An Algorithm for Calculating Terms of a Number Sequence using an Auxiliary Sequence

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Abstract

A formula giving the n:th number of a sequence defined by a recursion formula plus initial value is deduced using generating functions. Of particular interest is the possibility to get an exact expression for the nth term by means a recursion formula of the same type as the original one. As for the sequence itself it is of some interest that the original recursion is non-linear and the fact that the sequence grows very fast, the number of digits increasing more or less exponentially. Other sequences with the same rekursion span can be treated similarly.

The numbers are denoted $a_n, n = 0, 1, 2, ...$ and are defined by

$$a_{n+3} = a_{n+2} \cdot a_n$$

and $a_0 = 1, a_1 = 2, a_2 = 3$. With other initial values we get similar sequences. The "span" of the recursion, in this case $[n, n + 3]$, is essential since it determines the degree of a polynomial whose roots occur in the formula for the nth term in the sequence.

The sequence grows very fast. As an example we present the first 15 terms and the 20th, calculated by brute force from the defining recursion relation using the CAS *Derive*[1] to take care of the many digits.

[1]Derive is a symbol handling software, developed by Soft Warehouse on Hawaii and taken over by Hewlett-Packard who unfortunately stopped maintance.

$a_0 - a_{14}$:

1, 2, 3, 3, 6, 18, 54, 324, 5832,314928,102036672,595077871104,
187406683791040512, 19122354324594117261656064,
11379289901975835088948428694571974656,

a_{19}:

40355115689415229911776335471083161526375224235958675590564116402
22972891112867883288097096709484155916317024983814628771564740313
41700466532124498258165138396370160723936446677131795900233889842
71721403167027070264796258196638075360510963802490011648,

or, approximately,

$4,0355115689415229911776335 \cdot 10^{250}$.

To find a formula for a_n, let $b_n = \ln a_n$. Then we get a linear recursion

$$b_{n+3} = b_{n+2} + b_n \tag{1}$$

and initial values $b_0 = 0, b_1 = \ln 2, b_2 = \ln 3$.

1 Generating function

A generating function for the sequence $\{b_n\}_{n=0}^{\infty}$ is $f(x) = \sum_{n=0}^{\infty} b_n x^n$. Some manipulation of indices and use of the recursion formula yields (the second and third terms are denoted α and β, i e $b_1 = \alpha \ (= \ln 2)$ and $b_2 = \beta \ (= \ln 3)$ and the first term $b_0 = 0$)

$$f(x) = \sum_{n=0}^{\infty} b_{n+3} x^{n+3} + \alpha x + \beta x^2 \tag{2}$$

$$f(x) = \sum_{n=0}^{\infty} b_{n+2} x^{n+2} + \alpha x \tag{3}$$

Next, we use the recursion relation in (2).

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} (b_{n+2} + b_n) x^{n+3} + \alpha x + \beta x^2 \\ &= x \cdot \sum_{n=0}^{\infty} b_{n+2} x^{n+2} + x^3 \cdot \sum_{n=0}^{\infty} b_n x^n + \alpha x + \beta x^2 \ . \end{aligned}$$

Inserting the sum from (3) we get

$$f(x) = x(f(x) - \alpha x) + x^3 f(x) + \alpha x + \beta x^2$$

from which $f(x)$ is found in closed form,

$$f(x) = \frac{\alpha x + (\beta - \alpha)x^2}{1 - x - x^3}$$

Now, let

$$r(n, x) = \frac{1}{n!} \cdot \frac{\mathrm{d}^n}{\mathrm{d}x^n} \frac{\alpha x + (\beta - \alpha)x^2}{1 - x - x^3} \ .$$

Then $b_n = r(n, 0)$ and $a_n = \exp(b_n)$.

Expansion of the expression for $f(x)$ into partial fractions gives

$$\frac{\alpha x + (\beta - \alpha)x^2}{1 - x - x^3} = \sum_{k=1}^{3} \frac{r_k}{x - x_k} \ .$$

Here x_1, x_2, x_3 are the zeros of the polynomial $x^3 + x - 1$ and $r_k = g(x_k)$ where

$$g(x) = \frac{\alpha x + (\beta - \alpha)x^2}{\frac{\mathrm{d}}{\mathrm{d}x}(1 - x - x^3)} = \frac{x \cdot [x \cdot (\alpha - \beta) - \alpha]}{3x^2 + 1}$$

so

$$r_k = \frac{x_k \cdot [(\alpha - \beta)x_k - \alpha]}{3x_k^2 + 1} \ .$$

The differentiation can now be performed which gives

$$r(n, x) = \frac{1}{n!} \cdot \sum_{k=1}^{3} \left\{ \frac{(-1)^n \cdot n!}{(x - x_k)^{n+1}} \cdot r_k \right\}$$

and so

$$\begin{aligned} r(n, 0) &= -\sum_{k=1}^{3} \frac{r_k}{x_k^{n+1}} = \\ &= \sum_{k=1}^{3} \frac{1}{x_k^{n-1}} \cdot \frac{\alpha + (\beta - \alpha)x_k}{3x_k^3 + x_k} \\ &= \sum_{k=1}^{3} \frac{\alpha + (\beta - \alpha)x_k}{(3 - 2x_k)x_k^{n-1}} \ , \end{aligned}$$

since $x_n^3 + x_n - 1 = 0$.

Zeroes of the polynomial $x^3 + x - 1$:

$$\begin{cases} x_1 &= \delta \\ x_2 &= -\frac{1}{2}\delta + i \cdot \sigma \\ x_3 &= -\frac{1}{2}\delta - i \cdot \sigma \end{cases}$$

where δ and σ are real, so δ is the one real zero.

Now, let

$$D_n = \sum_{k=1}^{3} \frac{1}{(3 - 2x_k)x_k^n} \ . \tag{4}$$

Then

$$r(n, 0) = \alpha D_{n-1} + (\beta - \alpha) D_{n-2} \ . \tag{5}$$

Using the connection between roots and coefficients,

$$\begin{cases} x_1 + x_2 + x_3 &= 0 \\ x_1x_2 + x_2x_3 + x_3x_1 &= 1 \\ x_1x_2x_3 &= 1 \end{cases} ,$$

we can calculate the LCD[2] in the sum (4) for D_n,

$$\begin{aligned} \prod_{k=1}^{3}(3 - 2x_k)x_k^n &= (3 - 2x_1)(3 - 2x_2)(3 - 2x_3)(x_1x_2x_3)^n \\ &= 27 - 18x_1 - 18x_2 - 18x_3 + 12x_1x_2 + 12x_1x_3 + 12x_2x_3 - 8x_1x_2x_3 \\ &= 27 + 12 \cdot 1 - 8 \cdot 1 = 31 \ , \quad \text{ie} \end{aligned}$$

$$\prod_{k=1}^{3}(3 - 2x_k)x_k^n = 31 \ .$$

[2] Least Common Divisor (or Denominator).

Then,

$$\begin{aligned}
31D_n &= (3-2x_2)(3-2x_3)(x_2x_3)^n + (3-2x_3)(3-2x_1)(x_3x_1)^n \\
&\quad + (3-2x_1)(3-2x_2)(x_1x_2)^n \\
&= (9-6x_2-6x_3+4x_2x_3)(x_2x_3)^n + (9-6x_3-6x_1+4x_3x_1)(x_3x_1)^n \\
&\quad + (9-6x_1-6x_2+4x_1x_2)(x_1x_2)^n \\
&= 9(x_2x_3)^n - 6x_2^{n+1}x_3^n - 6x_2^n x_3^{n+1} + 4(x_2x_3)^{n+1} \\
&\quad + 9(x_3x_1)^n - 6x_3^{n+1}x_1^n - 6x_3^n x_1^{n+1} + 4(x_3x_1)^{n+1} \\
&\quad + 9(x_1x_2)^n - 6x_1^{n+1}x_2^n - 6x_1^n x_2^{n+1} + 4(x_1x_2)^{n+1} \ , \quad \text{ie}
\end{aligned}$$

$$31D_n = 9p_n + 4p_{n+1} - 6q_n \ ,$$

where we have defined two quantities p_n and q_n,

$$\begin{cases} p_n = x_1^n x_2^n + x_2^n x_3^n + x_3^n x_1^n \\ q_n = x_1^{n+1}x_2^n + x_1^n x_2^{n+1} + x_2^{n+1}x_3^n + x_2^n x_3^{n+1} + x_3^{n+1}x_1^n + x_3^n x_1^{n+1} \end{cases}$$

We will need

$$p_{n+1} = p_n + p_{n-2} \tag{6}$$

$$q_n = -p_{n-1} \tag{7}$$

which are proved using the connection between roots and coefficients and $1/x_k = x_k^2 + 1$ from $x_k^3 + x_k - 1 = 0$:

$$\begin{aligned}
p_{n+1} &= (x_1x_2)^n \cdot \frac{1}{x_3} + (x_2x_3)^n \cdot \frac{1}{x_1} + (x_3x_1)^n \cdot \frac{1}{x_2} \\
&= (x_1x_2)^n \cdot (x_3^2+1) + (x_2x_3)^n \cdot (x_1^2+1) + (x_3x_1)^n \cdot (x_2^2+1) \\
&= (x_1x_2)^n \cdot (x_1x_2)^{-2} + (x_2x_3)^n \cdot (x_2x_3)^{-2} + (x_3x_1)^n \cdot (x_3x_1)^{-2} \\
&\quad + (x_1x_2)^n + (x_2x_3)^n + (x_3x_1)^n = p_{n-2} + p_n
\end{aligned}$$

and

$$\begin{aligned}
q_n &= (x_1x_2)^n(x_1+x_2) + (x_2x_3)^n(x_2+x_3) + (x_3x_1)^n(x_3+x_1) \\
&= (x_1x_2)^n(-x_3) + (x_2x_3)^n(-x_1) + (x_3x_1)^n(-x_2) \\
&= -(x_1x_2)^n \cdot \frac{1}{x_1x_2} - (x_2x_3)^n \cdot \frac{1}{x_2x_3} - (x_3x_1)^n \cdot \frac{1}{x_3x_1} \\
&= -(x_1x_2)^{n-1} - (x_2x_3)^{n-1} - (x_3x_1)^{n-1} = -p_{n-1} \ .
\end{aligned}$$

Finally, from (5),

$$\boxed{a_n = 2^{B_n} \cdot 3^{C_n}} \ , \tag{8}$$

where we have introduced the notation

$$\begin{cases} B_n &= D_{n-1} - D_{n-2} \\ C_n &= D_{n-2} \end{cases} \ ,$$

2 Exact calculation of B_n and C_n

We will need some identities following from the defining ekvation for δ ($x^3 + x = 1$ and δ real), mostly

$$\delta^3 = 1 - \delta \ , \ \delta^4 = \delta - \delta^2 \ , \ \delta^2 + 1 = 1/\delta \ .$$

Consider

$$(\delta^2 + 1)^n = k_n \delta^2 + l_n \delta + m_n$$

where the coefficients k_n, l_n, m_n are integers. Further,

$$(\delta^2 + 1)^{n+1} = (\delta^2 + 1)^n \cdot (\delta^2 + 1) = m_n \delta^2 + k_n \delta + (l_n + m_n) \ .$$

But we also have

$$(\delta^2 + 1)^{n+1} = k_{n+1} \delta^2 + l_{n+1} \delta + m_{n+1}$$

and, thus,

$$\begin{cases} k_{n+1} &= m_n \\ l_{n+1} &= k_n \\ m_{n+1} &= l_n + m_n \end{cases}$$

From this,

$$k_{n+3} = m_{n+2} = l_{n+1} + m_{n+1} = k_n + k_{n+2} \ , \quad \text{ie}$$

$$k_{n+3} = k_{n+2} + k_n \ . \tag{9}$$

The recursion formula (9) is exactly the same as (1) for b_n so the same techique using generating function can be used, just observing the different initial values:

$k_0 = 0,\ k_1 = 1,\ k_2 = 1$

Just the coefficients of δ^2, i e the k_n, will be needed.

$$h(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{n+3} x^{n+3} + x + x^2 \ ,$$

$$h(x) = \frac{x}{1 - x - x^3} \ ,$$

$$k_n = \frac{1}{n!} \cdot \frac{\mathrm{d}^n}{\mathrm{d}x^n} \frac{x}{1 - x - x^3}\Big|_{x=0} \ ,$$

$$k_n = \sum_{k=1}^{3} \frac{1}{(3 - 2x_k) x_k^{n-1}} \ .$$

Thus $k_n = D_{n-1}$ from which we get relations between B_n, C_n, and k_n:

$$\begin{aligned} B_n + C_n &= k_n \\ C_n &= k_{n-1} \end{aligned}$$

$$\boxed{\begin{aligned} B_n &= k_n - k_{n-1} \\ C_n &= k_{n-1} \end{aligned}} \tag{10}$$

3 Examples

1. a_9:

We need k_n for $n = 8, 9$. To get these we expand $(\delta^2 + 1)^n$ using $\delta^3 = 1 - \delta$ and $\delta^4 = \delta - \delta^2$ which gives a second-degree polynomial where the coefficients are the numbers k_n (and l_n, m_n which we do not need).

$$
\begin{aligned}
(\delta^2+1)^2 &= \delta^2+\delta+1 \\
(\delta^2+1)^4 &= (\delta^2+\delta+1)^2 = 2\delta^2+\delta+13 \\
(\delta^2+1)^8 &= (2\delta^2+\delta+13)^2 = 9\delta^2+6\delta+13 \\
(\delta^2+1)^9 &= (9\delta^2+6\delta+13)(\delta^2+1) = 13\delta^2+9\delta+19
\end{aligned}
$$

The coefficients of δ^2 in the last two results give $k_8 = 9$, $k_9 = 13$.
Inserting this into (10) we get $B_9 = 13 - 9 = 4$, $C_9 = 9$,
and so

$$a_{10} = 2^4 \cdot 3^9 = 314928 \ .$$

in accordance with the value given at the beginning.

2. a_{19}:

We need k_n for $n = 18, 19$.

$$
\begin{aligned}
(\delta^2+1)^{18} &= (13\delta^2+9\delta+19)^2 = 406\delta^2+277\delta+595 \\
(\delta^2+1)^{19} &= (406\delta^2+277\delta+595)(\delta^2+1) \rightsquigarrow 595\delta^2
\end{aligned}
$$

where, in the last term, we just kept δ^2-terms.

The coefficients of δ^2 in the last two results give $k_{18} = 406$, $k_{19} = 595$.
Inserting this into (10) we get $B_{19} = 595 - 406 = 189$, $C_{19} = 406$,
and so

$$a_{19} = 2^{189} \cdot 3^{406}.$$

Expanding this we will get the digits shown at the beginning.
The number of digits equals $\lfloor \lg a_{20} \rfloor + 1 = 251$.

3. a_{49}:

We just give the main points.

The coefficients k_n needed are $k_{48} = 38789712$, $k_{49} = 56849086$, and we get

$$a_{49} = 2^{18059374} \cdot 3^{38789712} \ .$$

The number of digits is $\lfloor \lg a_{49} \rfloor + 1 = 23943810$.

4. a_{41}:

Just for fun we mention that the first term having more than 1 million decimal digits when written out in full is a_{41} with 1124962 digits.

5. a_{609}:

The first term having more than 1 googol (10^{100}) digits is a_{609} with 1025796070121477282014076737779472715918010553956063662897382761810734 8560001693953375691105965265292$\approx 1.03 \cdot 10^{100}$ digits.

6. a_{999}:

Number of digits $\sim 10^{165}$.

The number of digits grows more or less exponentially.