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On the properties of k-Fibonacci and k-Lucas numbers

Research Article

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Abstract: In this paper, some properties of $k-$Fibonacci and $k-$Lucas numbers are derived and proved by using matrices $S = \begin{pmatrix} \frac{k}{2} & \frac{k^2+4}{2} \\ \frac{1}{2} & \frac{k}{2} \end{pmatrix}$ and $M = \begin{pmatrix} 0 & k^2+4 \\ 1 & 0 \end{pmatrix}$. The identities we proved are not encountered in the $k-$Fibonacci and $k-$Lucas numbers literature.

MSC: 11B39 • 11B83

Keywords: $k-$Fibonacci numbers • $k-$Lucas numbers • Fibonacci Matrix

1. Introduction

This paper represents an interesting investigation about some special relations between matrices and $k-$Fibonacci numbers, $k-$Lucas numbers. This investigation is valuable to obtain new $k-$Fibonacci , $k-$Lucas identities by different methods. This paper contributes to $k-$Fibonacci, $k-$Lucas numbers literature, and encourage many researchers to investigate the properties of such number sequences.

Definition 1.1.

The $k-$Fibonacci sequence $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined as, $F_{k,0}=0, F_{k,1}=1$ and $F_{k,n+1}=kF_{k,n}+F_{k,n-1}$for $n\geq 1$

Definition 1.2.

The $k-$Lucas sequence $\{L_{k,n}\}_{n\in\mathbb{N}}$ is defined as, $L_{k,0}=2, L_{k,1}=k$ and $L_{k,n+1}=kL_{k,n}+L_{k,n-1}$ for $n\geq 1$

2. Main theorems

Lemma 2.1.

If X is a square matrix with $X^2=kX+I$, then $X^n=F_{k,n}X+F_{k,n-1}I$, for all $n\in Z$

Proof. If $n=0$ then result is obivious,
If $n=1$ then

$$\begin{aligned}(X)^1 &= F_{k,1}X+F_{k,0}I\\ &= 1X+I\\ &= X\end{aligned}$$

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Hence result is true for $n = 1$
It can be shown by induction that,

$$X^n = F_{k,n}X + F_{k,n-1}I, \text{for all } n \in Z$$

Assume that, $X^n = F_{k,n}X + F_{k,n-1}I$, and prove that, $X^{n+1} = F_{k,n+1}X + F_{k,n}I$,
Consider,

$$\begin{aligned} F_{k,n+1}X + F_{k,n}I &= (F_{k,n}X + F_{k,n-1}I)X + F_{k,n}I \\ &= (kX + I)F_{k,n} + XF_{k,n-1} \\ &= X^2F_{k,n} + XF_{k,n-1} = X(XF_{k,n} + F_{k,n-1}) \\ &= X(X^n) \\ &= X^{n+1} \end{aligned}$$

Hence, $X^{n+1} = F_{k,n+1}X + F_{k,n}I$,
By Induction, $X^n = F_{k,n}X + F_{k,n-1}I$, for all $n \in Z$
We now show that, $X^{-(n)} = F_{k,-n}X + F_{k,-n-1}I$, for all $n \in Z^+$
Let, $Y = kI - X$, then

$$\begin{aligned} Y^2 &= (kI - X)^2 \\ &= k^2I - 2kX + X^2 \\ &= k^2I - 2kX + kX + I \\ &= k^2I - kX + I \\ &= k(kI - X) + X + I \\ &= kY + I \end{aligned}$$

Therefore, $Y^2 = kY + I$,
This shows that,

$$\begin{aligned} &Y^n = F_{k,n}Y + F_{k,n-1}I, \\ &i.e. \quad (-X^{-1})^n = F_{k,n}(kI - X) + F_{k,n-1}I(-1)^nX^{-n} \\ &\qquad\qquad = -F_{k,n}X + F_{k,n+1}I \\ &X^{-n} = (-1)^{n+1}F_{k,n}X + (-1)^nF_{k,n+1}I \end{aligned}$$

Since, $F_{k,-n} = (-1)^{n+1}F_{k,n}, F_{k,-n-1} = (-1)^nF_{k,n+1}$, therefore $X^{-n} = F_{k,-n}X + F_{k,-n-1}I$, gives $X^{-(n)} = F_{k,-n}X + F_{k,-n+1}I$, for all $n \in Z^+$.
Hence proof. □

Corollary 2.1.

Let, $M = \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix}$*, then* $M^n = \begin{pmatrix} F_{k,n+1}F_{k,n} \\ F_{k,n}F_{k,n-1} \end{pmatrix}$

Proof. Since,

$$\begin{aligned} M^2 &= kM + I = F_{k,n}M + F_{k,n-1}I \,\big(\text{Using Lemma 2.1}\big) \\ &= \begin{pmatrix} kF_{k,n} & F_{k,n} \\ F_{k,n} & 0 \end{pmatrix} + \begin{pmatrix} F_{k,n-1} & 0 \\ 0 & F_{k,n-1} \end{pmatrix} \\ &= \begin{pmatrix} F_{k,n+1} & F_{k,n} \\ F_{k,n} & F_{k,n-1} \end{pmatrix}, \end{aligned}$$

for all $n \in Z$
Hence proof. □

Corollary 2.2.

Let, $S = \begin{pmatrix} \frac{k}{2} & \frac{k^2+4}{2} \\ \frac{k}{2} & \frac{k}{2} \end{pmatrix}$*, then* $S^n = \begin{pmatrix} \frac{L_{k,n}}{2} & \frac{(k^2+4)F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{pmatrix}$*, for every* $n \in Z$

Lemma 2.2.

$L_{k,n}^2 - (k^2+4)F_{k,n}^2 = 4(-1)^n$*, for all* $n \in Z$

Proof. Since, $det(S) = -1, \quad det(S^n) = [det(S)]^n = (-1)^n,$
Moreover since,

$$S^n = \begin{pmatrix} \frac{L_{k,n}}{2} & \frac{(k^2+4)F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{pmatrix},$$

We get

$$det(S^n) = \frac{L_{k,n}^2}{4} - \frac{(k^2+4)F_{k,n}^2}{4},$$

Thus it follows that $L_{k,n}^2 - (k^2+4)F_{k,n}^2 = 4(-1)^n$,for all $n \in Z$
Hence proof. □

Lemma 2.3.

$$2L_{k,n+m} = L_{k,n}L_{k,m} + (k^2+4)F_{k,n}F_{k,m} \quad \textit{and} \quad 2F_{k,n+m} = F_{k,n}L_{k,m} + L_{k,n}F_{k,m}$$

for all $n, m \in Z$

Proof. Since,

$$\begin{aligned} S^{n+m} &= S^n \cdot S^m \\ &= \begin{pmatrix} \frac{L_{k,n}}{2} & \frac{(k^2+4)F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{L_{k,m}}{2} & \frac{(k^2+4)F_{k,m}}{2} \\ \frac{F_{k,m}}{2} & \frac{L_{k,m}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{L_{k,n}L_{k,m}+(k^2+4)F_{k,n}F_{k,m}}{4} & \frac{(k^2+4)[L_{k,n}F_{k,m}+F_{k,n}L_{k,m}}{4} \\ \frac{L_{k,n}F_{k,m}+F_{k,n}L_{k,m}}{4} & \frac{L_{k,n}L_{k,m}+(k^2+4)F_{k,n}F_{k,m}}{4} \end{pmatrix} \end{aligned}$$

But,

$$S^{n+m} = \begin{pmatrix} \frac{L_{k,n+m}}{2} & \frac{(k^2+4)F_{k,n+m}}{2} \\ \frac{F_{k,n+m}}{2} & \frac{L_{k,n+m}}{2} \end{pmatrix},$$

Gives,

$$2L_{k,n+m} = L_{k,n}L_{k,m} + (k^2+4)F_{k,n}F_{k,m}$$

and

$$2F_{k,n+m} = F_{k,n}L_{k,m} + L_{k,n}F_{k,m}$$

for all $n, m \in Z$
Hence proof. □

Lemma 2.4.

$$2(-1)^m L_{k,n-m} = L_{k,n}L_{k,m} - (k^2+4)F_{k,n}F_{k,m}$$

and

$$2(-1)^m F_{k,n-m} = F_{k,n}L_{k,m} - L_{k,n}F_{k,m}$$

for all $n, m \in Z$.

Proof. Since,

$$\begin{aligned} S^{n-m} &= S^n \cdot S^{-m} \\ &= S^n \cdot [S^m]^{-1} \\ &= S^n \cdot (-1)^m \begin{pmatrix} \frac{L_{k,m}}{2} & \frac{-(k^2+4)F_{k,m}}{2} \\ \frac{-F_{k,m}}{2} & \frac{L_{k,m}}{2} \end{pmatrix} \\ &= (-1)^m \begin{pmatrix} \frac{L_{k,n}}{2} & \frac{(k^2+4)F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{L_{k,m}}{2} & \frac{-(k^2+4)F_{k,m}}{2} \\ \frac{-F_{k,m}}{2} & \frac{L_{k,m}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{L_{k,n}L_{k,m}-(k^2+4)F_{k,n}F_{k,m}}{4} & \frac{(k^2+4)[L_{k,n}F_{k,m}-F_{k,n}L_{k,m}}{4} \\ \frac{L_{k,n}F_{k,m}-F_{k,n}L_{k,m}}{4} & \frac{L_{k,n}L_{k,m}-(k^2+4)F_{k,n}F_{k,m}}{4} \end{pmatrix} \end{aligned}$$

But,

$$S^{n-m}=\begin{pmatrix} \frac{L_{k,n-m}}{2} & \frac{(k^2+4)F_{k,n-m}}{2} \\ \frac{F_{k,n-m}}{2} & \frac{L_{k,n-m}}{2} \end{pmatrix},$$

Gives,

$$2(-1)^m L_{k,n-m}=L_{k,n}L_{k,m}-(k^2+4)F_{k,n}F_{k,m}$$

and

$$2(-1)^m F_{k,n-m}=F_{k,n}L_{k,m}-L_{k,n}F_{k,m}$$

for all $n,m\in Z$. □

Lemma 2.5.

$$(-1)^m L_{k,n-m}+L_{k,n+m}=L_{k,n}L_{k,m}$$

and

$$(-1)^m F_{k,n-m}+F_{k,n+m}=F_{k,n}L_{k,m}$$

for all $n,m\in Z$.

Proof. By definition of the matrix S^n, it can be seen that

$$S^{n+m}+(-1)^m S^{n-m}=\begin{pmatrix} \frac{L_{k,n+m}+(-1)^m L_{k,n-m}}{2} & \frac{(k^2+4)[F_{k,n+m}+(-1)^m F_{k,n-m}}{2} \\ \frac{F_{k,n+m}+(-1)^m F_{k,n-m}}{2} & \frac{L_{k,n+m}+(-1)^m L_{k,n-m}}{2} \end{pmatrix}$$

On the other hand,

$$\begin{aligned} S^{n+m}+(-1)^m S^{n-m} &= S^n S^m+(-1)^m S^n S^{-m} \\ &= S^n[S^m+(-1)^m S^{-m}] \\ &= \begin{pmatrix} \frac{L_{k,n}}{2} & \frac{(k^2+4)F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{pmatrix}\cdot\left[\begin{pmatrix} \frac{L_{k,m}}{2} & \frac{(k^2+4)F_{k,m}}{2} \\ \frac{F_{k,m}}{2} & \frac{L_{k,m}}{2} \end{pmatrix}+(-1)^m\begin{pmatrix} \frac{L_{k,m}}{2} & \frac{-(k^2+4)F_{k,m}}{2} \\ \frac{-F_{k,m}}{2} & \frac{L_{k,m}}{2} \end{pmatrix}\right] \\ &= \begin{pmatrix} \frac{L_{k,n}}{2} & \frac{(k^2+4)F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{pmatrix}\cdot\begin{pmatrix} L_{k,m} & 0 \\ 0 & L_{k,m} \end{pmatrix} \\ &= \begin{pmatrix} \frac{L_{k,m}L_{k,n}}{2} & \frac{(k^2+4)F_{k,n}L_{k,m}}{2} \\ \frac{F_{k,n}L_{k,m}}{2} & \frac{L_{k,m}L_{k,n}}{2} \end{pmatrix} \end{aligned}$$

Gives,

$$(-1)^m L_{k,n-m}+L_{k,n+m}=L_{k,n}L_{k,m}$$

and

$$(-1)^m F_{k,n-m}+F_{k,n+m}=F_{k,n}L_{k,m}$$

for all $n,m\in Z$. □

Lemma 2.6.

$$8F_{k,x+y+z}=L_{k,x}L_{k,y}F_{k,z}+F_{k,x}L_{k,y}L_{k,z}+L_{k,x}F_{k,y}L_{k,z}+(k^2+4)F_{k,x}F_{k,y}F_{k,z}$$

and

$$8L_{k,x+y+z}=L_{k,x}L_{k,y}L_{k,z}+(k^2+4)[L_{k,x}F_{k,y}F_{k,z}+F_{k,x}L_{k,y}F_{k,z}+F_{k,x}F_{k,y}L_{k,z}$$

for all $x,y,z\in Z$.

Proof. By definition of the matrix S^n,it can be seen that

$$S^{x+y+z}=\begin{pmatrix} \frac{L_{k,x+y+z}}{2} & \frac{(k^2+4)F_{k,x+y+z}}{2} \\ \frac{F_{k,x+y+z}}{2} & \frac{L_{k,x+y+z}}{2} \end{pmatrix}$$

On the other hand,

$$\begin{aligned} S^{x+y+z} &= S^{x+y}S^z \\ &= \begin{pmatrix} \frac{L_{k,x+y}}{2} & \frac{(k^2+4)F_{k,x+y}}{2} \\ \frac{F_{k,x+y}}{2} & \frac{L_{k,x+y}}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{L_{k,z}}{2} & \frac{(k^2+4)F_{k,z}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{L_{k,x+y}L_{k,z}+(k^2+4)F_{k,x+y}F_{k,z}}{4} & \frac{(k^2+4)[L_{k,x+y}F_{k,z}+F_{k,x+y}L_{k,z}}{4} \\ \frac{L_{k,z}F_{k,x+y}+F_{k,z}L_{k,x+y}}{4} & \frac{L_{k,x+y}L_{k,z}+(k^2+4)F_{k,x+y}F_{k,z}}{4} \end{pmatrix} \end{aligned}$$

Using,

$$\begin{aligned} 2L_{k,x+y} &= L_{k,x}L_{k,y}+(k^2+4)F_{k,x}F_{k,y} \\ 2F_{k,x+y} &= L_{k,y}F_{k,x}+(k^2+4)F_{k,y}L_{k,x} \end{aligned}$$

Gives,

$$8F_{k,x+y+z}=L_{k,x}L_{k,y}F_{k,z}+F_{k,x}L_{k,y}L_{k,z}+L_{k,x}F_{k,y}L_{k,z}+(k^2+4)F_{k,x}F_{k,y}F_{k,z}$$

and

$$8L_{k,x+y+z}=L_{k,x}L_{k,y}L_{k,z}+(k^2+4)[L_{k,x}F_{k,y}F_{k,z}+F_{k,x}L_{k,y}F_{k,z}+F_{k,x}F_{k,y}L_{k,z}$$

for all $x,y,z\in Z$. □

Theorem 2.1.

$$L^2_{k,x+y}-(k^2+4)(-1)^{x+y+1}F_{k,z-x}L_{k,x+y}F_{k,y+z}-(k^2+4)(-1)^{x+z}F^2_{k,y+z}=(-1)^{y+z}L^2_{k,z-x}$$

for all $x,y,z\in Z$.

Proof. Consider matrix multiplication given below.
That is,

$$\begin{pmatrix} \frac{L_{k,x}}{2} & \frac{(k^2+4)F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{pmatrix} \cdot \begin{pmatrix} L_{k,y} \\ F_{k,y} \end{pmatrix} = \begin{pmatrix} L_{k,x+y} \\ F_{k,y+z} \end{pmatrix}$$

Now,

$$\begin{aligned} det\begin{pmatrix} \frac{L_{k,x}}{2} & \frac{(k^2+4)F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{pmatrix} &= \frac{L_{k,x}L_{k,z}-(k^2+4)F_{k,x}F_{k,z}}{4} \\ &= \frac{(-1)^x L_{k,z-x}}{2} \\ &= Q\neq 0 \end{aligned}$$

Therefore we can write

$$\begin{aligned} \begin{pmatrix} L_{k,y} \\ F_{k,y} \end{pmatrix} &= \begin{pmatrix} \frac{L_{k,x}}{2} & \frac{(k^2+4)F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{pmatrix}^{-1} \cdot \begin{pmatrix} L_{k,x+y} \\ F_{k,y+z} \end{pmatrix} \\ &= \frac{1}{Q}\begin{pmatrix} \frac{L_{k,z}}{2} & \frac{-(k^2+4)F_{k,x}}{2} \\ \frac{-F_{k,z}}{2} & \frac{L_{k,x}}{2} \end{pmatrix} \cdot \begin{pmatrix} L_{k,x+y} \\ F_{k,y+z} \end{pmatrix} \end{aligned}$$

Gives,

$$L_{k,y}=\frac{(-1)^x[L_{k,z}L_{k,x+y}-(k^2+4)F_{k,x}F_{k,y+z}]}{L_{k,z-x}}$$

and

$$F_{k,y} = \frac{(-1)^x [L_{k,x} F_{k,z+y} - F_{k,z} L_{k,y+x}]}{L_{k,z-x}}$$

Since,

$$L_{k,y}^2 - (k^2+4) F_{k,y}^2 = 4(-1)^y$$

We get,

$$[L_{k,z} L_{k,x+y} - (k^2+4) F_{k,x} F_{k,y+z}]^2 - (k^2+4)^2 [L_{k,x} F_{k,z+y} - F_{k,z} L_{k,y+x}]^2 = 4(-1)^y L_{k,z-x}^2$$

Using Lemma 2.4 and Lemma 2.6,

$$\left(L_{k,z}^2 L_{k,x+y}^2 - 2(k^2+4) L_{k,z} F_{k,x+y} F_{k,y+z} + (k^2+4)^2 F_{k,x}^2 F_{k,y+z}^2\right) \\ -(k^2+4)(L_{k,x}^2 F_{k,y+z}^2 - 2L_{k,x} F_{k,z} F_{k,y+z} L_{k,x+y} + F_{k,z}^2 L_{k,x+y}^2) = 4(-1)^y L_{k,z-x}^2$$

Gives,

$$L_{k,x+y}^2 - (k^2+4)(-1)^{x+y+1} F_{k,z-x} L_{k,x+y} F_{k,y+z} - (k^2+4)(-1)^{x+z} F_{k,y+z}^2 = (-1)^{y+z} L_{k,z-x}^2$$

for all $x, y, z \in Z$. □

Theorem 2.2.

$$L_{k,x+y}^2 - (-1)^{x+z} L_{k,z-x} L_{k,x+y} L_{k,y+z} + (-1)^{x+z} L_{k,y+z}^2 = (-1)^{y+z+1} (k^2+4) F_{k,z-x}^2$$

for all $x, y, z \in Z$, $x \neq z$.

Proof. Consider matrix multiplication,

$$\begin{pmatrix} \frac{L_{k,x}}{2} & \frac{(k^2+4)F_{k,x}}{2} \\ \frac{L_{k,z}}{2} & \frac{(k^2+4)F_{k,z}}{2} \end{pmatrix} \cdot \begin{pmatrix} L_{k,y} \\ F_{k,y} \end{pmatrix} = \begin{pmatrix} L_{k,x+y} \\ L_{k,y+z} \end{pmatrix}$$

Now,

$$det \begin{pmatrix} \frac{L_{k,x}}{2} & \frac{(k^2+4)F_{k,x}}{2} \\ \frac{L_{k,z}}{2} & \frac{(k^2+4)F_{k,z}}{2} \end{pmatrix} = \frac{(k^2+4)(-1)^x F_{k,z-x}}{2} \\ = P \neq 0, \quad (\text{ if } x \neq z)$$

Therefore for $x \neq z$, we can write

$$\begin{pmatrix} L_{k,y} \\ F_{k,y} \end{pmatrix} = \begin{pmatrix} \frac{L_{k,x}}{2} & \frac{(k^2+4)F_{k,x}}{2} \\ \frac{L_{k,z}}{2} & \frac{(k^2+4)F_{k,z}}{2} \end{pmatrix}^{-1} \cdot \begin{pmatrix} L_{k,x+y} \\ L_{k,y+z} \end{pmatrix} \\ = \frac{1}{P} \begin{pmatrix} \frac{(k^2+4)F_{k,z}}{2} & \frac{-(k^2+4)F_{k,x}}{2} \\ \frac{-L_{k,z}}{2} & \frac{L_{k,x}}{2} \end{pmatrix} \cdot \begin{pmatrix} L_{k,x+y} \\ L_{k,y+z} \end{pmatrix}$$

Gives,

$$L_{k,y} = \frac{(-1)^x [F_{k,z} L_{k,x+y} - F_{k,x} L_{k,y+z}]}{F_{k,z-x}}$$

and

$$F_{k,y} = \frac{(-1)^x [L_{k,x} L_{k,z+y} - L_{k,z} L_{k,y+x}]}{(k^2+4) F_{k,z-x}}$$

Since,

$$L_{k,y}^2 - (k^2+4) F_{k,y}^2 = 4(-1)^y$$

We get,

$$(k^2+4)[F_{k,z} L_{k,x+y} - F_{k,x} L_{k,y+z}]^2 - [L_{k,x} L_{k,z+y} - L_{k,z} L_{k,y+x}]^2 = 4(k^2+4)(-1)^y F_{k,z-x}^2$$

Using Lemma 2.4 and Lemma 2.6, We obtain

$$L_{k,x+y}^2 - (-1)^{x+z} L_{k,z-x} L_{k,x+y} L_{k,y+z} + (-1)^{x+z} L_{k,y+z}^2 = (-1)^{y+z+1} (k^2+4) F_{k,z-x}^2$$

for all $x, y, z \in Z$, $x \neq z$. □

Theorem 2.3.

$$F_{k,x+y}^2 - L_{k,x-z}F_{k,x+y}F_{k,y+z} + (-1)^{x+z}F_{k,y+z}^2 = (-1)^{y+z}F_{k,z-x}^2$$

for all $x, y, z \in Z, x \neq z$.

Proof. Consider matrix multiplication,

$$\begin{pmatrix} \frac{F_{k,x}}{2} & \frac{F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{pmatrix} \cdot \begin{pmatrix} L_{k,y} \\ F_{k,y} \end{pmatrix} = \begin{pmatrix} F_{k,x+y} \\ F_{k,y+z} \end{pmatrix}$$

Now,

$$\begin{aligned} det\begin{pmatrix} \frac{F_{k,x}}{2} & \frac{F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{pmatrix} &= \frac{(-1)^z F_{k,x-z}}{2} \\ &= R \neq 0, \quad (\text{ if } x \neq z) \end{aligned}$$

Therefore for $x \neq z$, we get,

$$\begin{aligned} \begin{pmatrix} L_{k,y} \\ F_{k,y} \end{pmatrix} &= \begin{pmatrix} \frac{F_{k,x}}{2} & \frac{F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{pmatrix}^{-1} \cdot \begin{pmatrix} F_{k,x+y} \\ F_{k,y+z} \end{pmatrix} \\ &= \frac{1}{R}\begin{pmatrix} \frac{L_{k,z}}{2} & \frac{-L_{k,x}}{2} \\ \frac{-F_{k,z}}{2} & \frac{F_{k,x}}{2} \end{pmatrix} \cdot \begin{pmatrix} F_{k,x+y} \\ F_{k,y+z} \end{pmatrix} \end{aligned}$$

Gives,

$$L_{k,y} = \frac{(-1)^z[L_{k,z}F_{k,x+y} - L_{k,x}F_{k,y+z}]}{F_{k,x-z}}$$

and

$$F_{k,y} = \frac{(-1)^z[F_{k,x}F_{k,z+y} - F_{k,z}F_{k,y+x}]}{F_{k,x-z}}$$

Now consider,

$$[L_{k,z}F_{k,x+y} - L_{k,x}F_{k,y+z}]^2 - (k^2+4)[F_{k,x}F_{k,z+y} - F_{k,z}F_{k,y+x}]^2 = 4(-1)^y F_{k,x-z}^2$$

Using Lemma 2.4 and Lemma 2.6, We get

$$F_{k,x+y}^2 - L_{k,x-z}F_{k,x+y}F_{k,y+z} + (-1)^{x+z}F_{k,y+z}^2 = (-1)^{y+z}F_{k,z-x}^2$$

for all $x, y, z \in Z, x \neq z$. □

3. Conclusions

The conclusions arising from the work are as follows:
Some new identities have been obtained for the $k-$Fibonacci and $k-$Lucas sequences.

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