On Quadrics and Pseudoquadrics Inversions in Hyperpseudospheres

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Abstract

This note on quadrics and pseudoquadrics inversions in hyperpseudospheres shows that the inversions produce different results in a three-dimensional spacetime. Using Geometric Algebra, all quadric and pseudoquadric entities and operations are in the $\mathbb{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA). Quadrics at zero velocity are purely spatial entities in $xyz$-space that are hyper-cylinders in $wxyz$-spacetime. Pseudoquadrics represent quadrics in a three-dimensional (3D) $xyw$, $yzw$, or $zxw$ spacetime with the pseudospatial $w$-axis that is associated with time $w = ct$. The inversion of a quadric in a hyperpseudosphere can produce a Darboux pseudocyclide in a 3D spacetime that is a quartic hyperbolic (infinite) surface, which does not include the point at infinity. The inversion of a pseudoquadric in a hyperpseudosphere can produce a Darboux pseudocyclide in a 3D spacetime that is a quartic finite surface. A quadric and pseudoquadric can represent the same quadric surface in space, and their two different inversions in a hyperpseudosphere represent the two types of reflections of the quadric surface in a hyperboloid.

1 Introduction

In the $\mathbb{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA) [1], using the DCSTA extraction elements, it is possible to form quadric surface entities in $xyz$-space, or pseudoquadric surface entities in any one of $xyw$-spacetime, $yzw$-spacetime, or $zxw$-spacetime. For example, a quadric ellipsoid entity $E$ is

$$E = T_{xx}/a^2 + T_{yy}/b^2 + T_{zz}/c^2 - T_1,$$

And, for example, a pseudoquadric ellipsoid entity $E^+$ in $xyw$-spacetime is

$$E^{+z} = T_{xx}/a^2 + T_{yy}/b^2 + T_{ww}/c^2 - T_1,$$

where the $z$-axis has been replaced with the pseudospatial $w$-axis that is associated with time $w = ct$.

The quadric and pseudoquadric both represent the same quadric surface in space, but they are different entities in spacetime that have different properties.

2 Results

2.1 Pseudoquadric inversion in hyperpseudosphere

As an example, let the pseudoquadric ellipsoid be

$$E^{+z} = T(T_{xx}/5^2 + T_{yy}/20^2 + T_{ww}/5^2 - T_1) T^{-1},$$

with translator $T$ for translation by

$$d = w\gamma_0 + x\gamma_1 = 2\gamma_0 + 12\gamma_1,$$

such that $E^{+z}$ is centered at position $(w, x, y, z) = (2, 12, 0, 0)$. Let the hyperpseudosphere be at the origin $e_o$ with radius $r_0 = 8$ as

$$\Sigma = \Sigma_{c1} \wedge \Sigma_{c2},$$

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where
\[
\Sigma_{c_1} = e_{o_1} + \frac{1}{2} s^2 e_{\infty 1}
\]
\[
\Sigma_{c_2} = e_{o_2} + \frac{1}{2} s^2 e_{\infty 2}.
\]

The 3D spacetime is \( xyw \)-spacetime, where the pseudospatial \( w \)-axis can be treated as the new \( z \)-axis. The hyperpseudosphere at \( z = 0 \) is a spacetime hyperboloid in \( xyw \)-spacetime. The hyperboloid is to be graphed with \( z = 0 \), and the \( w \)-axis acts as the new \( z \)-axis in the graphing space.

The inversion of \( E^{+z} \) in \( \Sigma \) is
\[
\Omega = \Sigma E^{+z} \Sigma.
\]

The inversion \( \Omega \) is to be graphed at \( z = 0 \), and the \( w \)-axis is graphed as if it is the \( z \)-axis.
Figures 1-4 show the inversion $\Omega = \Sigma E^{+\ast} \Sigma$ of the pseudoquadric $E^{+\ast}$ in the hyperpseudosphere $\Sigma$. The result is a finite quartic surface $\Omega$, which has been called a Darboux pseudocyclide, that does not include the infinity point $e_\infty$. This may be a realistic reflection of the ellipsoid in the hyperboloid.

2.2 Quadric inversion in hyperpseudosphere

Similar to the pseudoquadric, let the quadric ellipsoid be

$$E = T(T_{xy}/5^2 + T_{yz}/20^2 + T_{zx}/5^2 - T_1)T^\ast,$$

with translator $T$ for translation by

$$d = x\gamma_1,$$

$$= 12\gamma_1,$$

such that $E$ is centered at position $(w, x, y, z) = (w, 12, 0, 0)$. The quadric is independent of time $w$ and is a hypercylinder in spacetime. Let the hyperpseudosphere again be at the origin $e_0$ with radius $r_0 = 8$ as

$$\Sigma = \Sigma_{e_1} \wedge \Sigma_{e_2},$$
where
\[ \Sigma_{C_1} = e_{o_1} + \frac{1}{2} s^2 e_{\infty 1} \]
\[ \Sigma_{C_2} = e_{o_2} + \frac{1}{2} s^2 e_{\infty 2}. \]

The 3D spacetime is \( xyw \)-spacetime, where the pseudospatial \( w \)-axis can be treated as the new \( z \)-axis. The hyperpseudosphere at \( z = 0 \) is a spacetime hyperboloid in \( xyw \)-spacetime. The hyperboloid is to be graphed with \( z = 0 \), and the \( w \)-axis acts as the new \( z \)-axis in the graphing space.

The inversion of \( E \) in \( \Sigma \) is
\[ \Omega = \Sigma E \Sigma. \]

The inversion \( \Omega \) is to be graphed at \( z = 0 \), and the \( w \)-axis is graphed as if it is the \( z \)-axis.

As shown in Figure 5, the hyperpseudosphere \( \Sigma \) at \( z = 0 \) is the hyperboloid, and the quadric ellipsoid at \( z = 0 \) is a \( xy \)-plane ellipse that is an elliptical cylinder in \( xyw \)-space. The result \( \Omega \) is the inversion of the elliptic cylinder in the hyperboloid, all in \( xyw \)-spacetime.

### 3 Conclusion

The inversion of a pseudoquadric \( E^{+ \pm} \), representing a quadric \( E \), in a pseudosphere (spacetime circular hyperboloid), which is the hyperpseudosphere \( \Sigma \) at \( z = 0 \), is the entity \( \Omega = \Sigma E^{+ \pm} \Sigma \) that appears to be the correct inversion of a quadric \( E \) in a circular hyperboloid.

On the other hand, a quadric \( E \) in \( xyw \)-space is a kind of cylinder in \( wxyz \)-spacetime that is symmetrical around the pseudospatial time \( (w = ct) \)-axis, and its inversion in a spacetime circular hyperboloid is the inversion of an elliptical cylinder in the hyperboloid.

The \( G_{4,8} \) Double Conformal Space-Time Algebra (DCSTA) has entities for both spatial quadrics in \( xyz \)-space, and also similar pseudoquadric entities that are formed in a 3D spacetime that uses the pseudospatial time \( (w = ct) \)-axis as a drop-in replacement for one of the usual \( x, y, \) or \( z \) spatial axes. The inversion of the pseudoquadrics in a pseudosphere (circular hyperboloid) appears to be a correct reflection or inversion. DCSTA provides new representations of quadrics as pseudoquadrics, and a new inversion operation for their inversions in circular hyperboloids.

### Bibliography