The Cartan Model for Equivariant Cohomology

Xu Chen *

Abstract

In this article, we will discuss a new operator $d_C$ on $W(\mathfrak{g}) \otimes \Omega^*(M)$ and to construct a new Cartan model for equivariant cohomology. We use the new Cartan model to construct the corresponding BRST model and Weil model, and discuss the relations between them.

1 Introduction

The standard Cartan model for equivariant cohomology is construct on the algebra $W(\mathfrak{g}) \otimes \Omega^*(M)$ with operator

$$d_C \phi^i = 0, \phi^i \in S(\mathfrak{g}^*), i = 1, \cdots, n;$$

$$d_C \eta = (1 \otimes d - \sum_{b=1}^n \phi^b \otimes \iota_b) \eta, \eta \in \Omega^*(M),$$

where $\iota_b$ is $\iota_{e_b}$(see [4],[5],[7],[8]). We can also introduce a new operator on $W(\mathfrak{g}) \otimes \Omega^*(M)$ by

$$d_C \phi^i = 0, \phi^i \in S(\mathfrak{g}^*), i = 1, \cdots, n;$$

$$d_C \eta = (1 \otimes d - \sum_{b=1}^n \phi^b \otimes (\iota_b + \sqrt{-1} f_b^a \iota_a)) \eta, \eta \in \Omega^*(M) \otimes \mathbb{C},$$

where $\iota_b$ is $\iota_{e_b}$. In this article we construct the new model for equivariant cohomology which also called Cartan model. The idea comes form the article [3]. We also use the new Cartan model to construct the corresponding BRST model and Weil model.

2 Cartan model

Let $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. We known the Weil algebra is

$$W(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*).$$

The contraction $i_X$ and the exterior derivative $d_W$ on $W(\mathfrak{g})$ defined as follow:

Choose a basis $e_1, \cdots, e_n$ for $\mathfrak{g}$ and let $e^*_1, \cdots, e^*_n$ be the dual basis of $\mathfrak{g}^*$. Let $\theta^1, \cdots, \theta^n$ be the dual basis of $\mathfrak{g}^*$ generating the exterior algebra $\Lambda(\mathfrak{g}^*)$ and let $\phi^1, \cdots, \phi^n$ be the dual basis of $\mathfrak{g}^*$ generating the symmetric algebra $S(\mathfrak{g}^*)$. Let $c^i_{jk}$ be the structure constants of

*Email: xiaorenwu08@163.com. ChongQing, China
\[ \mathfrak{g} \text{ (see [6]), that is } [e_j, e_k] = \sum_{i=1}^{n} c_{jk}^i e_i. \] We know that \( S(\mathfrak{g}^*) \) is identified with the polynomial ring \( \mathbb{C}[\phi^1, \ldots, \phi^n] \).

Define the contraction \( i_X \) on \( W(\mathfrak{g}) \) for any \( X \in \mathfrak{g} \) by
\[
 i_{e_r}(\theta^s) = \delta_r^s, \ i_{e_r}(\phi^s) = 0
\]
for all \( r, s = 1, \ldots, n \) and extending by linearity and as a derivation.

Define \( d_W \) by
\[
d_W \theta^i = -\frac{1}{2} \sum_{j,k} c_{jk}^i \theta^j \wedge \theta^k + \theta^i
\]
and
\[
d_W \phi^i = -\sum_{j,k} c_{jk}^i \theta^j \phi^k
\]
and extending \( d_W \) to \( W(\mathfrak{g}) \) as a derivation.

The Lie derivative on \( W(\mathfrak{g}) \) is defined by
\[
 L_X = d_W \cdot i_X + i_X \cdot d_W.
\]

**Lemma 1.** \( L_{e_i} \theta^j = -\sum_k c_{ik}^j \theta^k \) and \( L_{e_i} \phi^j = -\sum_k c_{ik}^j \phi^k \).

**Proof.** Because
\[
 L_{e_i} \theta^j = (d_W \cdot i_{e_i} + i_{e_i} \cdot d_W) \theta^j = i_{e_i}(-\frac{1}{2} \sum_{i,k} c_{ik}^j \theta^i \wedge \theta^k + \theta^j) = -\sum_k c_{ik}^j \theta^k,
\]
\[
 L_{e_i} \phi^j = (d_W \cdot i_{e_i} + i_{e_i} \cdot d_W) \phi^j = i_{e_i}(-\sum_{i,k} c_{ik}^j \theta^i \phi^k) = -\sum_k c_{ik}^j \phi^k
\]
\[ \square \]

**Lemma 2.** The operators \( i_X, d_W, L_X \) on \( W(\mathfrak{g}) \) satisfy the following identities:

1. \( d_W^2 = 0 \);
2. \( L_X \cdot d_W - d_W \cdot L_X = 0 \), for any \( X \in \mathfrak{g} \);
3. \( i_X i_Y + i_Y i_X = 0 \), for any \( X, Y \in \mathfrak{g} \);
4. \( L_X i_Y - i_Y L_X = i_{[X,Y]} \), for any \( X, Y \in \mathfrak{g} \);
5. \( L_X L_Y - L_Y L_X = L_{[X,Y]} \), for any \( X, Y \in \mathfrak{g} \);
6. \( d_W i_X + i_X d_W = L_X \), for any \( X \in \mathfrak{g} \).

**Proof.** see [4]. \[ \square \]

So, there is a complex \((W(\mathfrak{g}), d_W)\), the cohomology of \((W(\mathfrak{g}), d_W)\) is trivial (see [5]), i.e. \( H^*(W(\mathfrak{g})) \cong \mathbb{R} \).

Let \( M \) be a smooth closed manifold with \( G \) acting smoothly on the left. Let \( X^M \) be the vector field generated by the Lie algebra element \( X \in \mathfrak{g} \) given by
\[
(X^M f)(x) = \frac{d}{dt} f(\exp(-tX) \cdot x) \bigg|_{t=0}.
\]
Set \( d, i_X^M, L_X^M \) be the exterior derivative, contraction and Lie derivative on \( \Omega^*(M) \). Denote \( i_X = i_X^M \) and \( L_X = L_X^M \) acting on \( \Omega^*(M) \).
Definition 1. The Cartan model is defined by the algebra

\[ S(\mathfrak{g}^*) \otimes \Omega^*(M) \]

and the differential

\[ d_C \phi^i = 0, \phi^i \in S(\mathfrak{g}^*), i = 1, \cdots, n; \]

\[ d_C \eta = (1 \otimes d - \sum_{i=1}^{n} \phi^i \otimes (t_i + \sqrt{-1} f_i^j t_j)) \eta, \eta \in \Omega^*(M) \otimes \mathbb{C}, \]

where \( t_i \) is \( t_{e_i} \) and \( f_i^j \in \mathbb{R} \). The operator \( d_C \) is called the equivariant exterior derivative.

Its action on forms \( \alpha \in S(\mathfrak{g}^*) \otimes \Omega^*(M) \) is

\[ (d_C \alpha)(X) = (d - t_X \cdot - \sqrt{-1} t_Y \cdot)(\alpha(X)) \]

where \( X^M = c^i X_i^M \) is the vector field on \( M \) generated by the Lie algebra element \( X = c^i e_i \in \mathfrak{g}, Y^M = f^i_j c^j X_i^M \) (see [2]). In the article [3] we use the operator \( d + t_X + \sqrt{-1} t_Y \cdot \) to construct an complex \((\Omega^*(M) \otimes \mathbb{C}, d + t_X + \sqrt{-1} t_Y \cdot)\) and cohomology group \( H^*_X + \sqrt{-1} t_Y \cdot (M) \), we can do it in the same way by the operator \( d - t_X - \sqrt{-1} t_Y \cdot \).

Lemma 3.

\[ d_C^2 = - \sum_{i=1}^{n} \phi^i \otimes (\mathcal{L}_i + \sqrt{-1} f_i^j \mathcal{L}_j) \]

Proof. By the lemma 2. we have

\[ d_C^2 = (1 \otimes d - \sum_{i=1}^{n} \phi^i \otimes (t_i + \sqrt{-1} f_i^j t_j))(1 \otimes d - \sum_{i=1}^{n} \phi^i \otimes (t_i + \sqrt{-1} f_i^j t_j)) \]

\[ = - \sum_{i=1}^{n} \phi^i \otimes [d(t_i + \sqrt{-1} f_i^j t_j)] + (t_i + \sqrt{-1} f_i^j t_j) d] \]

\[ = - \sum_{i=1}^{n} \phi^i \otimes (\mathcal{L}_i + \sqrt{-1} f_i^j \mathcal{L}_j) \]

\[ \square \]

Let \((S(\mathfrak{g}^*) \otimes \Omega^*(M))^G\) be the subalgebra of \( S(\mathfrak{g}^*) \otimes \Omega^*(M) \) which satisfied

\[ (\sum_{i=1}^{n} \phi^i \otimes (\mathcal{L}_i + \sqrt{-1} f_i^j \mathcal{L}_j)) \alpha = 0, \forall \alpha \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G \]

So we get the complex \(((S(\mathfrak{g}^*) \otimes \Omega^*(M))^G, d_C)\). The equivariantly closed form is \( \forall \alpha \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G \) with \( d_C \alpha = 0 \), the equivariantly exact form is \( \forall \alpha \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G \) there is \( \beta \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G \) with \( \alpha = d_C \beta \).

As in [8] we can define the equivariant connection

\[ \nabla_\mathfrak{g} = 1 \otimes \nabla - \sum_{i=1}^{n} \phi^i \otimes (t_i + \sqrt{-1} f_i^j t_j) \]

and the equivariant curvature of the connection

\[ F_\mathfrak{g} = (\nabla_\mathfrak{g})^2 + \sum_{i=1}^{n} \phi^i \otimes (\mathcal{L}_i + \sqrt{-1} f_i^j \mathcal{L}_j) \]
3 BRST model

This section is inspired by [5]. First, we will to construct the BRST differential algebra. The algebra is

\[ B = W(g) \otimes \Omega^*(M). \]

The BRST operator is

\[
\delta = d_W \otimes 1 + 1 \otimes d + \sum_{i=1}^{n} \theta^i \otimes (L_i + \sqrt{-1} f_i^j \mathcal{L}_j) - \sum_{a=1}^{n} \partial^a \otimes (\iota_a + \sqrt{-1} f_{aL} f_{Lj}) + \frac{1}{2} \sum_{j<k} c^i_{jk} \theta^j \theta^k \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j)
\]

\[
- \sum_{j<k} \theta^j \theta^k \otimes ((L_j + \sqrt{-1} f_j^h \mathcal{L}_h)(\iota_k + \sqrt{-1} f_k^g \iota_g) - (\iota_j + \sqrt{-1} f_j^h \iota_h)(\mathcal{L}_k + \sqrt{-1} f_k^g \mathcal{L}_g))
\]

where \( L_i \) is \( \mathcal{L}_{e_i} \) and \( \iota_a \) is \( \iota_{e_a} \).

Lemma 4. On the algebra \( W(g) \otimes \Omega^*(M) \), we have \( \delta^2 = 0 \).

Proof. By computation, we have

\[
\delta = \exp(\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))(d_W \otimes 1 + 1 \otimes d) \exp(-\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))
\]

where \( \iota_a \) is \( \iota_{e_a} \). So we have

\[
\delta^2 = \exp(\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))(d_W \otimes 1 + 1 \otimes d) \exp(-\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j)) \cdot \\
\exp(\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))(d_W \otimes 1 + 1 \otimes d) \exp(-\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))
\]

\[
= \exp(\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))(d_W \otimes 1 + 1 \otimes d)^2 \exp(-\sum_{i=1}^{n} \theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))
\]

\[
= 0 \quad \square
\]

So we get the BRST differential algebra \( (W(g) \otimes \Omega^*(M), \delta) \).

Lemma 5. Fixing the index \( i \) and \( k \)

\[
(\theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))(\theta^k \otimes (\iota_k + \sqrt{-1} f_k^j \iota_j)) = (\theta^k \otimes (\iota_k + \sqrt{-1} f_k^j \iota_j))(\theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))
\]

Proof. If \( i = k \), we have

\[
(\theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))(\theta^k \otimes (\iota_k + \sqrt{-1} f_k^j \iota_j)) = 0 = (\theta^k \otimes (\iota_k + \sqrt{-1} f_k^j \iota_j))(\theta^i \otimes (\iota_i + \sqrt{-1} f_i^j \iota_j))
\]

If \( i \neq k \), then because

\[
(\theta^i \otimes \iota_i)(\theta^k \otimes \iota_k) = -\theta^i \wedge \theta^k \otimes \iota_i \iota_k = -\theta^k \wedge \theta^i \otimes \iota_i \iota_k = (\theta^k \otimes \iota_k)(\theta^i \otimes \iota_i)
\]

\[
(\theta^i \otimes (\sqrt{-1} f_i^j \iota_j))(\theta^k \otimes \iota_k) = -\theta^i \wedge \theta^k \otimes (\sqrt{-1} f_i^j \iota_j) \iota_k = -\theta^k \wedge \theta^i \otimes \iota_k (\sqrt{-1} f_i^j \iota_j) = (\theta^k \otimes \iota_k)(\theta^i \otimes (\sqrt{-1} f_i^j \iota_j))
\]

So we get the result. \( \square \)
Let $\psi : W(\mathfrak{g}) \otimes \Omega^*(M) \to W(\mathfrak{g}) \otimes \Omega^*(M)$ be the map
\[
\psi = \prod_i (1 - \theta^i \otimes (\iota_i + \sqrt{-1} f^i_j \iota_j)).
\]

By computation
\[
(1 - \theta^1 \otimes (\iota_1 + \sqrt{-1} f^1_j \iota_j))(1 - \theta^2 \otimes (\iota_2 + \sqrt{-1} f^2_j \iota_j)) \cdots (1 - \theta^n \otimes (\iota_n + \sqrt{-1} f^n_j \iota_j))
\]
we have
\[
\psi = \exp(-\sum_{i=1}^n \theta^i \otimes (\iota_i + \sqrt{-1} f^i_j \iota_j)).
\]

In the section 5. we will discuss the map $\psi$.

### 4 Weil model

The exterior derivative operator on $W(\mathfrak{g}) \otimes \Omega^*(M)$ is defined by
\[
D = d_W \otimes 1 + 1 \otimes d,
\]
the contraction operator is defined by
\[
\tilde{i}_X = i_X \otimes 1 + 1 \otimes \iota_X
\]
and Lie derivative be defined by
\[
\tilde{L}_X = L_X \otimes 1 + 1 \otimes \mathcal{L}_X
\]

**Lemma 6.** The operators $\tilde{i}_X, D, \tilde{L}_X$ on $W(\mathfrak{g}) \otimes \Omega^*(M)$ satisfy the following identities:

1. $D^2 = 0$;
2. $\tilde{L}_X \cdot D - D \cdot \tilde{L}_X = 0$, for any $X \in \mathfrak{g}$;
3. $\tilde{i}_X \tilde{i}_Y + \tilde{i}_Y \tilde{i}_X = 0$, for any $X, Y \in \mathfrak{g}$;
4. $\tilde{L}_X \tilde{i}_Y - \tilde{i}_Y \tilde{L}_X = \tilde{i}_{[X,Y]}$, for any $X, Y \in \mathfrak{g}$;
5. $\tilde{L}_X \tilde{L}_Y - \tilde{L}_Y \tilde{L}_X = \tilde{L}_{[X,Y]}$, for any $X, Y \in \mathfrak{g}$;
6. $\tilde{L}_X = D \cdot \tilde{i}_X + \tilde{i}_X \cdot D$, for any $X \in \mathfrak{g}$.

**Proof.** see [4].

Set
\[
\tilde{i}_{X+\sqrt{-1}Y} = i_X \otimes 1 + 1 \otimes (\iota_X + \sqrt{-1} \iota_Y)
\]
be the contraction operator on $W(\mathfrak{g}) \otimes \Omega^*(M)$ induced by the contraction of $X + \sqrt{-1}Y$.

Set
\[
\tilde{L}_{X+\sqrt{-1}Y} = L_X \otimes 1 + 1 \otimes (\mathcal{L}_X + \sqrt{-1} \mathcal{L}_Y)
\]
be the Lie derivative on $W(\mathfrak{g}) \otimes \Omega^*(M)$ about $X + \sqrt{-1}Y$.  

\[\]
Theorem 1. \( \psi \) is an isomorphism of differential algebra, i.e., the diagram

\[
\begin{array}{ccc}
W(g) \otimes \Omega^*(M) & \xrightarrow{\psi} & W(g) \otimes \Omega^*(M) \\
\delta & & \downarrow D \\
W(g) \otimes \Omega^*(M) & \xrightarrow{\psi} & W(g) \otimes \Omega^*(M)
\end{array}
\]

commutes.
Proof. By computation in lemma 4., we have
\[ \delta = \psi \cdot D \cdot \psi^{-1} \]
\[ \square \]

**Theorem 2.** We have the following commutative diagram:
\[
\begin{array}{ccc}
(W(\mathfrak{g}) \otimes \Omega^*(M), \delta) & \xrightarrow{\psi} & (W(\mathfrak{g}) \otimes \Omega^*(M), D) \\
\uparrow{id} & & \uparrow{id} \\
(S(\mathfrak{g}^*) \otimes \Omega^*(M))^\tilde{G} & \xrightarrow{\psi} & (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}
\end{array}
\]

Proof. For \( \forall \alpha \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^\tilde{G} \), by
\[
\prod_a (1 - \theta^a \otimes (t_a + \sqrt{-1} f^b_a t_b)) \cdot (i_k \otimes 1) = (i_k \otimes 1 + 1 \otimes (i_k + \sqrt{-1} f^j_k t_j)) \cdot \prod_a (1 - \theta^a \otimes (t_a + \sqrt{-1} f^b_a t_b))
\]
we have
\[
(i_k \otimes 1 + 1 \otimes (t_k + \sqrt{-1} f^j_k t_j)) (\psi(\alpha)) = 0.
\]
Because
\[
[i, i_k \otimes 1] = L_k \otimes 1 + 1 \otimes (L_k + \sqrt{-1} f^j_k L_j)
\]
and
\[
\prod_a (1 - \theta^a \otimes (t_a + \sqrt{-1} f^b_a t_b)) \cdot (L_k \otimes 1 + 1 \otimes (L_k + \sqrt{-1} f^j_k L_j))
\]
\[
= (L_k \otimes 1 + 1 \otimes (L_k + \sqrt{-1} f^j_k L_j)) \cdot \prod_a (1 - \theta^a \otimes (t_a + \sqrt{-1} f^b_a t_b))
\]
so we have
\[
(L_k \otimes 1 + 1 \otimes (L_k + \sqrt{-1} f^j_k L_j)) (\psi(\alpha)) = 0
\]
Then we get \( \psi(\alpha) \in (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas} \). So we get the commutative diagram. \( \square \)

The theorem 2. tell us the relation about BRST model and Cartan model.

**Theorem 3.**
\[
(S(\mathfrak{g}^*) \otimes \Omega^*(M))^\tilde{G} \xrightarrow{\psi} (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}
\]
is a isomorphism.

Proof. For \( \forall \eta \in (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas} \), \( \psi^{-1} \eta = \prod_a (1 + \theta^a \otimes (t_a + \sqrt{-1} f^b_a t_b)) \eta \). By
\[
\prod_a (1 + \theta^a \otimes (t_a + \sqrt{-1} f^b_a t_b)) |_{(W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}} = \prod_a (1 - \theta^a i_a \otimes 1) |_{(W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}}
\]
and
\[
\text{Im}(1 - \theta^a i_a \otimes 1) = \text{Ker}(i_a \otimes 1)
\]
So
\[
\psi^{-1} \eta \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))_{bas}.
\]
Then
\[
(\sum_{i=1}^n \phi^i \otimes (L_i + \sqrt{-1} f^j_i L_j)) \psi^{-1} \eta = 0
\]
i.e., \( \psi^{-1} \eta \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^\tilde{G} \). And by the proof in theorem 2, we get that \( \psi \) is a isomorphism. \( \square \)

The theorem 3. tell us the relation about Cartan model and Weil model.
References


