Infinite arctangent sums involving Fibonacci and Lucas numbers

Kunle Adegoke†

Department of Physics and Engineering Physics, Obafemi Awolowo University, Ile-Ife, 220005 Nigeria

Saturday 23rd July, 2016, 16:43

Abstract

Using a straightforward elementary approach, we derive numerous infinite arctangent summation formulas involving Fibonacci and Lucas numbers. While most of the results obtained are new, a couple of celebrated results appear as particular cases of the more general formulas derived here.

Contents

1 Introduction 2

2 Preliminary result 4

3 Main Results 5

3.1 $G \equiv F$ in identity (2.3), that is, $G_0 = 0$, $G_1 = 1$ . . . . . . . . . . 5

3.2 $G \equiv L$ in identity (2.3), that is, $G_0 = 2$, $G_1 = 1$ . . . . . . . . . . 6

MSC 2010: 11B39, 11Y60

†adegoke00@gmail.com

Keywords: Fibonacci numbers, Lucas numbers, Lehmer formula, arctangent sums, Infinite sums
4 Corollaries and special values

4.1 Results from Theorem 3.1

4.1.1 $\lambda = F_j, p = 1$ and $k = 0$ in identity (3.1)

4.1.2 $\lambda = L_j, p = 1$ and $k = 0$ in identity (3.1)

4.1.3 $\lambda = F_{2j}, k = j$ and $p = 0$ in identity (3.1)

4.1.4 $\lambda = F_{2j}$ and $p = 1$ in identity (3.1)

4.1.5 $5\lambda^2 = L_{4j}, p = 0$ and $k = j$ in identity (3.1)

4.1.6 $5\lambda^2 = L_{4j}, p = 0$ and $k = 2j$ in identity (3.1)

4.1.7 $\lambda = L_{2j}/\sqrt{5}$ and $k = j$ in identity (3.1)

4.1.8 $\lambda = L_{2j}/\sqrt{5}, p = 0$ and $k = 2j \neq 0$ in identity (3.1)

4.2 Results from Theorem 3.2

4.2.1 $\lambda = F_{2j-1}$ and $p = 1$ in identity (3.2)

4.2.2 $\lambda = L_{2j-1}/\sqrt{5}$ and $k = j$ in identity (3.2)

4.2.3 $5\lambda^2 = L_{4j-2}$ and $k = j$ in identity (3.2)

4.3 Results from Theorem 3.3

4.3.1 $\lambda = \sqrt{L_{4j}}, k = 0$ and $p = 1$ in identity (3.3)

4.3.2 $\lambda = L_{2j}$, and $p = 1$ in identity (3.3)

4.3.3 $\lambda = \sqrt{5}F_{2j}, p = 1$ and $k = 0$ in identity (3.3)

4.4 Results from Theorem 3.4

4.4.1 $\lambda = \sqrt{L_{4j-2}}$ and $j = 0 = k$ in identity (3.4)

4.4.2 $\lambda = L_{2j-1}$ and $p = 1$ in identity (3.4)

4.4.3 $\lambda = L_{2j-1}$ and $j = 0 = k$ in identity (3.4)

4.4.4 $\lambda = \sqrt{5}F_{2j-1}$ and $j = 0 = k$ in identity (3.4)

5 Conclusion

1 Introduction

It is our goal, in this work, to derive infinite arctangent summation formulas involving Fibonacci and Lucas numbers. The results obtained will be found to be of a more general nature than one finds in earlier literature.

Previously known results containing arctangent identities and/or infinite summation involving Fibonacci numbers can be found in references [1, 2, 3, 4, 5] and references therein.
In deriving the results in this paper, the main identities employed are the trigonometric addition formula

\[
\tan^{-1} \left( \frac{\lambda (y - x)}{xy + \lambda^2} \right) = \tan^{-1} \frac{\lambda}{x} - \tan^{-1} \frac{\lambda}{y},
\]

which holds for \( \lambda \in \mathbb{R} \) such that either \( xy > 0 \) or \( xy < 0 \) and \( \lambda^2 < -xy \), and the following identities which resolve products of Fibonacci and Lucas numbers

1. \[
F_{u-v}F_{u+v} = F_u^2 - (-1)^{(u-v)}F_v^2, \tag{1.2a}
\]
2. \[
L_{u-v}L_{u+v} = L_{2u} + (-1)^{(u-v)}L_{2v}, \tag{1.2b}
\]
3. \[
L_uF_v = F_{v+u} + (-1)^uF_{v-u}, \tag{1.2c}
\]
4. \[
F_uL_v = F_{v+u} - (-1)^uF_{v-u}, \tag{1.2d}
\]
5. \[
L_uL_v = L_{u+v} + (-1)^uL_{v-u}, \tag{1.2e}
\]
6. \[
5F_{u-v}F_{u+v} = L_{2u} - (-1)^{(u-v)}L_{2v}. \tag{1.2f}
\]

Also we shall make repeated use of the following identities connecting Fibonacci and Lucas numbers:

1. \[
F_{2u} = F_uL_u, \tag{1.3a}
\]
2. \[
L_{2u} - 2(-1)^u = 5F_u^2, \tag{1.3b}
\]
3. \[
5F_u^2 - L_u^2 = 4(-1)^{(u+1)}, \tag{1.3c}
\]
4. \[
L_{2u} + 2(-1)^u = L_u^2. \tag{1.3d}
\]

Identities (1.2) and (1.3) or their variations can be found in [6, 7, 8].

On notation, \( G_i, i \) integers, denotes generalized Fibonacci numbers defined through the second order recurrence relation \( G_i = G_{i-1} + G_{i-2} \), where two boundary terms, usually \( G_0 \) and \( G_1 \), need to be specified. When \( G_0 = 0 \) and \( G_1 = 1 \), we have the Fibonacci numbers, denoted \( F_i \), while when \( G_0 = 2 \) and \( G_1 = 1 \), we have the Lucas numbers, denoted \( L_i \).

Throughout this paper, the principal value of the arctangent function is assumed.
Interesting results obtained in this paper, for integers $k, j \neq 0$ and $p$ include

$$
\sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{F_r^2 L_{2r} L_{4r}}{F_{2r}^2 - F_{2j}^2 + F_{2r}^2} \right\} = \tan^{-1}\left( \frac{1}{L_j} \right), \quad \sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{L_j^2 F_r^2 L_{4r}}{F_{4r}^2 - F_{2j}^2 + L_j^2} \right\} = \tan^{-1}\left( \frac{1}{F_j} \right),
$$

$$
\sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{F_r^2 L_{4r+2j}}{F_{4r+2j}^2 - L_{2j}^2} \right\} = \tan^{-1}\left( \frac{1}{L_{2j}} \right), \quad \sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{F_{2(2j-1)}^2}{F_{4(2j-2)+2j}^2 - L_{2j-1}^2} \right\} = \tan^{-1}\left( \frac{1}{F_{2j-1}} \right),
$$

$$
\sum_{r=p}^{\infty} \tan^{-1}\left\{ \frac{1}{L_{2r}} \right\} = \tan^{-1}\left( \frac{1}{L_{2p-1}} \right), \quad \sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{F_{4j}}{F_{4r-1}} \right\} = \tan^{-1}\left( \frac{L_{2j}}{L_{2j-1}} \right).
$$

We also obtained the following special values

$$
\sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{L_{4r-2}}{F_{4r-2}} \right\} = \frac{\pi}{2}, \quad \sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{L_{4r}}{F_{4r}} \right\} = \frac{\pi}{4}, \quad \sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{\sqrt{35} L_{4r-2}}{L_{2(4r-2)}} \right\} = \frac{\pi}{2},
$$

$$
\sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{\sqrt{3} L_{2r}}{L_{4r}} \right\} = \frac{\pi}{3}, \quad \sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{1}{5} \frac{L_{2r}}{F_{2r}} \right\} = \frac{\pi}{4}, \quad \sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{\sqrt{5}}{L_{2r}} \right\} = \tan^{-1}\sqrt{5},
$$

$$
\sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{\sqrt{35} L_{4r}}{L_{8r}} \right\} = \sqrt{7}, \quad \sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{\sqrt{5} F_{2r-1}}{L_{2r-1}^2} \right\} = \frac{\pi}{2},
$$

$$
\sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{5\sqrt{7} F_{4r-1}}{L_{2(4r-1)}} \right\} = \tan^{-1}\sqrt{7}, \quad \sum_{r=1}^{\infty} \tan^{-1}\left\{ \frac{L_{4r+2}}{F_{2r+2}^2} \right\} = \tan^{-1}\left( \frac{1}{3} \right).
$$

2 Preliminary result

Taking $x = G_{mr+n-m}$ and $y = G_{mr+n}$ in the arctangent addition formula, identity ([1,1]), gives

$$
\tan^{-1}\left\{ \frac{\lambda (G_{mr+n} - G_{mr+n-m})}{G_{mr+n} G_{mr+n-m} + \lambda^2} \right\} = \tan^{-1}\left( \frac{\lambda}{G_{mr+n-m}} \right) - \tan^{-1}\left( \frac{\lambda}{G_{mr+n}} \right).
$$

(2.1)
Summing each side of identity (2.1) from \( r = p \in \mathbb{Z} \) to \( r = N \in \mathbb{Z}^+ \) and noting that the summation of the terms on the right hand side telescopes, we obtain

\[
\sum_{r=p}^{N} \tan^{-1} \left( \frac{\lambda(G_{mr+n} - G_{mr+n-m})}{G_{mr+n}G_{mr+n-m} + \lambda^2} \right) = \tan^{-1} \left( \frac{\lambda}{G_{mp+n-m}} \right) - \tan^{-1} \left( \frac{\lambda}{G_{mN+n}} \right).
\]

(2.2)

Now taking limit as \( N \to \infty \), we have

**Lemma.**

For \( \lambda \in \mathbb{R}, n, m, p \in \mathbb{Z}, m \neq 0 \) holds

\[
\sum_{r=p}^{\infty} \tan^{-1} \left( \frac{\lambda(G_{mr+n} - G_{mr+n-m})}{G_{mr+n}G_{mr+n-m} + \lambda^2} \right) = \tan^{-1} \left( \frac{\lambda}{G_{mp+n-m}} \right). \tag{2.3}
\]

### 3 Main Results

#### 3.1 \( G \equiv F \) in identity (2.3), that is, \( G_0 = 0, G_1 = 1 \)

Choosing \( m = 4j \) and \( n = 2k + 2j \) and using identities (1.2a) and (1.2d) we prove

**THEOREM 3.1.** For \( \lambda \in \mathbb{R}, j, k, p \in \mathbb{Z} \) and \( j \neq 0 \) holds

\[
\sum_{r=p}^{\infty} \tan^{-1} \left( \frac{\lambda F_{2j}L_{4jr+2k}}{F_{4jr+2k}^2 - F_{2j}^2 + \lambda^2} \right) = \tan^{-1} \left( \frac{\lambda}{F_{4jp+2k-2j}} \right). \tag{3.1}
\]

while taking \( m = 4j - 2 \) and \( n = 2k + 2j - 2 \) and using identities (1.2a) and (1.2c) we prove

**THEOREM 3.2.** For \( \lambda \in \mathbb{R} \) and \( j, k, p \in \mathbb{Z} \) holds

\[
\sum_{r=p}^{\infty} \tan^{-1} \left( \frac{\lambda L_{2j-1}F_{4jr-2r+2k-1}}{F_{4jr-2r+2k-1}^2 - F_{2j-1}^2 + \lambda^2} \right) = \tan^{-1} \left( \frac{\lambda}{F_{4jp-2p+2k-2j}} \right). \tag{3.2}
\]
3.2 \( G \equiv L \) in identity (2.3), that is, \( G_0 = 2, G_1 = 1 \)

Choosing \( m = 4j \) and \( n = 2k + 2j - 1 \) and using identities (1.2b) and (1.2f) we prove

THEOREM 3.3. For \( \lambda \in \mathbb{R}, j, k, p \in \mathbb{Z} \) and \( j \neq 0 \) holds

\[
\sum_{r=p}^{\infty} \tan^{-1}\left( \frac{5\lambda F_{2j} F_{4jr+2k-1}}{L_{8jr+4k-2} - L_{4j} + \lambda^2} \right) = \tan^{-1}\left( \frac{\lambda}{L_{4jp+2k-2j-1}} \right),
\]

while taking \( m = 4j - 2 \) and \( n = 2k + 2j - 1 \) and using identities (1.2b) and (1.2e) we prove

THEOREM 3.4. For \( \lambda \in \mathbb{R} \) and \( j, k, p \in \mathbb{Z} \) holds

\[
\sum_{r=p}^{\infty} \tan^{-1}\left( \frac{\lambda L_{2j-1} L_{4jr-2r+2k}}{L_{8jr-4r+4k} - L_{4j-2} + \lambda^2} \right) = \tan^{-1}\left( \frac{\lambda}{L_{4jp-2p+2k-2j+1}} \right).
\]

4 Corollaries and special values

Different combinations of the parameters \( \lambda, j, k \) and \( p \) in the above theorems yield a variety of interesting particular cases. In this section we will consider some of the possible choices.

4.1 Results from Theorem 3.1

4.1.1 \( \lambda = F_j, p = 1 \) and \( k = 0 \) in identity (3.1)

The choice \( \lambda = F_j, p = 1 \) and \( k = 0 \) in identity (3.1) gives

\[
\sum_{r=1}^{\infty} \tan^{-1}\left( \frac{F_j^2 L_j L_{4jr}}{F_{4jr}^2 - F_{2j}^2 + F_j^2} \right) = \tan^{-1}\left( \frac{1}{L_j} \right).
\]

Thus, at \( j = 1 \), we obtain the special value

\[
\sum_{r=1}^{\infty} \tan^{-1}\left( \frac{L_{4r}}{F_{4r}^2} \right) = \frac{\pi}{4}.
\]
4.1.2 \( \lambda = L_j, \ p = 1 \) and \( k = 0 \) in identity (3.1)

The above choice gives

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{L_j^2 F_j L_{4jr}}{F_{2j}^2 - F_{2j}^2 + L_j^2} \right\} = \tan^{-1} \left( \frac{1}{F_j} \right). \tag{4.3}
\]

At \( j = 1 \), identity (4.2) is reproduced, while at \( j = 2 \) we have the special value

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{9 L_{8r}}{F_{8r}^2} \right\} = \frac{\pi}{4}. \tag{4.4}
\]

Note that identities (4.2) and (4.4) are special cases of identity (4.8) below, at \( j = 1 \) and \( j = 2 \), respectively.

4.1.3 \( \lambda = F_{2j}, \ k = j \) and \( p = 0 \) in identity (3.1)

This choice gives

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{2j}^2 L_{4jr - 2j}}{F_{2j}^2 - F_{4jr - 2j}} \right\} = \frac{\pi}{2}. \tag{4.5}
\]

which, at \( j = 1 \), gives the special value

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{L_{4r - 2}}{F_{4r - 2}^2} \right\} = \frac{\pi}{2}. \tag{4.6}
\]

4.1.4 \( \lambda = F_{2j} \) and \( p = 1 \) in identity (3.1)

This choice gives

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{2j}^2 L_{4jr + 2k}}{F_{2j}^2 - F_{4jr + 2k}} \right\} = \tan^{-1} \left( \frac{F_{2j}}{F_{2j + 2k}} \right). \tag{4.7}
\]

At \( k = 0 \) in identity (4.7) we have

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{2j}^2 L_{4jr}}{F_{4jr}^2} \right\} = \frac{\pi}{4}. \tag{4.8}
\]
Note that identities (4.2) and (4.4) are special cases of identity (4.8) at \( j = 1 \) and \( j = 2 \), respectively.

At \( k = j \neq 0 \) in identity (4.7) we have

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{2j}^2}{L_{4jr+2j}} \right\} = \tan^{-1} \left( \frac{1}{L_{2j}} \right),
\]

yielding at \( j = 1 \), the special value

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{L_{4r+2}}{F_{2r+2}^2} \right\} = \tan^{-1} \left( \frac{1}{3} \right).
\]

Finally, taking limit of identity (4.7) as \( j \to \infty \), we obtain

\[
\lim_{j \to \infty} \sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{2j}^2 L_{4jr+2k}}{F_{4jr+2k}^2} \right\} = \tan^{-1} \left( \frac{1}{\varphi^2k} \right).
\]

**4.1.5 5\( \lambda^2 = L_{4j}, \ p = 0 \) and \( k = j \) in identity (3.1)**

Another interesting particular case of identity (3.1) is obtained by setting \( 5\lambda^2 = L_{4j}, \ p = 0 \) and \( k = j \) to obtain

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{2j} \sqrt{5L_{4j}}}{L_{2(4jr-2j)}} \right\} = \frac{\pi}{2},
\]

which at \( j = 1 \) gives the special value

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{35} L_{4r-2}}{L_{2(4r-2)}} \right\} = \frac{\pi}{2}.
\]

**4.1.6 5\( \lambda^2 = L_{4j}, \ p = 0 \) and \( k = 2j \) in identity (3.1)**

In this case Theorem 3.1 reduces to

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{2j} \sqrt{5L_{4j}}}{L_{8jr}} \right\} = \tan^{-1} \left( \frac{1}{\sqrt{5} \sqrt{L_{4j}}} \right).
\]
At \( j = 1 \), we have the special value

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{35} L_{4r}}{L_{8r}} \right\} = \sqrt{\frac{7}{5}}.
\] (4.15)

### 4.1.7 \( \lambda = L_{2j}/\sqrt{5} \) and \( k = j \) in identity (3.1)

Setting \( \lambda = L_{2j}/\sqrt{5} \) and \( k = j \) in identity (3.1) we have

\[
\sum_{r=p}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{5} F_{4j}}{L_{4jr+2j}} \right\} = \tan^{-1} \left( \frac{L_{2j}}{F_{4jp} \sqrt{5}} \right),
\] (4.16)

which at \( p = 1 \) gives

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{5} F_{4j}}{L_{4jr+2j}} \right\} = \tan^{-1} \left( \frac{1}{F_{2j} \sqrt{5}} \right)
\] (4.17)

and at \( p = 0 \) yields

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \sqrt{5} F_{4j} \right\} = \frac{\pi}{2}.
\] (4.18)

### 4.1.8 \( \lambda = L_{2j}/\sqrt{5}, \ p = 0 \) and \( k = 2j \neq 0 \) in identity (3.1)

The above choice yields

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \sqrt{5} F_{4j} \right\} = \tan^{-1} \left( \frac{L_{2j}}{F_{2j} \sqrt{5}} \right).
\] (4.19)

### 4.2 Results from Theorem 3.2

#### 4.2.1 \( \lambda = F_{2j-1} \) and \( p = 1 \) in identity (3.2)

The above choice gives

\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{2(2j-1)}}{F_{4jr-2r+2k-1}} \right\} = \tan^{-1} \left( \frac{F_{2j-1}}{F_{2j+2k-2}} \right).
\] (4.20)

At \( k = j \) in identity (4.20) we have the interesting formula
\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{2(2j-1)}}{F_{4jr-2r+2j-1}} \right\} = \tan^{-1} \left( \frac{1}{L_{2j-1}} \right) . \tag{4.21}
\]

Note that identity (4.21), at \(j = 1\), includes Lehmer’s result (cited in [3, 5]) as a particular case.

Setting \(j = 1\) in identity (4.20) we obtain
\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{1}{F_{2r+2k-1}} \right\} = \tan^{-1} \left( \frac{1}{F_{2k}} \right) . \tag{4.22}
\]

Note again that identity (4.22) subsumes Lehmer’s formula and the result of Melham (\(p = 1\) in identity (3.5) of [5]), at \(k = 1\) and at \(k = 0\) respectively.

Finally, taking limit \(j \to \infty\) in identity (4.20), we obtain
\[
\lim_{j \to \infty} \sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{2(2j-1)}}{F_{4jr-2r+2k-1}} \right\} = \tan^{-1} \left( \frac{1}{\sqrt{5} L_{2j-1}} \right) . \tag{4.23}
\]

**4.2.2 \(\lambda = L_{2j-1}/\sqrt{5}\) and \(k = j\) in identity (3.2)**

The above choice gives
\[
\sum_{r=p}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{5} L_{2j-1}^2 F_{4jr-2r+2j-1}}{L_{4jr-2r+2j-1}} \right\} = \tan^{-1} \left( \frac{1}{\sqrt{5} F_{2p-2p}} \right) . \tag{4.24}
\]

Setting \(p = 1\) in identity (4.24), we find
\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{5} L_{2j-1}^2 F_{4jr-2r+2j-1}}{L_{4jr-2r+2j-1}} \right\} = \tan^{-1} \left( \frac{1}{\sqrt{5} F_{2j-1}} \right) , \tag{4.25}
\]

while choosing \(j = 1\) leads to
\[
\sum_{r=p}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{5} F_{2r+1}}{L_{2r+1}} \right\} = \tan^{-1} \left( \frac{1}{\sqrt{5} F_{2p}} \right) , \tag{4.26}
\]

which at \(p = 0\) gives the special value.
\[ \sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{5}F_{2r-1}}{L_{2r-1}^2} \right\} = \frac{\pi}{2}. \] (4.27)

4.2.3  \( 5\lambda^2 = L_{4j-2} \) and \( k = j \) in identity \([3.2]\)

The above substitutions give

\[ \sum_{r=p}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{5}L_{4j-2}L_{2j-1}F_{4jr-2r+2j-1}}{L_{2(4jr-2r+2j-1)}} \right\} = \tan^{-1} \left( \frac{\sqrt{5}L_{4j-2}}{5F_{4jp-2p}} \right). \] (4.28)

At \( p = 0 \) in identity \([4.28]\) we have, for positive integers \( j \),

\[ \sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{5}L_{4j-2}L_{2j-1}F_{4jr-2r-2j+1}}{L_{2(4jr-2r-2j+1)}} \right\} = \frac{\pi}{2}, \] (4.29)

giving, at \( j = 1 \), the special value

\[ \sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{15}F_{2r-1}}{L_{2(2r-1)}} \right\} = \frac{\pi}{2}. \] (4.30)

At \( p = 2 \) in identity \([4.28]\) we have, for positive integers \( j \),

\[ \sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{5}L_{4j-2}L_{2j-1}F_{4jr-2r+6j-3}}{L_{2(4jr-2r+6j-3)}} \right\} = \tan^{-1} \left( \frac{1}{\sqrt{5}F_{4j-2}F_{8j-4}} \right), \] (4.31)

which gives, at \( j = 1 \), the special value

\[ \sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{15}F_{2r+3}}{L_{2(2r+3)}} \right\} = \tan^{-1} \left( \frac{1}{\sqrt{15}} \right). \] (4.32)

4.3 Results from Theorem 3.3

4.3.1  \( \lambda = \sqrt{L_{4j}}, k = 0 \) and \( p = 1 \) in identity \([3.3]\)

The above choice gives
\[
\sum_{r=1}^{\infty} \tan^{-1} \left( \frac{5\sqrt{L_{4j} F_{2j} F_{4jr-1}}}{L_{8jr-2}} \right) = \tan^{-1} \left( \frac{\sqrt{L_{4j}}}{L_{2j-1}} \right), \quad (4.33)
\]
which, at \( j = 1 \), gives
\[
\sum_{r=1}^{\infty} \tan^{-1} \left( \frac{5\sqrt{7} F_{4r-1}}{L_{2(4r-1)}} \right) = \tan^{-1} \sqrt{7}. \quad (4.34)
\]

### 4.3.2 \( \lambda = L_{2j} \) and \( p = 1 \) in identity (3.3)

Setting \( \lambda = L_{2j} \) and \( p = 1 \) in identity (3.3) gives
\[
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{4j}}{F_{4jr+2k-1}} \right\} = \tan^{-1} \left( \frac{L_{2j}}{L_{2j+2k-1}} \right). \quad (4.35)
\]
Taking limit as \( j \to \infty \) in identity (4.35) gives
\[
\lim_{j \to \infty} \sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{F_{4j}}{F_{4jr+2k-1}} \right\} = \tan^{-1} \left( \frac{1}{\phi^{2k-1}} \right). \quad (4.36)
\]

### 4.3.3 \( \lambda = \sqrt{5} F_{2j}, \ p = 1 \) and \( k = 0 \) in identity (3.3)

Setting \( \lambda = \sqrt{5} F_{2j}, \ p = 1 \) and \( k = 0 \) in identity (3.3) we obtain
\[
\sum_{r=1}^{\infty} \tan^{-1} \left( \frac{5\sqrt{5} F_{2j}^2 F_{4jr-1}}{L_{4jr-1}^2} \right) = \tan^{-1} \left( \frac{\sqrt{5} F_{2j}}{L_{2j-1}} \right), \quad (4.37)
\]
which gives the special value
\[
\sum_{r=1}^{\infty} \tan^{-1} \left( \frac{5\sqrt{5} F_{4r-1}}{L_{4r-1}^2} \right) = \tan^{-1} \sqrt{5}, \quad (4.38)
\]
at \( j = 1 \).
4.4 Results from Theorem 3.4

4.4.1 $\lambda = \sqrt{L_{4j-2}}$ and $j = 0 = k$ in identity (3.4)

With the above choice we obtain

$$\sum_{r=p}^{\infty} \tan^{-1} \left( \frac{\sqrt{3}L_{2r}}{L_{4r}} \right) = \tan^{-1} \left( \frac{\sqrt{3}}{L_{2p-1}} \right),$$

which gives rise, at $p = 1$, to the special value

$$\sum_{r=1}^{\infty} \tan^{-1} \left( \frac{\sqrt{3}L_{2r}}{L_{4r}} \right) = \frac{\pi}{3}.$$  \hspace{1cm} (4.40)

4.4.2 $\lambda = L_{2j-1}$ and $p = 1$ in identity (3.4)

With the above choice we have

$$\sum_{r=1}^{\infty} \tan^{-1} \left( \frac{L_{2j-1} L_{4j-2r+2k}}{5 \ F_{4j-2r+2k}^2} \right) = \tan^{-1} \left( \frac{L_{2j-1}}{L_{2j+2k-1}} \right).$$ \hspace{1cm} (4.41)

$k = 0$ in identity (4.41) gives

$$\sum_{r=1}^{\infty} \tan^{-1} \left( \frac{L_{2j-1} L_{4j-2r}}{5 \ F_{4j-2r}^2} \right) = \frac{\pi}{4},$$ \hspace{1cm} (4.42)

which at $j = 1$ gives the special value

$$\sum_{r=1}^{\infty} \tan^{-1} \left( \frac{1 \ L_{2r}}{5 \ F_{2r}^2} \right) = \frac{\pi}{4}.$$ \hspace{1cm} (4.43)

$j = 1$ in identity (4.41) leads to

$$\sum_{r=1}^{\infty} \tan^{-1} \left( \frac{1 \ L_{2r+2k}}{5 \ F_{2r+2k}^2} \right) = \tan^{-1} \left( \frac{1}{L_{2k+1}} \right),$$ \hspace{1cm} (4.44)

which gives the special value

$$\sum_{r=1}^{\infty} \tan^{-1} \left( \frac{1 \ L_{2r+2k}}{5 \ F_{2r+2k}^2} \right) = \tan^{-1} \left( \frac{1}{4} \right).$$ \hspace{1cm} (4.45)
at $k = 1$.

Taking limit $j \to \infty$ in identity (4.41), we obtain

$$
\lim_{j \to \infty} \sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{L_{2j-1}^2 L_{4jr-2r+2k}}{5 F_{4jr-2r+2k}^2} \right\} = \tan^{-1} \left( \frac{1}{\phi^{2k}} \right). \tag{4.46}
$$

### 4.4.3 $\lambda = L_{2j-1}$ and $j = 0 = k$ in identity (3.4)

This choice gives

$$
\sum_{r=p}^{\infty} \tan^{-1} \left\{ \frac{1}{5} \frac{L_{2r}}{F_{2r}^2} \right\} = \tan^{-1} \left( \frac{1}{L_{2p-1}} \right), \tag{4.47}
$$

Note that identities (4.43) and (4.45) are special cases of (4.47) at $p = 1$ and at $p = 2$.

### 4.4.4 $\lambda = \sqrt{5} F_{2j-1}$ and $j = 0 = k$ in identity (3.4)

The above choice gives

$$
\sum_{r=p}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{5}}{L_{2r}} \right\} = \tan^{-1} \left( \frac{\sqrt{5}}{L_{2p-1}} \right), \tag{4.48}
$$

which at $p = 1$ gives the special value

$$
\sum_{r=1}^{\infty} \tan^{-1} \left\{ \frac{\sqrt{5}}{L_{2r}} \right\} = \tan^{-1} \sqrt{5}. \tag{4.49}
$$

### 5 Conclusion

Using a fairly straightforward technique, we have derived numerous infinite arctangent summation formulas involving Fibonacci and Lucas numbers. While most of the results obtained are new, a couple of ‘celebrated’ results appear as particular cases of more general formulas derived in this paper.
References


