_{\boldsymbol{\alpha\beta}}n^{\boldsymbol{\alpha}}n^{\boldsymbol{\beta}})^2\right)&.\end{aligned} \tag{130}$$

Integrating this expression over a sphere of radius r, we obtain Einstein's well-known formula for the gravitational energy radiated by a system of bodies in unit time:

$$I=\frac{k}{5}\dddot{D}_{\boldsymbol{\alpha\beta}}\dddot{D}^{\boldsymbol{\alpha\beta}}. \tag{131}$$

IX. ENERGY OF THE GRAVITATIONAL FIELD

For solution (62)–(66), (79)–(93) obtained in the third-order approximation, the energy density of the GF is

$$\begin{aligned}\left(T^{0(g)}_0+T^{0(c)}_0\right)\sqrt{\frac{g}{\eta}}&=\frac{E_0}{16\pi r^4}\left(p-\frac{6p^2}{5}-\frac{3447p^3}{700}\right)\\&+\frac{E_0}{16\pi r^4}\left(\frac{R_b}{r}(\frac{21p^2}{5}-\frac{189p^3}{25})+\frac{26p^3R_b^2}{5r^2}\right)\end{aligned} \tag{132}$$

outside the body and

$$\begin{aligned}&\left(T^{0(g)}_0+T^{0(c)}_0\right)\sqrt{\frac{g}{\eta}}\\&=\frac{E_0}{16\pi R_b^4}\left(\frac{27p}{5}-\frac{1233p^2}{700}-\frac{50207p^3}{2100}\right)\\&+\frac{E_0}{16\pi R_b^4}\left(\frac{r^2}{R_b^2}(-p+\frac{87p^2}{20}+\frac{17919p^3}{350})\right)\\&+\frac{E_0}{16\pi R_b^4}\left(\frac{r^4}{R_b^4}(\frac{9p^2}{20}-\frac{5961p^3}{200})+\frac{599r^6p^3}{175R_b^6}\right)\end{aligned} \tag{133}$$

inside the body. The energy density of the GF is positive both outside and inside the body. The GF energy outside the body is

$$E_{gfout}=E_0\left(\frac{p}{4}+\frac{9p^2}{40}-\frac{14639p^3}{8400}\right) \tag{134}$$

and the GF energy inside the body is

$$E_{gfin}=E_0\left(\frac{7p}{20}+\frac{243p^2}{1400}-\frac{33757p^3}{42000}\right). \tag{135}$$

The total energy of the GF of the body is

$$E_{gf}=E_0\left(\frac{3p}{5}+\frac{279p^2}{700}-\frac{13369p^3}{5250}\right). \tag{136}$$

The energy density of the matter of the body is

$$\begin{aligned}T^{0(m)}_0\sqrt{\frac{g}{\eta}}&=\varepsilon\sqrt{\frac{g}{\eta}}=\varepsilon_0\sqrt{g_{00}}\\&=\varepsilon_0\left(1-\frac{3p}{2}-\frac{3p^2}{20}+\frac{1291p^3}{700}\right)\\&+\varepsilon_0\frac{r^2}{R_b^2}\left(\frac{p}{2}+\frac{27p^2}{40}+\frac{1557p^3}{2800}\right)\\&+\varepsilon_0\left(-\frac{r^4}{R_b^4}(\frac{3p^2}{10}+\frac{117p^3}{200})+\frac{283p^3r^6}{1680R_b^6}\right).\end{aligned} \tag{137}$$

The energy of the matter of the body is

$$E_m=E_0\left(1-\frac{6p}{5}+\frac{177p^2}{1400}+\frac{15619p^3}{7875}\right). \tag{138}$$

The total energy of the body is

$$E=E_m+E_{gf}=E_0\left(1-\frac{3p}{5}+\frac{21p^2}{40}-\frac{1267p^3}{2250}\right). \tag{139}$$

The energy E of the body with regard to the GF is less than the original energy E_0 of the matter of the body by the value of the gravitational energy defect, which is given by

$$\Delta_k=(E_0-E)/E_0=\frac{3p}{5}-\frac{21p^2}{40}+\frac{1267p^3}{2250}+O(p^4). \tag{140}$$

For comparison, in Newton's theory of gravity, the potential gravitational energy of a homogeneous sphere is given by [27]

$$\frac{1}{2}\int\varepsilon_0\varphi\sqrt{-\eta}d^3r=-\frac{3}{5}pE_0. \tag{141}$$

The mass of the body, $m=E/c^2$, is different from its gravitational mass (94)

$$m_g=\left(1-\frac{2217}{700}p^2\right)m+O(p^3). \tag{142}$$

However, this difference is of the second order of smallness, and the equivalence principle is satisfied to a sufficiently high accuracy.

X. LIGHT RAYS, PARTICLES, AND BODIES IN GRAVITATIONAL FIELDS

In the first-order approximation, the metric structure of space–time in the theory of gravity proposed coincides with the metric structure of space–time in GR. Hence, in this approximation, both theories identically describe the phenomena related to the null geodesic lines in space–time, in particular, the gravitational red shift of a light wave and the effect of the GF on the direction and time of propagation of light. Experimental investigations of these phenomena [21] lead to the following value of the post-Newtonian parameter: $\gamma = 1$.

In the second-order approximation, the metric structure of space-time in the theory of gravity proposed differs from the metric structure of space-time in GR. This difference manifests itself in the values of the observed effects. In particular, in GR the deflection angle of a light beam passing close to a body of mass m (in the isotropic coordinates) is

$$\alpha = 4h + h^2(\frac{15\pi}{4} - 8), \tag{143}$$

where $h = \frac{km}{c^2 r_0}$, r_0 is the minimum distance between the beam path and the body. In the theory proposed,

$$\alpha = 4h + h^2(\frac{23\pi}{4} - 8). \tag{144}$$

The delay time of the electromagnetic pulse during propagation from a point located at a distance r from the centre of the body to the point of closest approach r_0 is

$$\begin{aligned} ct(r, r_0) = \sqrt{r^2 - r_0^2} + 2r_g\sqrt{\frac{r - r_0}{r + r_0}} \\ +2r_g \ln\frac{r + \sqrt{r^2 - r_0^2}}{r_0} \\ +\frac{2r_g(3r_0^2 - 2r^2 - rr_0)}{r_0(r + r0)\sqrt{r^2 - r_0^2}} \\ +\frac{\kappa_2 r_g^2}{r_0}(\pi - 2\arctan\frac{r_0}{\sqrt{r^2 - r_0^2}}), \end{aligned} \tag{145}$$

where $r_g = \frac{km}{c^2}$, $\kappa_2 = \frac{15}{4}$ in GR, and $\kappa_2 = \frac{23}{4}$ in the theory proposed.

For a neutron star with a radius of 12-15 km and a mass of 2 solar masses, the greatest gravitational deflection angle in GR is $70 - 54°$, whereas, in the theory proposed, it is $93 - 69°$.

The energy–momentum density tensor $T^{\boldsymbol{\mu}}_{\boldsymbol{\nu}}(r^i)$ describes matter distributed over the entire space. Part of matter manifests itself as "clusters" that form compact objects such as elementary particles, atomic nuclei, atoms, molecules, planets, and stars. From the gravitational viewpoint, we can classify material objects into two groups: particles and bodies. A particle is the simplest object whose size and the internal structure can be neglected. In the limit case, this is an elementary particle. Particles can be combined into larger objects. By a body we mean an object, consisting of a set of particles, that is situated at a certain distance from other objects and moves as a unit whole. The position of a particle in space is described by coordinates R^i. The position of a body is described by the coordinates of its center of inertia. The coordinate basis e^{μ}_i is associated with the coordinate system. When considering the motion of bodies, we will neglect their size.

In special relativity theory, the motion of a particle is described by the principle of stationary action [27, 28]

$$S_p = -\int_a^b m_p c ds_p, \tag{146}$$

where a and b are the initial and final points of the trajectory, m is the mass of the particle, and $ds = \sqrt{g_{ij} dR^i dR^j}$ is an invariant interval. Here g_{ij} are metric coefficients of flat space–time. The particle has the velocity 4-vector $u^{\boldsymbol{\mu}} = \frac{dR^i}{ds} e^{\boldsymbol{\mu}}_i$ and the energy–momentum 4-vector

$$p_{\boldsymbol{\mu}} = m_p c g_{\boldsymbol{\mu\nu}} u^{\boldsymbol{\nu}}. \tag{147}$$

This description is also used in GR [27, 28]. The effect of gravity on a particle manifests itself in that the metric coefficients of the flat space are replaced by the metric coefficients of a space curved due to the entire external (with respect to the particle) matter. In addition, it is assumed that the mass m_p of a particle is independent of the external GFs. The energy of the particle is

$$cp_0 = m_p c^2 u_0 = \frac{m_p c^2 (g_{00} + g_{0a}\frac{v^a}{c})}{\sqrt{g_{00} + 2g_{0a}\frac{v^a}{c} + g_{ab}\frac{v^a v^b}{c^2}}}, \tag{148}$$

where v^a are the velocity components of the particle. The energy of the particle at rest is

$$cp_0 = m_p c^2 \sqrt{g_{00}}. \tag{149}$$

The variation of the energy of a particle in external GFs is attributed solely to the variation of the metric coefficient g_{00}. According to (146), the particle moves along a geodesic. In GR, it is assumed that the action (146) can also be used to describe the motion of bodies [6, 10, 27]. In this case, m_p is replaced by the mass m_b of the body. The mass of the body is independent of external GFs. The body moves in the GF of the entire external (with respect to the body) matter. Hence, just as in the description of the motion of a particle, when describing the motion of a body, one should pass from the real metric structure of space to a space such that the proper GF of the body is eliminated from its metric coefficients. For example, in free space, the body moves at constant velocity in the space with the background metric coefficients, whereas the space in the vicinity of the body is curved due to its GF.

When solving the GF equations (54), as a model of matter and, hence, as the source of the GF, the authors used the energy–momentum density tensor of an ideal fluid (57). There is no Lagrangian density for this tensor, whereby the Lagrange–Euler equations for the matter fields are replaced by their corollary—the conservation equation (43). In the second-order approximation, this approach correctly describes the emergence of GFs in space, depending on matter distributed in it. However, it does not fully describe the effect of the GF on matter. To establish the effect of the GF on matter, one needs a model of matter in the form of fields distributed in space and the joint solution of the system of equations (30, 31) for the fields of matter and the GF. To this end, one should represent matter as a collection of particles and each particle, as a particle-like solution of the corresponding field equation in curved space.

Consider this problem in greater detail by an example of a particle described by a nonlinear scalar field with the Lagrangian density

$$L_\phi = -\frac{1}{2}\partial_{\boldsymbol{\sigma}}\phi\partial^{\boldsymbol{\sigma}}\phi + U(\phi). \tag{150}$$

The Lagrangian corresponds to the field equation

$$\partial_{\boldsymbol{\sigma}}\partial^{\boldsymbol{\sigma}}\phi + \frac{\partial}{\partial\phi}U(\phi) = 0 \tag{151}$$

and the energy–momentum density tensor

$$T^{\boldsymbol{\pi}}_{\boldsymbol{\sigma}} = -\partial_{\boldsymbol{\sigma}}\phi\partial^{\boldsymbol{\pi}}\phi - L_\phi\delta^{\boldsymbol{\pi}}_{\boldsymbol{\sigma}}. \tag{152}$$

Suppose that there is a localized static particle-like solution $\phi(r)$ for the field equations. For this solution, $\partial_0\phi(r) = 0$, and

$$E_p = -\int L_\phi\sqrt{-g}d^3r. \tag{153}$$

For a scalar particle with the mass of a proton localized on the scale of its Compton wavelength, the gravitational potential is on the order of 6×10^{-39}. The gravitational potential on the Earth's surface is 6.96×10^{-10}. We can see that, for scalar particles, just as, seemingly, for fermions, in most cases one can neglect the proper GF of a particle because of the weak gravitational interaction. We neglect the proper GF. In this case, $\sqrt{-g} = \sqrt{-\eta}$ in empty space, and

$$E_p = m_pc^2 = -\int L_\phi\sqrt{-\eta}d^3r. \tag{154}$$

If a particle is in an external GF, then the GF can be assumed homogeneous within the distribution of the main mass of the particle: $F(r) = F_0, A(r) = A_0, g_{00} = 1+F_0, \sqrt{-g} = (1+F_0)^{1/2}(1-A_0)^{3/2}\sqrt{-\eta}$. For the energy and the mass of the particle, we obtain

$$E_p = m_pc^2\sqrt{-g_{00}}, \tag{155}$$

$$m_p = -\int L_\phi(1-A_0)^{3/2}\sqrt{-\eta}d^3r. \tag{156}$$

According to (151), a static particle-like solution $\phi(r)$ that takes place in the space without external GF turns into $\phi(r\sqrt{1-A_0})$ in the case of external GF. The value of the integral (156) for such a solution is independent of the value of A_0, and, hence, the mass m of a scalar particle is independent of the potential of the external GF. The rest energy of a particle, in contrast to its mass, is changed in external GFs. In the case of a moving particle, its energy is

$$E_p = m_pc^2u_0 = m_pc^2g^{(0)}_{0k}u^k, \tag{157}$$

where u_0 and u^k are the components of the velocity 4-vector of the particle. The relation obtained between the energy and mass of a particle justifies the relation (148) used in GR [27], but only under the assumption that the proper GF of the particle is negligible.

Having determined the energy and the mass of a particle, we generalize them to bodies made up of particles that interact only through the GF. For a body, in contrast to particles, one cannot neglect the proper GF. First, consider a stationary body in empty space. The rest energy of a body representing a system of particles and their joint GF should be equal to the sum of the energies of the particles and the GF energy:

$$E = c^2\sum_p m_pg_{0k}u^k_{(p)} + E_{gf}, \tag{158}$$

where E_{gf} is the gravitational energy of the body (136). For a stationary body composed of motionless particles, we have

$$E = c^2\sum_p m_p\sqrt{g_{00}} + E_{gf}. \tag{159}$$

Comparing (159) and (70), we can see that these relations are consistent only when the variation of the spatial volume in the energy density of the ideal fluid (69) is taken into account.

Now, consider the effect of the external GF on the body. Let $g^{(0)}_{00} = 1 + F_0$ for the field of external bodies and $g_{00} = 1 + F_{in} + F_0$ for the total field, where F_{in} is the internal $F(r)$ field of the body. The total field can be represented as

$$1 + F_{in} + F_0 = (1 + F_{in} - \frac{F_{in}F_0}{1+F_0})(1+F_0). \tag{160}$$

Hence we obtain

$$m_{mb} = \sum_p m_p\sqrt{1 + F_{in} - \frac{F_{in}F_0}{1+F_0}}, \tag{161}$$

$$E_{mb} = m_{mb}c^2\sqrt{g^{(0)}_{00}}. \tag{162}$$

From the relation, (161), we can see that a part of the mass of the body associated with the mass of its constituent particles depends on the external GF. However, it varies only by a value of the second order of smallness.

XI. TWO-BODY PROBLEM

It is convenient to represent the action for a body as

$$S_b = \int_a^b \Lambda_b dt, \tag{163}$$

where $\Lambda_b = -m_b c \frac{ds_b}{dt}$ is the Lagrangian of the body. Eliminating the constant component, we obtain

$$\Lambda_b = m_b c^2 (1 - \frac{ds_b}{cdt}). \tag{164}$$

Consider the motion of two bodies with regard to their gravitational interaction. The bodies have masses m_1 and m_2, which determine their inertial properties, and gravitational masses m_{g1} and m_{g2}, which characterize the bodies as sources of the GF. We obtain equations of motion in the second-order approximation with respect to the gravitation constant, assuming that the interaction is weak and the bodies move with velocities small compared with the velocity of light. The motion in the system of two bodies and the GF is described by the Lagrangian

$$\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_{int}, \tag{165}$$

where Λ_1 and Λ_2 are the Lagrangians of the individual bodies, including their proper GFs. Λ_{int} describes the interaction of bodies by means of their GFs. As the Lagrangian of bodies, we take (164), replacing the index b in it by on n, the number of a body. For body 1, we have

$$\left(\frac{ds_1}{dt}\right)^2 = g^{(2)}_{\boldsymbol{\alpha\beta}} v_1^{\boldsymbol{\alpha}} v_1^{\boldsymbol{\beta}}, \tag{166}$$

where $g^{(2)}_{\boldsymbol{\alpha\beta}}$ is the metric tensor corresponding to the GF of body 2, $v_1^{\boldsymbol{\alpha}} = e_0^{\boldsymbol{\alpha}} + v_1^a e_a^{\boldsymbol{\alpha}}, v_1^i = (v_1^1, v_1^2, v_1^3)$ are the velocity components of body 1, and similarly for body 2. Denote by $\mathbf{v}_n^2 = (\mathbf{v}_n \mathbf{v}_n) = (v_n^1)^2 + (v_n^2)^2 + (v_n^3)^2$ the squared velocity of body n. In the second-order approximation, we obtain the following Lagrangians of bodies 1 and 2:

$$\Lambda_1 = m_1 \left(\frac{\mathbf{v}_1^2}{2} + \frac{\mathbf{v}_1^4}{8}\right) + \frac{k m_1 m_{g2}}{R} - (\frac{15}{2} + \beta)\frac{k^2 m_1 m_{g2}^2}{R^2} + \frac{k m_1 m_{g2}}{2R}\left(3\mathbf{v}_1^2 + 5\mathbf{v}_2^2 - 8(\mathbf{v}_1\mathbf{v}_2) - (\mathbf{v}_2\mathbf{n})^2\right), \tag{167}$$

$$\Lambda_2 = m_2 \left(\frac{\mathbf{v}_2^2}{2} + \frac{\mathbf{v}_2^4}{8}\right) + \frac{k m_2 m_{g1}}{R} - (\frac{15}{2} + \beta)\frac{k^2 m_2 m_{g1}^2}{R^2} + \frac{k m_2 m_{g1}}{2R}\left(5\mathbf{v}_1^2 + 3\mathbf{v}_2^2 - 8(\mathbf{v}_1\mathbf{v}_2) - (\mathbf{v}_1\mathbf{n})^2\right), \tag{168}$$

where β is a post-Newtonian parameter and $R = \mid \mathbf{r}_2 - \mathbf{r}_1 \mid$ is the coordinate distance between the bodies, $\mathbf{n} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{R}$. To obtain Λ_{int}, we substitute the GF of two bodies (118) into the Lagrangian density of the GF (53). A part of the Lagrangian L_g depends only on the field $G^{(1)}_{\boldsymbol{\mu\nu}}$ of body 1, and another part, only on the field $G^{(2)}_{\boldsymbol{\mu\nu}}$ of body 2. These quantities should be included in the action of the bodies. Let L_{g12} be a part of L_g that depends on the field $\Phi_{\boldsymbol{\mu\nu}}$ or the fields $G^{(1)}_{\boldsymbol{\mu\nu}}$ and $G^{(2)}_{\boldsymbol{\mu\nu}}$ simultaneously. As the Lagrangian Λ_{int}, we take

$$\Lambda_{int} = \int L_{g12} \sqrt{-g} d^3 r. \tag{169}$$

Integrating over the volume, we obtain

$$\Lambda_{int} = -\frac{k m_{g1} m_{g2}}{R} + (7 + \beta)\frac{k^2 m_{g1} m_{g2}}{R^2}(m_{g1} + m_{g2}) - \frac{k m_{g1} m_{g2}}{2R}\left(5\mathbf{v}_1^2 + 5\mathbf{v}_2^2 - 9(\mathbf{v}_1\mathbf{v}_2)\right) + \frac{k m_{g1} m_{g2}}{2R}\left((\mathbf{v}_1\mathbf{n})^2 + (\mathbf{v}_2\mathbf{n})^2 - (\mathbf{v}_1\mathbf{n})(\mathbf{v}_2\mathbf{n})\right). \tag{170}$$

To obtain (170), we used formulas (A183)-(A185) listed in the Appendix. The contribution of the field $\Phi_{\boldsymbol{\mu\nu}}$, which depends on the distance to both bodies, to Λ_{int} can be represented as

$$\frac{1}{4}\int \Phi_{\boldsymbol{\mu\nu}} T^{\boldsymbol{\mu\nu}}_{(m)} \sqrt{-\eta} d^3 r. \tag{171}$$

In the second-order approximation, this contribution vanishes, because $\Phi_{00} = 0$ (119). Summing up (167), (168), and (170), we obtain the following Lagrangian, which describes the motion of two gravitationally interacting bodies in the second-order approximation:

$$\begin{aligned}\Lambda = {} & m_1\left(\frac{\mathbf{v}_1^2}{2} + \frac{\mathbf{v}_1^4}{8}\right) + m_2\left(\frac{\mathbf{v}_2^2}{2} + \frac{\mathbf{v}_2^4}{8}\right) + \frac{k}{R}\left(m_1 m_{g2} + m_2 m_{g1} - m_{g1} m_{g2}\right) - \\ & - \frac{k^2}{R^2}\left[(\frac{15}{2} + \beta)(m_1 m_{g2}^2 + m_2 m_{g1}^2) - (7 + \beta) m_{g1} m_{g2}(m_{g1} + m_{g2})\right] \\ & + \frac{k}{2R}\left[m_1 m_{g2}(3\mathbf{v}_1^2 + 5\mathbf{v}_2^2 - 8(\mathbf{v}_1\mathbf{v}_2) - (\mathbf{v}_2\mathbf{n})^2) + m_2 m_{g1}(5\mathbf{v}_1^2 + 3\mathbf{v}_2^2 - 8(\mathbf{v}_1\mathbf{v}_2) - (\mathbf{v}_1\mathbf{n})^2)\right] \\ & - \frac{k m_{g1} m_{g2}}{2R}\left[5\mathbf{v}_1^2 + 5\mathbf{v}_2^2 - 9(\mathbf{v}_1\mathbf{v}_2) - (\mathbf{v}_1\mathbf{n})^2 - (\mathbf{v}_2\mathbf{n})^2 + (\mathbf{v}_1\mathbf{n})(\mathbf{v}_2\mathbf{n})\right].\end{aligned} \tag{172}$$

It follows from (172) that there is a force of mutual attraction between two bodies. This force depends both on the masses of the bodies m_1 and m_2 and their gravitational masses m_{g1} and m_{g2}. A part of the force that is linear in the gravitation constant and independent of velocities,

$$F_{12} = \frac{k}{R^2}\left(m_1 m_{g2} + m_2 m_{g1} - m_{g1} m_{g2}\right), \qquad (173)$$

underlies Newton's law of universal gravitation. The substitution of ordinary masses for gravitational masses leads to the standard expression for the law of universal gravitation:

$$F_{12} = \frac{k}{R^2} m_1 m_2. \qquad (174)$$

Replacing gravitational masses by ordinary masses in (172), we obtain the Lagrangian, which was first proposed by Fichtenholz [5] (see also [6, 10, 27])

$$\Lambda = m_1\left(\frac{\mathbf{v}_1^2}{2} + \frac{\mathbf{v}_1^4}{8}\right) + m_2\left(\frac{\mathbf{v}_2^2}{2} + \frac{\mathbf{v}_2^4}{8}\right) + \frac{k m_1 m_2}{R} - \frac{k^2 m_1 m_2}{2R^2}(m_1 + m_2) + \frac{k m_1 m_2}{2R}\left[3\mathbf{v}_1^2 + 3\mathbf{v}_2^2 - 7(\mathbf{v}_1\mathbf{v}_2) - (\mathbf{v}_1\mathbf{n})(\mathbf{v}_2\mathbf{n})\right]. \qquad (175)$$

The Lagrangian (175) differs from (172) by a quantity of the fourth-order of smallness. We can also see that this Lagrangian, in contrast to GR, is independent of the post-Newtonian parameter β. The Lagrangian (175) corresponds to the equations of motion obtained by Einstein, Infeld, and Hoffman [2] and Eddington and Clarke [3]; the solution of these equations yields the following secular displacement of the periastron of two bodies of comparable masses: [4]

$$\delta\varphi = \frac{6\pi k^2(m_1^2 + m_2^2)}{c^2 J^2} = \frac{6\pi k(m_1 + m_2)}{c^2 a(1 - e^2)}, \qquad (176)$$

where J is the angular momentum, a is the length of the major semiaxis, and e is the eccentricity of the elliptic orbit. In GR,

$$\delta\varphi = \frac{4 - \beta}{3} \cdot \frac{6\pi k(m_1 + m_2)}{c^2 a(1 - e^2)}, \qquad (177)$$

and $\beta = 1$.

XII. CONCLUSIONS

The theory of gravity proposed here, just as GR, implements Einstein's idea about the effect of matter on the metric structure of space–time. However, in spite of the generality of the original idea, the implementations are fundamentally different. In GR, the energy–momentum density tensor of matter is related to the tensor of curvature of space–time through Einstein's equation. Gravitational phenomena are the manifestation of variation in the geometry of space–time due to the presence of matter. There is no gravity field in the classical sense.

In the theory proposed, the GF $G_{\mu\nu}$ is a classical tensor field, in complete analogy with the vector potential field in electromagnetic theory. The energy–momentum density tensor also has a positive energy density. A source of the GF is the energy–momentum density tensor of all kinds of matter, including the GF itself. For an appropriate source, the GF can be emitted in the form of gravitational waves and take away energy from the radiating system. A positive energy density should lead to repulsion. However, there is no repulsion, because the GF has one more property: it is related to the metric tensor. Therefore, all phenomena occur in curved pseudo-Riemannian space–time. A variation in the metric structure of space–time leads to additional interaction between bodies—attraction. In the first-order approximation (expansion of the solution in the gravitation constant), which corresponds to the linearized theory of gravity, the metric structure of space–time is the same in both theories. Accordingly, both theories identically describe the propagation of light along null geodesic lines. In the second-order approximation, the metric structure of space–time is different in these theories (see (105), (106)), which, in particular, is characterized by the post-Newtonian parameter $\beta = 1$ in GR and $\beta = \frac{1}{8}$ in the theory proposed. The only observable second-order effect—the periastron advance of the mutual orbit of two bodies—depends on β in GR and correctly describes the periastron advance for $\beta = 1$. In our theory of gravity, a motion of two bodies, just as the periastron advance, are independent of β; hence, the established value of $\beta = \frac{1}{8}$ is admissible. Experiments are needed that would give estimates for β and other post-Newtonian parameters from the viewpoints of the theory of gravity proposed here and from the viewpoint of GR.

The possibility of manipulating the energy of the GF has allowed us to obtain estimates that are of fundamental importance—the gravitational energy defect of bodies, the difference between the inertial and gravitational masses of bodies, and the effect of the external GF on the mass of a body.

The results obtained demonstrate the possibility of a new, sufficiently classical, approach to the construction of a theory of gravity in curved space–time. The development of the theory offers hope of further progress in understanding the structure of space–time and the structure and evolution of stars, galaxies, and the Universe.

APPENDIX

Here we present the GF equations in component representation for a spherically symmetric source in a spherical system of coordinates. To shorten expressions, we use the condition $B(r)=0$ and the following notations: $F=1+F(r), A=-1+A(r), D=D(r), U=U(r), \Pi=\Pi(r)$.

$$\frac{F'}{F}+\frac{2\Pi'}{\kappa^{-1}+\Pi}=0, \tag{A178}$$

$$\begin{aligned}&\frac{3\left(A'\right)^3}{A^4}+\frac{3U\left(A'\right)^2}{A^2}+\frac{2D\left(A'\right)^2}{A^3}-\frac{3F'\left(A'\right)^2}{A^3F}-\frac{12\left(A'\right)^2}{A^3r}-\frac{2\left(F'\right)^2A'}{A^3F}-\frac{\left(F'\right)^2A'}{A^2F^2}-\frac{8DA'}{A^2r}-\frac{24F'A'}{A^2Fr}-\frac{6A''A'}{A^3}\\&-\frac{6F''A'}{A^2F}-\frac{DA'}{A^2}-\frac{3U'A'}{A}-\frac{3\left(F'\right)^3}{AF^3}-\frac{2\left(F'\right)^3}{F^4}+\frac{U\left(F'\right)^2}{F^2}+\frac{4\left(F'\right)^2}{AF^2r}+\frac{4U}{r^2}-\frac{4D}{Ar^2}-\frac{3UA''}{A}-\frac{3DA''}{A^2}\\&-\frac{6F'A''}{A^2F}-\frac{UF''}{F}-\frac{DF''}{AF}+\frac{2F'F''}{A^2F}+\frac{6F'F''}{AF^2}+\frac{2F'F''}{F^3}-2U''+\frac{F'D'}{AF}+\frac{4D'}{Ar}-\frac{F'U'}{F}-\frac{4U'}{r}+\frac{2D''}{A}=0,\end{aligned} \tag{A179}$$

$$\begin{aligned}&\frac{3\left(A'\right)^2}{8A^3F}+\frac{3\left(A'\right)^2}{4A^3}-\frac{3UA'}{8AF}-\frac{2F'A'}{A^2F}+\frac{3F'A'}{4A^2F^2}-\frac{DA'}{8A^2F}-\frac{3A'}{A^2r}-\frac{\left(F'\right)^2}{4AF^2}-\frac{\left(F'\right)^2}{8A^2F^2}-\frac{3\left(F'\right)^2}{8AF^3}-\frac{\left(F'\right)^2}{2F^3}\\&-\frac{\left(F'\right)^2}{8F^4}-\frac{U}{2Fr}-\frac{UF'}{8F^2}+\frac{F'}{AFr}+\frac{DF'}{8AF^2}-\frac{U'}{4F}-\frac{3A''}{2A^2}+\frac{F''}{2AF}-\frac{D'}{4AF}-\frac{D}{2AFr}+\frac{12p}{F\kappa}=0,\end{aligned} \tag{A180}$$

$$\begin{aligned}&\frac{\left(A'\right)^2}{4A^3}-\frac{3\left(A'\right)^2}{8A^4}-\frac{3DA'}{8A^3}-\frac{3UA'}{8A^2}+\frac{F'A'}{A^2F}-\frac{3F'A'}{4A^3F}+\frac{A'}{A^2r}+\frac{\left(F'\right)^2}{2A^2F}+\frac{\left(F'\right)^2}{8A^3F}+\frac{\left(F'\right)^2}{4AF^2}+\frac{3\left(F'\right)^2}{8A^2F^2}\\&+\frac{\left(F'\right)^2}{8AF^3}-\frac{U}{2Ar}-\frac{UF'}{8AF}-\frac{DF'}{8A^2F}+\frac{F'}{AFr}+\frac{A''}{2A^2}+\frac{F''}{2AF}+\frac{D'}{4A^2}-\frac{U'}{4A}-\frac{D}{2A^2r}-\frac{12p\Pi}{A}=0,\end{aligned} \tag{A181}$$

$$-\frac{3\left(A'\right)^2}{2A^3}-\frac{DA'}{A^2}-\frac{3F'A'}{A^2F}+\frac{\left(F'\right)^2}{AF}+\frac{\left(F'\right)^2}{2AF^2}+\frac{\left(F'\right)^2}{F^2}+\frac{D'}{A}-\frac{D}{Ar}=0. \tag{A182}$$

The values of the integrals needed to obtain the Lagrangian of interaction of two bodies:

$$\int(\boldsymbol{\nabla}\varphi_1\boldsymbol{\nabla}\varphi_2)\sqrt{-\eta}d^3r=\frac{4\pi}{R}k^2m_{g1}m_{g2}\left(1+\frac{1}{2}(\mathbf{v}_1^2+\mathbf{v}_2^2-(\mathbf{v}_1\mathbf{n}_1)^2-(\mathbf{v}_2\mathbf{n}_2)^2)\right); \tag{A183}$$

$$\int(\partial_t\varphi_1\partial_t\varphi_2)\sqrt{-\eta}d^3r=\frac{2\pi}{R}k^2m_{g1}m_{g2}\left((\mathbf{v}_1\mathbf{v}_2)-(\mathbf{v}_1\mathbf{n})(\mathbf{v}_2\mathbf{n})\right); \tag{A184}$$

$$\int(\varphi_1+\varphi_2)(\boldsymbol{\nabla}\varphi_1\boldsymbol{\nabla}\varphi_2)\sqrt{-\eta}d^3r=-\frac{2\pi k^3m_{g1}m_{g2}(m_{g1}+m_{g2})}{R^2}; \tag{A185}$$

where $\boldsymbol{\nabla}$ is the gradient operator.

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