The Concept of Neutrosophic Less Than or Equal To: A New Insight in Unconstrained Geometric Programming

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Abstract

In this paper, we introduce the concept of *neutrosophic less than* or *equal to*. The neutrosophy considers every idea $< A >$ together with its opposite or negation $< \text{antiA} >$ and with their spectrum of neutralities $< \text{neutA} >$ in between them (i.e. notions or ideas supporting neither $< A >$ nor $< \text{antiA} >$). The $< \text{neutA} >$ and $< \text{antiA} >$ ideas together are referred to as $< \text{nonA} >$. Neutrosophic Set and Neutrosophic Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic) [5]. In neutrosophic logic, a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^{-}0, 1^{+}[$. Another purpose of this article is to explain the mathematical theory of *neutrosophic geometric programming* (the unconstrained posynomial case). It is necessary to work in fuzzy neutrosophic space $\text{FN}_s = [0,1] \cup [0, nI], n \in [0,1]$. The theory stated in this article aims to be a complementary theory of *neutrosophic geometric programming*.

Keywords

Neutrosophic Less Than or Equal To, Geometric Programming (GP), Signomial Geometric Programming (SGP), Fuzzy Geometric Programming (FGP), Neutrosophic Geometric Programming (NGP), Neutrosophic Function in Geometric Programming.

1 Introduction

The classical Geometric Programming (GP) is an optimization technique developed for solving a class of non-linear optimization problems in engineering design. GP technique has its origins in Zener's work (1961). Zener tried a new approach to solve a class of unconstrained non-linear optimization problems, where the terms of the objective function were posynomials. To solve these problems, he used the well-known arithmetic-geometric mean inequality (i.e. the arithmetic mean is greater than or equal to the geometric mean). Because of this, the approach came to be known as GP technique. Zener used this technique to solve only problems where the number of posynomial terms of the objective function was one more than the number of variables, and the function was not subject to any constraints. Later on (1962), Duffin extended the use of this technique to solve problems where the number of posynomial terms in the objective function is arbitrary. Peterson (1967), together with Zener and Duffin, extended the use of this technique to solve problems which also include the inequality constraints in the form of posynomials. As well, Passy and Wilde (1967) extended this technique further to solve problems in which some of the posynomial terms have negative coefficients. Duffin (1970) condensed the posynomial functions to a monomial form (by a logarithmic transformation, it became linear), and particularly showed that a "duality gap" function could not occur in geometric programming. Further, Duffin and Peterson (1972) pointed out that each of those posynomial programs GP can be reformulated so that every constraint function becomes posy-/bi-nomial, including at most two posynomial terms, where posynomial programming - with posy-/mo-nomial objective and constraint functions - is synonymous with linear programming.

As geometric programming became a widely used optimization technique, it was desirable that an efficient and highly flexible method of solutions were available. As the complexity of prototype geometric programs to be solved increased, several considerations became important. Canonically, the degree of problem difficulty and the inactive constraints reported an algorithm capable of dealing with these considerations. Consequently, McNamara (1976) proposed a solution procedure for geometric programming involving the formulation of an augmented problem that possessed zero degree of difficulty.

Accordingly, several algorithms have been proposed for solving GP (1980's). Such algorithms are somewhat more effective and reliable when they are applied to a convex problem, and also avoid difficulties with derivative singularities, as variables raised to fractional powers approach zero, since logs of such variables will approach $-\infty$, and large negative lower bounds should be placed on those variables.

In the 1990's, a strong interest in interior point (IP) algorithms has spawned several (IP) algorithms for GP. Rajgopal and Bricker (2002) produced an efficient procedure for solving posynomial geometric programming. The procedure, which used the concept of condensation, was embedded within an algorithm for a more general (signomial) GP problem. The constraint structure of the reformulation provides insight into why this algorithm is successful in avoiding all of the computational problems, traditionally associated with dual-based algorithms.

Li and Tsai (2005) proposed a technique for treating (positive, zero or negative) variables in SGP. Most existing methods of global optimization for SGP actually compute an approximate optimal solution of a linear or convex relaxation of the original problem. However, these approaches may sometimes provide an infeasible solution, or might form the true optimum to overcome these limitations.

A robust solution algorithm is proposed for global algorithm optimization of SGP by Shen, Ma and Chen (2008). This algorithm guarantees adequately to obtain a robust optimal solution which is feasible and close to the actual optimal solution, and is also stable under small perturbations of the constraints [6].

In the past 20 years, FGP has developed extensively. In 2002, B. Y. Cao published the first monography of fuzzy geometric programming as applied optimization. A large number of FGP applications have been discovered in a wide variety of scientific and non-scientific fields, since FGP is superior to classical GP in dealing with issues in fields like power system, environmental engineering, postal services, economical analysis, transportation, inventory theory; and so more to be discovered.

Arguably, fuzzy geometric programming potentially becomes a ubiquitous optimization technology, the same as fuzzy linear programming, fuzzy objective programming, and fuzzy quadratic programming [2].

This work is the first attempt to formulate the neutrosophic posynomial geometric programming (the simplest case, i.e. the unconstrained case). A previous work investigated the maximum and the minimum solutions to the neutrosophic relational GP [7,8].

2 Neutrosophic Less than or Equal To

In order to understand the concept of neutrosophic less than or equal to in optimization, we begin with some preliminaries which serve the subject.

Definition 2.1

Let X be the set of all fuzzy neutrosophic variable vectors x_i , $i = 1,2, \dots , m$, i.e. $X = \{(x_1, x_2, \dots , x_m)^{\mathrm{T}} \mid x_i \in \mathrm{FN_s}\}$. The function $\mathrm{g}(x)\colon X \to \mathrm{R} \cup \mathrm{I}$ is said to be the neutrosophic GP function of x, where $\mathrm{g}(x) = \sum_{\mathrm{k}=1}^{\mathrm{J}} \mathrm{c_k} \prod_{\mathrm{l}=1}^{\mathrm{m}} \mathrm{x_l}^{\gamma_{\mathrm{kl}}}$, $\mathrm{c_k} \geq 0$ are constants, γ_{kl} - are arbitrary real numbers.

Definition 2.2

Let $\mathrm{g}(x)$ be any linear or non-linear neutrosophic function, and let $\mathrm{A_0}$ be the neutrosophic set for all functions $\mathrm{g}(x)$ that are neutrosophically less than or equal to 1.

$$\begin{aligned}\mathrm{A_0} &= \{\, \mathrm{g}(x) < \text{₦} 1, x_i \in \mathrm{FN_s}\} \\ &= \{\, \mathrm{g}(x) < 1, \quad \mathrm{anti}(\, \mathrm{g}(x)) > 1, \quad \mathrm{neut}(\, \mathrm{g}(x)) = 1, x_i \in \mathrm{FN_s}\}.\end{aligned}$$

Definition 2.3

Let $\mathrm{g}(x)$ be any linear or non-linear neutrosophic function, where $x_i \in [0,1] \cup [0, \mathrm{nI}]$ and $x = (x_1, x_2, \dots , x_m)^{\mathrm{T}}$ a m-dimensional fuzzy neutrosophic variable vector.

We have the inequality

$$\mathrm{g}(x) < \text{₦}\ 1 \tag{1}$$

where " $<$ ₦" denotes the neutrosophied version for " $\leq$ " with the linguistic interpretation being "less than (the original claimed), greater than (the anti-claim of the original less than), equal (neither the original claim, nor the anti-claim)".

The inequality (1) can be redefined as follows:

$$\left.\begin{array}{r}\mathrm{g(x)} < 1 \\ \mathrm{anti\ (g(x))} > 1 \\ \mathrm{neut(\ g(x))} = 1\end{array}\right\} \tag{2}$$

Definition 2.4

Let $\mathrm{A_0}$ be the set of all neutrosophic non-linear functions that are neutrosophically less than or equal to 1.

$$\begin{aligned}\mathrm{A_0} &= \{\, \mathrm{g}(x) < \text{₦}\ 1, x_i \in \mathrm{FN_s}\} \\ &= \{\, \mathrm{g}(x) < 1, \quad \mathrm{anti}(\, \mathrm{g}(x)) > 1, \quad \mathrm{neut}(\, \mathrm{g}(x)) = 1, x_i \in \mathrm{FN_s}\}.\end{aligned}$$

It is significant to define the following membership functions:

$$\mu_{\mathrm{A_o}}(\, \mathrm{g(x)}) = \begin{cases} 1 & 0 \leq \mathrm{g(x)} \leq 1 \\ \left(\mathrm{e}^{\frac{-1}{\mathrm{d_o}}(\mathrm{g(x)}-1)} + \mathrm{e}^{\frac{-1}{\mathrm{d_o}}(\mathrm{anti(\ g(x))}-1)} - 1\right), & 1 < \mathrm{g(x)} \leq 1 - \mathrm{d_o} \ln 0.5 \end{cases} \tag{3}$$

$$\mu_{A_o}(\text{anti}(\, g(x))) = \begin{cases} 0 & 0 \le g(x) \le 1 \\ \left(1 - e^{\frac{-1}{d_o}(\text{anti}(\, g(x))-1)} - e^{\frac{-1}{d_o}(g(x)-1)}\right), & 1 - d_o \ln 0.5 \le g(x) \le 1 + d_o \end{cases} \quad (4)$$

It is clear that $\mu_{A_o}(\text{neut}(\, g(x)))$ consists of intersection the following functions:

$$e^{\frac{-1}{d_o}(g(x)-1)}, \quad 1 - e^{\frac{-1}{d_o}(\text{anti}(\, g(x))-1)}$$

i.e.

$$\mu_{A_o}(\text{neut}(\, g(x))) = \begin{cases} 1 - e^{\frac{-1}{d_o}(\text{anti}(\, g(x))-1)} & 1 \le g(x) \le 1 - d_o \ln 0.5 \\ e^{\frac{-1}{d_o}(g(x)-1)} & 1 - d_o \ln 0.5 < g(x) \le 1 + d_o \end{cases} \quad (5)$$

Note that $d_o > 0$ is a constant expressing a limit of the admissible violation of the neutrosophic non-linear function $g(x)$ [3].

2.1 The relationship between $g(x)$, anti $g(x)$ in NGP

1. At

$$1 < g(x) \le 1 - d_o \ln 0.5$$

$$\mu_{A_o}(\, g(x)\,) > \mu_{A_o}(\text{anti}(\, g(x)\,) \qquad \text{(see } \textit{Figure 1}\text{)}$$

$$e^{\frac{-1}{d_o}(g(x)-1)} > 1 - e^{\frac{-1}{d_o}(\text{anti}(\, g(x)\,)-1)}$$

$$e^{\frac{-1}{d_o}(\text{anti}(\, g(x)\,)-1)} > 1 - e^{\frac{-1}{d_o}(g(x)-1)}$$

$$\frac{-1}{d_o}(\text{anti}(\, g(x)\,) - 1) > \ln(1 - e^{\frac{-1}{d_o}(g(x)-1)})$$

$$\text{anti}(\, g(x)\,) < 1 - d_o \ \ln(1 - e^{\frac{-1}{d_o}(g(x)-1)})$$

2. Again at

$$1 - d_o \ln 0.5 < g(x) \le 1 + d_o$$

$$\mu_{A_o}\big(\, g(x)\big) < \mu_{A_o}(\text{anti}(\, g(x)))$$

$$\therefore \ \text{anti}(\, g(x)\,) > 1 - d_o \ \ln(1 - e^{\frac{-1}{d_o}(g(x)-1)})$$

3 Neutrosophic Geometric Programming (the unconstrained case)

Geometric programming is a relative method for solving a class of non-linear programming problems. It was developed by Duffin, Peterson, and Zener (1967) [4]. It is used to minimize functions that are in the form of posynomials, subject to constraints of the same type.

Inspired by Zadeh's fuzzy sets theory, fuzzy geometric programming emerged from the combination of fuzzy sets theory with geometric programming.

Fuzzy geometric programming was originated by B.Y. Cao in the Proceedings of the second IFSA conferences (Tokyo, 1987) [1].

In this work, the neutrosophic geometric programming (the unconstrained case) was established where the models were built in the form of posynomials.

Definition 3.1

Let

$$\left.\begin{array}{l}\overset{N}{(P)} \quad \overset{N}{\min}\ g(x) \\ \quad x_i \in FN_s\end{array}\right\}. \tag{6}$$

The neutrosophic unconstrained posynomial geometric programming, where $x = (x_1, x_2, \ldots, x_m)^{\mathrm{T}}$ is a m-dimensional fuzzy neutrosophic variable vector, "T" represents a transpose symbol, and $g(x) = \sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}$ is a neutrosophic posynomial GP function of x, $c_k \geq 0$ a constant, γ_{kl} an arbitrary real number, $g(x) < ₦\, z \to \overset{N}{\min} g(x)$; the objective function $g(x)$ can be written as a minimizing goal in order to consider z as an upper bound; z is an expectation value of the objective function $g(x)$, "$< ₦$" denotes the neutrosophied version of "$\leq$" with the linguistic interpretation (see Definition 2.3), and $d_o > 0$ denotes a flexible index of $g(x)$.

Note that the above program is undefined and has no solution in the case of $\gamma_{kl} < 0$ with some x_l's taking indeterminacy value, for example,

$$\overset{N}{\min} \qquad g(x) = 2x_1^{-.2}x_2^{.3}x_4^{1.5} + 7x_1^{3}x_2^{-.5}x_3,$$

where $x_i \in FN_s, i = 1,2,3,4$.

This program is not defined at $x = (.2I, .3, .25, I)^{\mathrm{T}}$, $g(x) = 2(.2I)^{-.2}(.3)^{.3}I^{1.5} + 7(.2I)^{3}(.3)^{-.5}(.25)$ is undefined at $x_1 = .2I$ with $\gamma_1 = -0.2$.

Definition 3.2

Let A_0 be the set of all neutrosophic non-linear functions $g(x)$ that are neutrosophically less than or equal to z, i.e.

$$A_0 = \{\ g(x) < ₦\, z, x_i \in FN_s\}.$$

The membership functions of $g(x)$ and $\operatorname{anti}(g(x))$ are:

$$\mu_{A_o}(\,g(x)) = \begin{cases} 1 & 0 \leq g(x) \leq z \\ \left(e^{\frac{-1}{d_o}(g(x)-z)} + e^{\frac{-1}{d_o}(\operatorname{anti}(g(x))-z)} - 1\right), & z < g(x) \leq z - d_o \ln 0.5 \end{cases} \tag{7}$$

$$\mu_{A_o}(\operatorname{anti}(\,g(x))) = \begin{cases} 0 & 0 \leq g(x) \leq z \\ \left(1 - e^{\frac{-1}{d_o}(\operatorname{anti}(g(x))-z)} - e^{\frac{-1}{d_o}(g(x)-z)}\right), & z - d_o \ln 0.5 \leq g(x) \leq z + d_o \end{cases} \tag{8}$$

Eq. (6) can be changed into

$$g(x) < ₦\ z, \qquad x = (x_1, x_2, \ldots, x_m), x_i \in FN_s \tag{9}$$

The above program can be redefined as follow:

$$\left.\begin{array}{l} g(x) < z \\ \text{anti}\big(g(x)\big) > z \\ \text{neut}\big(g(x)\big) = z \\ x = (x_1, x_2, \dots, x_m), x_i \in FN_s \end{array}\right\} \tag{10}$$

It is clear that $\mu_{A_o}(\text{neut}(g(x)))$ consists from the intersection of the following functions:

$$e^{\frac{-1}{d_o}(g(x)-z)} \quad \& \quad 1 - e^{\frac{-1}{d_o}(\text{anti}(g(x))-z)}$$

$$\mu_{A_o}(\text{neut}(g(x))) = \begin{cases} 1 - e^{\frac{-1}{d_o}(\text{anti}(g(x))-z)} & z \le g(x) \le z - d_o \ln 0.5 \\ e^{\frac{-1}{d_o}(g(x)-z)} & z - d_o \ln 0.5 < g(x) \le z + d_o \end{cases} \tag{11}$$

Definition 3.3

Let $\tilde{N}$ be a fuzzy neutrosophic set defined on $[0,1] \cup [0, nI], n \in [0,1]$; if there exists a fuzzy neutrosophic optimal point set A_o^* of $g(x)$ such that

$$\tilde{N}(x) = \begin{array}{c} \min\{\mu(\text{neut}\, g(x))\} \\ x = (x_1, x_2, \dots, x_m), x_i \in FN_s \end{array} \tag{12}$$

$$\tilde{N}(x) = e^{\frac{-1}{d_o}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}} - z\right)} \wedge \ 1 - e^{\frac{-1}{d_o}\left(\text{anti}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z\right)},$$

then $\max \tilde{N}(x)$ is said to be a neutrosophic geometric programming (the unconstrained case) with respect to $\tilde{N}(x)$ of $g(x)$.

Definition 3.4

Let x^* be an optimal solution to $\tilde{N}(x)$, i.e.

$$\tilde{N}(x^*) = \max \tilde{N}(x) \ , x = (x_1, x_2, \dots, x_m), x_i \in FN_s \ , \tag{13}$$

and the fuzzy neutrosophic set $\tilde{N}$ satisfying (12) is a fuzzy neutrosophic decision in (9).

Theorem 3.1

The maximum of $\tilde{N}(x)$ is equivalent to the program:

$$\left.\begin{array}{l} \max \alpha \\ g(x) < z - d_o \ln \alpha \\ \text{anti}\, g(x) > z - d_o \ln(1-\alpha) \\ x = (x_1, x_2, \dots, x_m), x_i \in FN_s \, , d_o > 0 \end{array}\right\} \tag{14}$$

Proof

It is known by definition (3.4) that x^* satisfied eq. (12), called an optimal solution to (9). Again, x^* bears the similar level for $g(x)$, $\text{anti}\big(g(x)\big)$ & $\text{neut}\big(g(x)\big)$. Particularly, x^* is a solution to neutrosophic posynomial geometric programming (6) at $\tilde{N}(x^*) = 1$. However, when $g(x) < z$ and $\text{anti}(g(x)) > z$, there exists

$$\tilde{N}(x) = e^{\frac{-1}{d_o}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}} - z\right)} \wedge \ 1 - e^{\frac{-1}{d_o}\left(\text{anti}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z\right)},$$

given $\alpha = \tilde{N}(x)$. Now, $\forall\ \alpha \in FN_s$; it is clear that

$$e^{\frac{-1}{d_o}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}} - z\right)} \geq \alpha \tag{15}$$

$$1 - e^{\frac{-1}{d_o}\left(\text{anti}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z\right)} \geq \alpha \tag{16}$$

From (15), we have

$$\frac{-1}{d_o}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}} - z\right) \geq \ln \alpha$$

$$g(x) = \left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) \leq z - d_o \ln \alpha \,. \tag{17}$$

From (16), we have

$$1 - \alpha \geq e^{\frac{-1}{d_o}\left(\text{anti}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z\right)}$$

$$\rightarrow \text{anti}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z \geq -d_o \ln(1-\alpha) \tag{18}$$

$$\text{anti}\left(g(x)\right) \geq z - d_o \ln(1-\alpha).$$

Note that, for the equality in (17) & (18), it is exactly equal to neut $g(x)$.

Therefore, the maximization of $\tilde{N}(x)$ is equivalent to (14) for arbitrary $\alpha \in FN_s$, and the theorem holds.

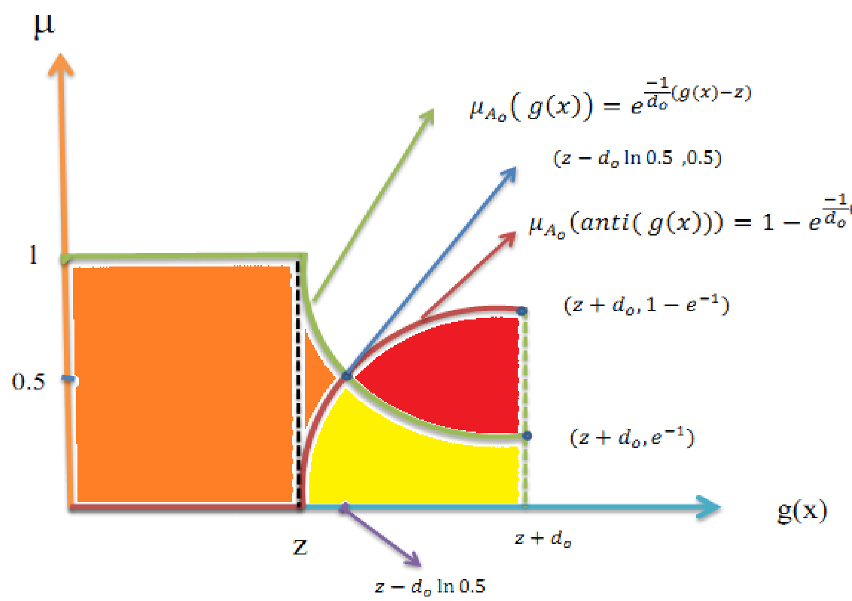

Figure 1. The orange color means the region covered by $\mu_{A_o}(g(x))$, the red color means the region covered by $\mu_{A_o}(\text{anti}(g(x)))$, and the yellow color means the region covered by $\mu_{A_o}(\text{neut}(g(x)))$.

4 Conclusion

The innovative concept and procedure explained in this article suit to the neutrosophic GP. A neutrosophic less than or equal to form can be completely turned into classical less than, greater than and equal forms. The feasible region for unconstrained neutrosophic GP can be determined by a fuzzy neutrosophic optimal point set in the fuzzy neutrosophic decision region $\tilde{N}(x^*)$.

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