Material bodies moving at superluminal speed

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Abstract

As seen from a spaceship accelerating towards a star, the star approaches the ship at a speed that can exceed c, the speed of light in vacuum. For the particular case of constant acceleration with given final speed kc at the star this speed is calculated as a function of the ship's proper time. It is found that the upper bound of this speed is 1,5c.

1 Introduction

According to Special Relativity a material body cannot be accelerated to or above c, the speed of light in vacuum. This is as measured from an inertialsystem. A well-known case of superluminal speed is furnished by distant galaxies, receeding from us at higher speed than c. This is caused by the expansion of space and goes beyond Special Relativity. According to General Relativity the local speed is still bounded by c. A lesser known case seems to be that the speed of material bodies, such as stars, can exceed c as measured from an accelerated system, such as a spaceship.

Qualitatively this is an obvious implication of Lorentz contraction in combination with time dilatation. If, for example, a spaceship accelerates towards a star in a short time to the speed $c\sqrt{3}/2 \approx 0.87c$ then the distance to the star as measured from the spaceship will be slightly less than half the remaining rest distance. Due to time dilatation the proper time elapsed in the spaceship can be very small. Consequently, the sped of the approaching star must have been larger than c. In what follows we calculate the speed for the case of uniform acceleration. – Essential is that we represent an accelerated reference system by a sequence of momentary rest systems.

2 General results

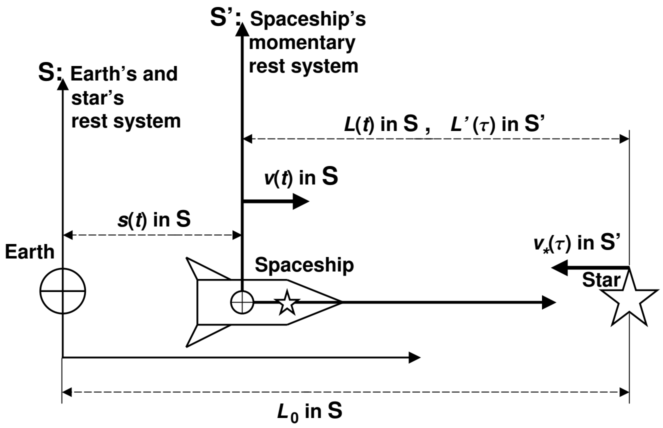

Figure 1: Spaceship moving towards a star at distance L_0

Speed of the star as measured from the spaceship:

$$|v_*(\tau)| = \left|\frac{\mathrm{d}\,L'(\tau)}{\mathrm{d}\,\tau}\right| \tag{1}$$

where

$$\begin{aligned} L'(\tau) &= L(t)\sqrt{1 - v(t)^2/c^2} \\ L(t) &= L_0 - \int_0^t v(\xi)\,\mathrm{d}\xi = L_0 - s(t) \end{aligned}$$

$$\tau = \int_0^t \sqrt{1 - v(\xi)^2/c^2}\,\mathrm{d}\xi \tag{2}$$

Total time **S** is denoted T, the final speed kc, $0 < k < 1$ and the velocity $v(t)$ in **S** is assumed positive ($v(t) > 0$), and the acceleration non-negative ($\mathrm{d}v(t)/\mathrm{d}t$) for all t, $0 < t \leq T$. Also, of course, $L_0 \geq s(t)$.

$$\begin{aligned} v_* &= \frac{\mathrm{d}\,L'(\tau)}{\mathrm{d}\,\tau} = \frac{\mathrm{d}\,L'(\tau)}{\mathrm{d}\,t} \cdot \left(\frac{\mathrm{d}\,\tau}{\mathrm{d}\,t}\right)^{-1} = \\ &= \frac{\mathrm{d}}{\mathrm{d}\,t}\{L(t)\sqrt{1 - v(t)^2/c^2}\} \cdot \frac{1}{\sqrt{1 - v(t)^2/c^2}} = \\ &= -\frac{v(t)}{1 - v(t)^2/c^2}\left\{1 - v(t)^2/c^2 + [L_0 - s(t)] \cdot \frac{1}{c^2} \cdot \frac{\mathrm{d}\,v(t)}{\mathrm{d}\,t}\right\} \end{aligned}$$

$$
\begin{aligned}
|v_*| &= \frac{v(t)}{1-v(t)^2/c^2}\left\{1-v(t)^2/c^2+[L_0-s(t)]\cdot\frac{1}{c^2}\cdot\frac{\mathrm{d}\,v(t)}{\mathrm{d}\,t}\right\} \\
&= v(t)+\frac{1}{c^2}[L_0-s(t)]\cdot\frac{v(t)}{1-v(t)^2/c^2}\frac{\mathrm{d}\,v(t)}{\mathrm{d}\,t} = \\
&= v(t)-\frac{1}{2}[L_0-s(t)]\cdot\frac{\mathrm{d}}{\mathrm{d}\,t}\ln(1-v(t)^2/c^2)
\end{aligned}
$$

3 Constant acceleration

We will now apply this to the case of *constant acceleration* $a = kc/T > 0$ as measured from **S**. The speed of the spaceship in **S** is $v(t) = a \cdot t$, $0 \le t \le T$, $v(T) = kc,\ 0 < k < 1$.

$$
\begin{aligned}
v(t) &= at = \frac{kc}{T}t\,,\ 0<k<1\,,\ 0\le t\le T \\
s(t) &= \frac{1}{2}at^2 = \frac{kc}{2T}t^2 \\
s(T) &= L_0 = \frac{kc}{2T}T^2 = \frac{1}{2}kcT
\end{aligned}
$$

$$
\begin{aligned}
|v_*(\tau)| &= \frac{at}{1-(at)^2/c^2}\left\{(1-(at)^2/c^2)+\frac{1}{c^2}\cdot(L_0-\frac{1}{2}at^2)\cdot a\right\} && (3) \\
& && (4) \\
&= c\cdot\frac{at/c}{1-(at/c)^2}\left\{1+\frac{1}{2}k^2-\frac{3}{2}(at/c)^2\right\} && (5)
\end{aligned}
$$

By the time dilatation, equation (2), t is implicitly defined as a function of τ.
Thus

$$|v_*(\tau)| = c\cdot f(at/c) = c\cdot f(kt/T)$$

where

$$f(x) = \frac{x}{1-x^2}\cdot\left(1+\frac{1}{2}k^2-\frac{3}{2}x^2\right)\ .$$

Since $0 \le at \le aT = kc$, then $0 \le x < 1$.

Calculation of the maximal speed:

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left\{\frac{x}{1-x^2}\cdot\left(1+\frac{1}{2}k^2-\frac{3}{2}x^2\right)\right\}=0\ ,$$

$$x=\frac{1}{\sqrt{6}}\cdot\sqrt{-\sqrt{k^4-26k^2+25}-k^2+7}\ .$$

Maximal speed of the star:

$$|v_*(\tau)|_{max}=\frac{\sqrt{6}}{4}\cdot\frac{(\sqrt{k^4-26k^2+25}+3(k^2-1))\sqrt{-\sqrt{k^4-26k^2+25}-k^2+7}}{\sqrt{k^4-26k^2+25}+k^2-1}$$

Below is shown $|v_*(\tau)|$ as a function of time in the Earth's system for $k=0.95$. At $t=T$ the spaceship arrives at the star with speed kc and then $|v_*|$ has the same value as it should since the spaceship and the star are then at the same location. When $k=0.95$, $|v_*|_{max}\approx 1.13$ så the speed exceeds the speed of ligt by around 13%. Also $\lim_{k\to 1^-}=3/2$ så the upper bound of $|v_*|$ is 50% above the speed of light.

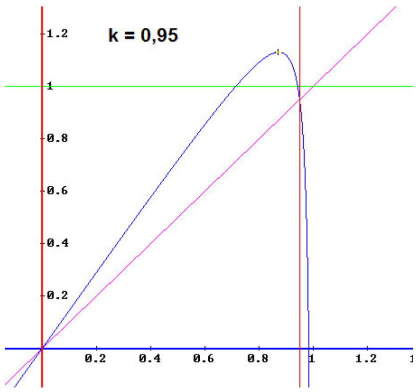

Figure 2: Speed of the star in unit c as measured from the spaceship as a function of time/speed in the outside system; at/c on the horizontal axis.

Relation between τ and t:

$$\tau=\frac{c}{2a}\cdot g(at/c)$$

where

$$g(x) = \arcsin x + x\sqrt{1-x^2}\ .$$

$$at = g^{-1}\left(\frac{2a\tau}{c}\right) \tag{6}$$

$$\left\{\begin{array}{rcl} |v_*(\tau)| & = & c\cdot f(\xi) \\ \tau & = & \dfrac{c}{2a}g(\xi) \end{array}\right.$$

and

$$\left\{\begin{array}{rcl} L'(\tau) & = & L_0 - \dfrac{1}{2a}\xi^2\sqrt{1-\xi^2} \\ \tau & = & \dfrac{1}{2a}g(\xi) \end{array}\right.$$

4 Non-constant acceleration I

$$\begin{array}{rcl} v(t) & = & \dfrac{2kct^2}{t^2+T^2},\ 0<k<1\,,\ 0\le t\le T \\ v(T) & = & kc \\ s(t) & = & 2kcT\cdot\left(\dfrac{t}{T} - \arctan\dfrac{t}{T}\right) \\ s(T) & = & L_0 = 2kcT\cdot\left(1-\dfrac{\pi}{4}\right) \end{array}$$

$$|v_*(\tau)| = c\cdot\frac{2x^2}{1+x^2}\cdot\left\{1 + 2k^2x\cdot\frac{4-\pi-4x+4\arctan x}{(1+x^2)^2-4k^2x^4}\right\} \tag{7}$$

where $x = t/T$, $0 \le x \le 1$.

Ex: $x = \tan\dfrac{2\pi}{9} \approx 0.84$, $k = 0.95$

$$\begin{array}{rcl} |v_*| & = & c\cdot\dfrac{2\cdot 0.95\cdot\tan^2(2\pi/9)}{1+\tan^2(2\pi/9)} \\ & \cdot & \left\{1+2\cdot 0.95^2\tan(2\pi/9)\cdot\dfrac{4-\pi-4\tan(2\pi/9)+8\pi/9}{(1+\tan^2(2\pi/9))^2-4\cdot 0.95^2\tan^4(2\pi/9)}\right\} \\ & \approx & 1.09c\ . \end{array}$$

Största $|v_*(\tau)| \approx 1.14c$ för $t \approx 0.91T$, motsvarande $\tau \approx 0.80T$.

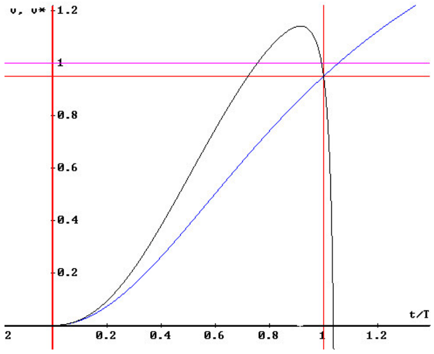

Figure 3: Speed of the star in unit c as measured from the spaceship as a function of time in the outside system; t/T on the horizontal axis.

5 Non-constant acceleration II

$$
\begin{aligned}
v(t) &= ce^{-\lambda T/t},\ \lambda > 0,\ 0 \leq t \leq T \\
v(T) &= ce^{-\lambda} = kc,\ 0 < k < 1, \quad e^{-\lambda} = k,\ \lambda = -\ln k \\
s(t) &= c \cdot \int_0^{t>0} e^{-\lambda T/\xi}\,\mathrm{d}\xi = \\
s(T) &= L_0 = c \cdot \int_0^T e^{-\lambda T/\xi} \\
a &= c \cdot \lambda T \frac{e^{-\lambda T/t}}{t^2}
\end{aligned}
$$

$$
|v_*(\tau)| = c \cdot e^{-\lambda T/t} \cdot \left\{1 + \frac{1}{c} \cdot \frac{\lambda T}{t^2} \cdot \frac{1}{\sinh(\lambda T/t)} \cdot \int_t^T e^{-\lambda T/\xi}\,\mathrm{d}\xi\right\}
$$

$$
|v_*(\tau)| = c \cdot e^{-\lambda/x} \cdot \left\{1 + \frac{\lambda}{2} \cdot \frac{1}{x^2 \sinh(\lambda/x)} \cdot \int_x^1 e^{-\lambda/\xi}\,\mathrm{d}\xi\right\}
$$

where $x = t/T$, $0 \leq x \leq 1$.

Ex:

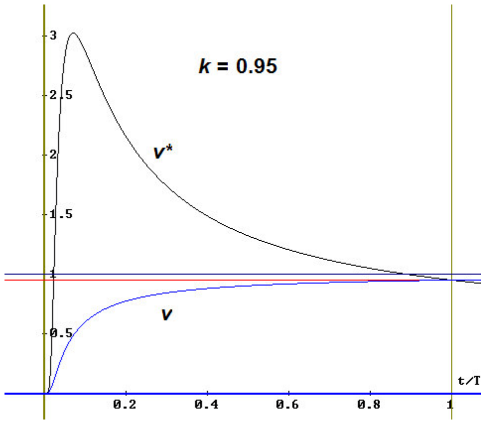

Figure 4: Speed v^* of the star in unit c as measured from the spaceship as a function of time/speed in the outside system; t/T on the horizontal axis.

6 Non-constant acceleration III

$$\begin{aligned} v(t) &= c\sin(\lambda t/T),\ \lambda > 0,\ 0 \le t \le T \\ v(T) &= = c\sin(\lambda) = kc,\ 0 < k < 1; \quad \lambda = \arcsin k \\ s(t) &= \frac{cT}{\lambda}(1-\cos(\lambda t/T)) \\ s(T) &= L_0 = \frac{cT}{\lambda}(1-\sqrt{1-k^2}) \end{aligned}$$

$$\begin{aligned} |v_*(\tau)| &= c\cdot\left\{2\sin(\lambda t/T) - \sqrt{1-k^2}\tan(\lambda t/T)\right\} && (8) \\ & && (9) \\ &= c\cdot\left\{2\sin(\lambda x) - \sqrt{1-k^2}\tan(\lambda x)\right\} && (10) \end{aligned}$$

where $x = t/T$, $0 \le x \le 1$.

Ex:

$$k = 0.95$$

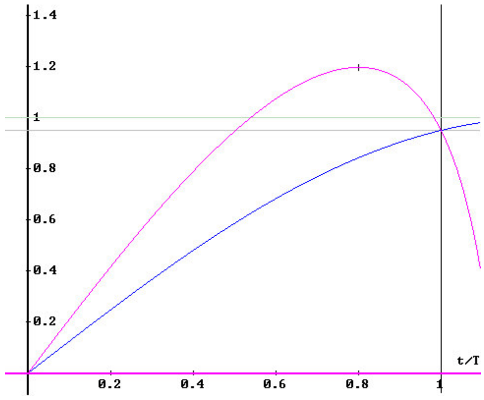

Figure 5: Speed of the star in unit c as measured from the spaceship as a function of time/speed in the outside system; t/T on the horizontal axis.

$\frac{\mathrm{d}}{\mathrm{d}x}|v_*(\tau)| = 0$ gives

$$x = \left(2 \arcsin \frac{24375^{1/6}}{10} - \pi\right) \cdot \left(4 \arctan \frac{1}{\sqrt{39}} - \pi\right)^{-1} \approx 0.800$$

and the maximal value of $|v_*(\tau)|$ is

$$\sqrt{4 - \frac{195^(1/3)}{5} - \frac{1}{20}\left(2374281 - 365040 \cdot 195^(1/3) + 18252 \cdot 195^(2/3)\right)} \approx 1.20$$

that is 20% above c.

7 Non-constant acceleration IV – hyperbolic motion

In this case the world-line of the spaceship is $(s+\lambda cT)^2 - c^2t^2 = (\lambda cT)^2$, λ, $T > 0$, which for $0 \leq t \leq T$ is a part of a hyperbola.

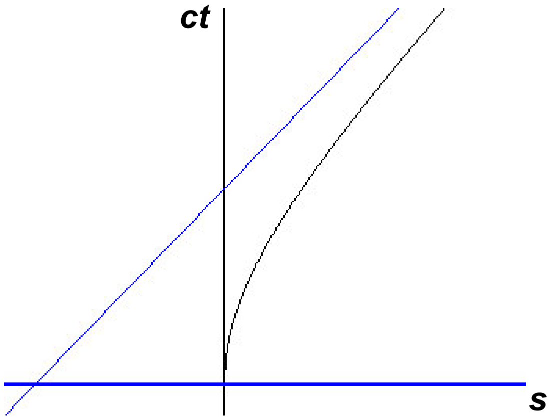

Figure 6: Worldline for hyperbolic motion with an asymptote

$$s(t) = c\sqrt{\lambda^2 T^2 + t^2} - c\lambda T$$

$$v(t) = c \cdot \left(1 + \frac{\lambda^2 T^2}{t^2}\right)^{-1/2}$$

$v(T) = kc$ where $0 < k < 1$ gives

$$kc = c \cdot \left(1 + \lambda^2\right)^{-1/2}, \text{ so } k = 1/\sqrt{1+\lambda^2} \text{ or } \lambda = \sqrt{1-k^2}/k\ .$$

Further,

$$L_0 = s(T) = c\sqrt{\lambda^2 T^2 + T^2} - c\lambda T = cT(\sqrt{1+\lambda^2} - \lambda)$$

The acceleration is

$$\dot{v}(t) = c \cdot \frac{\lambda^2 T^2}{(\sqrt{\lambda T^2 + t^2})^3}$$

and the speed is

$$
\begin{aligned}
|v_*| &= c \cdot \left(1 + \frac{\lambda T^2}{t^2}\right)^{-1/2} - \frac{1}{2}(L_0 - s(t)) \cdot \frac{\mathrm{d}}{\mathrm{d}\,t} \ln\left(1 - \left(1 + \frac{\lambda^2 T^2}{t^2}\right)^{-1}\right) \\
&= c \cdot \left(1 + \frac{\lambda T^2}{t^2}\right)^{-1/2} - \frac{1}{2}(L_0 - s(t)) \cdot \frac{\mathrm{d}}{\mathrm{d}\,t} \ln \frac{\lambda^2 T^2}{\lambda^2 T^2 + t^2} \\
&= c \cdot \frac{t}{\sqrt{\lambda^2 T^2 + t^2}} + \frac{1}{2}\left(cT\sqrt{1+\lambda^2} - cT\sqrt{\lambda^2 + t^2/T^2}\right) \cdot \frac{\mathrm{d}}{\mathrm{d}\,t} \ln(\lambda^2 + t^2/T^2) \\
&= \frac{1}{2}\left(cT\sqrt{1+\lambda^2}\right) \cdot \frac{2t}{\lambda^2 T^2 + t^2} = c \cdot \frac{T\sqrt{1+\lambda^2} \cdot t}{\lambda^2 T^2 + t^2}
\end{aligned}
$$

Now, let $t = xT$ where $0 \leq x \leq 1$. Then $v(t) = c \cdot \left(1 + \frac{\lambda^2}{x^2}\right)^{-1/2} = c \cdot x/\sqrt{x^2 + \lambda^2}$ and

$$
|v_*| = c \cdot \frac{x\sqrt{1+\lambda^2}}{x^2 + \lambda^2} = c \cdot \frac{kx}{1 - k^2(1 - x^2)} .
$$

$k = 0.9$, $\lambda = \sqrt{19}/9$, maximal $|v_*| = 5/\sqrt{19} \approx 1,15c$ for $x = \lambda$.

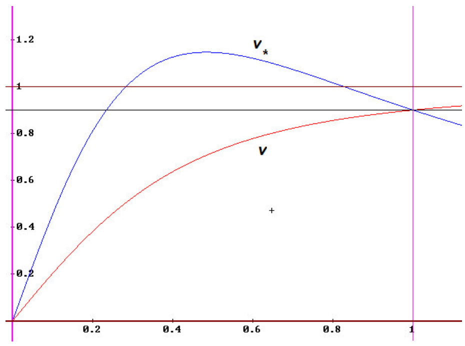

Figure 7: Speed of the star in unit c as measured from the spaceship as a function of time/speed in the outside system; t/T on the horizontal axis.

In hyperbolic motion the *acceleration* as measured in the accelerated system i *constant*. This follows from the Lorentz transformation of acceleration, $a' = a \cdot \gamma(v)^2$, in combination with $a = c^2/(c\lambda T) \cdot \gamma(v)^{-3}$.

8 A voyage to Arcturus

For a numerical example example let's consider a voyage to the giant star Arcturus[1], 36.66 light years from Earth.

[1]Idea from the science fiction novel *A Voyage to Arcturus* by David Lindsay, [1].

We assume constant acceleration $g \approx 9.81\,\mathrm{m/s^2}$ as measured from the space-ship, i e hyperbolic motion during the first half of the voyage.

Like in the previous section we set $s(t) = \sqrt{\lambda^2 c^2 T^2 + c^2 t^2} - \lambda ct$ where now T is the time for half voyage to a distance $L_0/2$. Then from $L_0/2 = s(T) = cT(\sqrt{\lambda^2+1} - \lambda)$ and $g = c/(\lambda T)$ we get

$$\begin{cases} \lambda & = & \dfrac{1}{\sqrt{gL_0/(2c^2)}\sqrt{gL_0/(2c^2)+2}} \approx 0.050243 \\[2ex] \dfrac{1}{\lambda} & = & \sqrt{gL_0/(2c^2)}\sqrt{gL_0/(2c^2)+2} \approx 19.903 \\[2ex] T & = & \dfrac{c}{g}\sqrt{gL_0/(2c^2)}\sqrt{gL_0/(2c^2)+2} \approx 19.27\,\text{years} \end{cases}$$

Then, from $v(t) = c/\sqrt{1+(\lambda T/t)^2} = c/\sqrt{1+(\frac{c}{gt})^2}$, we get

$$\gamma(v) = \sqrt{1+\left(\frac{gt}{c}\right)^2}$$

and the proper time for the entire voyage

$$\begin{aligned} \tau &= 2\cdot\int_0^T \frac{\mathrm{d}t}{\gamma(v(t))} = 2\cdot\int_0^T \frac{\mathrm{d}t}{\sqrt{1+\left(\frac{gt}{c}\right)^2}} \\ &= 2\cdot\frac{c}{g}\cdot\int_0^{gT/c}\frac{\mathrm{d}\xi}{\sqrt{1+\xi^2}} = 2\cdot\frac{c}{g}\cdot\ln\left\{\frac{gT}{c}+\sqrt{1+\left(\frac{gT}{c}\right)^2}\right\} \\ &= 2\lambda T\cdot\left\{\frac{1}{\lambda}+\sqrt{1+\left(\frac{1}{\lambda}\right)^2}\right\} \approx 6.19\,\text{years} \end{aligned}$$

Arcturus speed as measured from the space-ship:

$$|v_*| = c\cdot\frac{T(2\sqrt{1+\lambda^2}-\lambda)\cdot t}{\lambda^2 T^2 + t^2} = c\cdot\frac{x(2\sqrt{1+\lambda^2}-\lambda)}{x^2+\lambda^2}$$

where $\lambda = t/T$.
Maximal speed, at $t = \lambda T \approx 0,97\,$years corresponding to $x = \lambda$:

$$|v_*|_{max} = c\cdot\frac{\lambda(2\sqrt{1+\lambda^2}-\lambda)}{\lambda^2+\lambda^2} = c\cdot\frac{2\sqrt{1+\lambda^2}-\lambda}{2\lambda} \approx 19.43c$$

9 Conclusion

As mentioned in the introduction it is easy to realize *qualitatively* that the speed of a material body as measured from an accelerated/non-inertial reference system can exceed c. This is particulaly convincing in extreme situations like interstellar travel as in the section "A voyage to Arcturus". In this article we have derived a general *quantitative* formula, and applied it to some particular cases. To keep things easy we have used the well-known formulas for length contraction and time dilatation although a direct use of the Lorentz transformation is "safer" and should be preferred in more complex situations; se appendix below.

Everything rest upon the interpretation of the concept of an *accelerated refernce system*. We have understood this to mean a *sequence of local inertial systems*. This means that the results are unrefutable, just being mathematical consequences. However there is no reason to doubt that the results correpond to what should be measured in a real situation. For example, the same interpretation is used in the usual derivations of the time dilatation in a more or less arbitrary motion, and these results agree with experiment. A more complicated interpretation, developed by Møller is aimed at understanding a *rigid* accelerated frame of reference; see [2].

We have considered the speed of material bodies as measured from an accelerated system. It may be of interest to consider the speed of light and, in connection with this, how the motion of the material body is seen from the accelerated system. Further, similar considerations may give some new insights into the erroneously named *twin paradox.*[2], at least pedagogically.

10 Appendix

An alternative derivation of v_*

We have derived the expression for v_*, the velocity of a star as measured from an accelerated spsce-ship heading straight towards the star. The concept of *accelerated refence system* was then interpreted as a sequence of momentary inertial rest system. The well-known expressions for time dilatation and Lorentz contraction were then combined. However, it is safer to use the Lorentz transformation from scratch since all effects are then automatically taken care of. A well-known example of possible mistakes when not doing this is the "paradox of a falling bar". In that case a contradiction is obtained, using length contraction but forgeting about the relativity of simultaneity. – So we will rederive v_* from the Lorentz transformation.

[2] In fact, it is just a mistake!

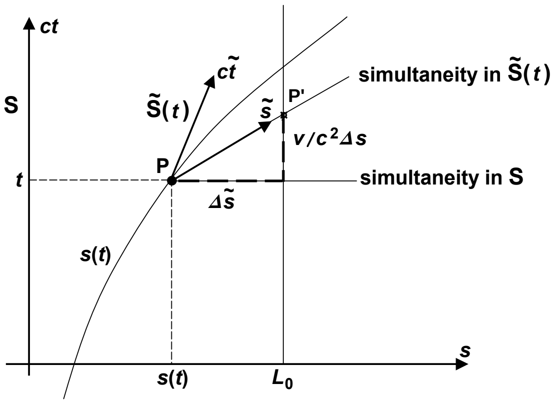

Figure 8: $0 = \Delta\tau = \gamma(\Delta t - v/c^2 \Delta s)$. $\Delta t = v/c^2 \Delta s$ in $\mathbf{S}$

$$\begin{cases} t_P &= t,\ s_P = s(t) \\ t_{P'} &= t + \Delta t = t + \dfrac{v}{c^2}\Delta s \\ \Delta s &= L_0 - s(t) \\ L_0 &= s_{P'} = s(t) + \Delta s \end{cases}$$

$$\begin{pmatrix} \Delta s \\ \Delta t \end{pmatrix} = \gamma \begin{pmatrix} 1 & v \\ v/c^2 & 1 \end{pmatrix} \begin{pmatrix} \Delta\tilde{s} \\ 0 \end{pmatrix}$$

$$\begin{cases} \Delta s &= \gamma(v)\Delta\tilde{s} \\ \Delta t &= \gamma(v)\dfrac{v}{c^2}\Delta\tilde{s} \end{cases}$$

$$\Delta\tilde{s} = \frac{1}{\gamma(v)}\Delta s = [L_0 - s(t)]\sqrt{1 - v^2/c^2}$$

$$\begin{aligned} v_* &= \frac{\mathrm{d}}{\mathrm{d}t}\left\{[L_0 - s(t)]\sqrt{1 - \dot{s}^2/c^2}\right\} \cdot \frac{1}{\sqrt{1 - \dot{s}^2/c^2}} \\ &= \cdots = -\dot{s}\left\{1 + \frac{1}{c^2} \cdot \frac{\ddot{s} \cdot (L_0 - s)}{1 - \dot{s}^2/c^2}\right\} \end{aligned}$$

The factor $\dfrac{1}{\sqrt{1 - \dot{s}^2/c^2}}$ is the time dilatation, and follows from the Lorentz transformation $\mathrm{d}t = \gamma(\dot{s})(\mathrm{d}\tau - \dot{s}/c^2 \cdot 0$.
So, we get the same result as before!

References

[1] D. Lindsay, *A Voyage to Arcturus* Wilder Publications, 2009

[2] C. Møller, *The Theory of Relativity* Oxford University Press, 1972

[3] I. Newton, *Principia* Prometheus Books, 1995

[4] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).