Lucas’s Inner Circles

In this article, we define the Lucas’s inner circles and we highlight some of their properties.

1. Definition of the Lucas’s Inner Circles

Let $ABC$ be a random triangle; we aim to construct the square inscribed in the triangle $ABC$, having one side on $BC$.

In order to do this, we construct a square $A'B'C'D'$ with $A' \in (AB)$, $B', C' \in (BC)$ (see Figure 1).

We trace the line $BD'$ and we note with $D_a$ its intersection with $(AC)$; through $D_a$ we trace the
parallel $D_aA_a$ to $BC$ with $A_a \in (AB)$ and we project onto $BC$ the points $A_a, D_a$ in $B_a$ respectively $C_a$.

We affirm that the quadrilateral $A_aB_aC_aD_a$ is the required square.

Indeed, $A_aB_aC_aD_a$ is a square, because $\frac{D_aC_a}{D\cdot C'} = \frac{B_aD_a}{B\cdot D'}$ and, as $D\cdot C' = A\cdot D'$, it follows that $A_aD_a = D_aC_a$.

Definition.

It is called A-Lucas’s inner circle of the triangle $ABC$ the circle circumscribed to the triangle $AAaD_a$.

We will note with $L_a$ the center of the A-Lucas’s inner circle and with $l_a$ its radius.

Analogously, we define the B-Lucas’s inner circle and the C-Lucas’s inner circle of the triangle $ABC$.

2. Calculation of the Radius of the A-Lucas Inner Circle

We note $A_aD_a = x$, $BC = a$; let $h_a$ be the height from $A$ of the triangle $ABC$.

The similarity of the triangles $AA_aD_a$ and $ABC$ leads to: $\frac{x}{a} = \frac{h_a - x}{h_a}$, therefore $x = \frac{ah_a}{a + h_a}$.

From $\frac{l_a}{R} = \frac{x}{a}$ we obtain $l_a = \frac{Rh_a}{a + h_a}$. (1)
Note.

Relation (1) and the analogues have been deduced by Eduard Lucas (1842-1891) in 1879 and they constitute the “birth certificate of the Lucas’s circles”.

1st Remark.

If in (1) we replace \( h_a = \frac{2S}{a} \) and we also keep into consideration the formula \( abc = 4RS \), where \( R \) is the radius of the circumscribed circle of the triangle \( ABC \) and \( S \) represents its area, we obtain:

\[
l_a = \frac{R}{1 + \frac{2aR}{bc}} \quad \text{[see Ref. 2].}
\]

3. Properties of the Lucas’s Inner Circles

1st Theorem.

The Lucas’s inner circles of a triangle are inner tangents of the circle circumscribed to the triangle and they are exteriorly tangent pairwise.

Proof.

The triangles \( AA_aD_a \) and \( ABC \) are homothetic through the homothetic center \( A \) and the rapport: \( \frac{h_a}{a+h_a} \).
Because \( \frac{l_a}{R} = \frac{h_a}{a + h_a} \), it means that the A-Lucas’s inner circle and the circle circumscribed to the triangle \( ABC \) are inner tangents in \( A \).

Analogously, it follows that the B-Lucas’s and C-Lucas’s inner circles are inner tangents of the circle circumscribed to \( ABC \).

We will prove that the A-Lucas’s and C-Lucas’s circles are exterior tangents by verifying

\[
L_a L_c = l_a + l_c. \tag{2}
\]

We have:

\[
OL_a = R - l_a;
\]

\[
OL_c = R - l_c
\]

and

\[
m(A0C) = 2B
\]

(if \( m(\hat{B}) > 90^\circ \) then \( m(A0C) = 360^\circ - 2B \).
The theorem of the cosine applied to the triangle $O L_a L_c$ implies, keeping into consideration (2), that:

$$(R - l_a)^2 + (R - l_a)^2 - 2(R - l_a)(R - l_c)\cos 2B = (l_a + l_c)^2.$$  

Because $\cos 2B = 1 - 2\sin^2 B$, it is found that (2) is equivalent to:

$$\sin^2 B = \frac{l_a l_c}{(R - l_a)(R - l_c)}. \quad (3)$$

But we have: $l_a l_c = \frac{R^2 ab^2 c}{(2aR + bc)(2cR + ab)}$, $l_a + l_c = R b \left( \frac{c}{2aR + bc} + \frac{a}{2cR + ab} \right)$.

By replacing in (3), we find that $\sin^2 B = \frac{ab^2 c}{4acR^2} = \frac{b^2}{4a^2} \iff \sin B = \frac{b}{2R}$ is true according to the sines theorem.

So, the exterior tangent of the A-Lucas’s and C-Lucas’s circles is proven.

Analogously, we prove the other tangents.

2\textsuperscript{nd} Definition.

It is called an A-Apollonius’s circle of the random triangle $ABC$ the circle constructed on the segment determined by the feet of the bisectors of angle $A$ as diameter.

Remark.

Analogously, the B-Apollonius’s and C-Apollonius’s circles are defined. If $ABC$ is an isosceles triangle with $AB = AC$ then the A-Apollonius’s circle
isn’t defined for $ABC$, and if $ABC$ is an equilateral triangle, its Apollonius’s circle isn’t defined.

2nd Theorem.

The A-Apollonius’s circle of the random triangle is the geometrical point of the points $M$ from the plane of the triangle with the property: $\frac{MB}{MC} = \frac{c}{b}$.

3rd Definition.

We call a fascicle of circles the bunch of circles that do not have the same radical axis.

a. If the radical axis of the circles’ fascicle is exterior to them, we say that the fascicle is of the first type.

b. If the radical axis of the circles’ fascicle is secant to the circles, we say that the fascicle is of the second type.

c. If the radical axis of the circles’ fascicle is tangent to the circles, we say that the fascicle is of the third type.

3rd Theorem.

The A-Apollonius’s circle and the B-Lucas’s and C-Lucas’s inner circles of the random triangle $ABC$ form a fascicle of the third type.
Proof.

Let \( \{O_A\} = L_bL_c \cap BC \) (see Figure 3).

Menelaus’s theorem applied to the triangle \( OBC \) implies that:
\[
\frac{O_A B}{O_A C} \cdot \frac{l_b B}{l_b O} \cdot \frac{L_c O}{L_c C} = 1,
\]
so:
\[
\frac{O_A B}{O_A C} \cdot \frac{l_b}{R-l_b} \cdot \frac{R-l_c}{l_c} = 1
\]
and by replacing \( l_b \) and \( l_c \), we find that:
\[
\frac{O_A B}{O_A C} = \frac{b^2}{c^2}.
\]

This relation shows that the point \( O_A \) is the foot of the exterior symmedian from \( A \) of the triangle \( ABC \) (so the tangent in \( A \) to the circumscribed circle), namely the center of the A-Apollonius’s circle.

Let \( N_1 \) be the contact point of the B-Lucas’s and C-Lucas’s circles. The radical center of the B-Lucas’s, C-Lucas’s circles and the circle circumscribed to the triangle \( ABC \) is the intersection \( T_A \) of the tangents traced in \( B \) and in \( C \) to the circle circumscribed to the triangle \( ABC \).

It follows that \( BT_A = CT_A = N_1 T_A \), so \( N_1 \) belongs to the circle \( C_A \) that has the center in \( T_A \) and orthogonally cuts the circle circumscribed in \( B \) and \( C \). The radical axis of the B-Lucas’s and C-Lucas’s circles is \( T_A N_1 \), and \( O_A N_1 \) is tangent in \( N_1 \) to the circle \( C_A \). Considering the power of the point \( O_A \) in relation to \( C_A \), we have:
\[
O_A N_1^2 = O_A B \cdot O_A C.
\]
Also, $O_A O^2 = O_A B \cdot O_A C$; it thus follows that $O_A A = O_A N_1$, which proves that $N_1$ belongs to the A-Apollonius’s circle and is the radical center of the A-Apollonius’s, B-Lucas’s and C-Lucas’s circles.

**Remarks.**

1. If the triangle $ABC$ is right in $A$ then $L_b L_c \parallel BC$, the radius of the A-Apollonius’s circle is equal to: $\frac{abc}{|b^2-c^2|}$. The point $N_1$ is the foot of the bisector from $A$. We find that $O_A N_1 = \frac{abc}{|b^2-c^2|}$, so the theorem stands true.
2. The A-Apollonius’s and A-Lucas’s circles are orthogonal. Indeed, the radius of the A-Apollonius’s circle is perpendicular to the radius of the circumscribed circle, \( OA \), so, to the radius of the A-Lucas’s circle also.

4th Definition.

The triangle \( T_AT_BT_C \) determined by the tangents traced in \( A,B,C \) to the circle circumscribed to the triangle \( ABC \) is called the tangential triangle of the triangle \( ABC \).

1st Property.

The triangle \( ABC \) and the Lucas’s triangle \( L_aL_bL_c \) are homological.

Proof.

Obviously, \( AL_a,BL_b,CL_c \) are concurrent in \( O \), therefore \( O \), the center of the circle circumscribed to the triangle \( ABC \), is the homology center.

We have seen that \( \{O_A\} = L_bL_c \cap BC \) and \( O_A \) is the center of the A-Apollonius’s circle, therefore the homology axis is the Apollonius’s line \( O_AO_BO_C \) (the line determined by the centers of the Apollonius’s circle).
2nd Property.

The tangential triangle and the Lucas’s triangle of the triangle $ABC$ are orthogonal triangles.

Proof.

The line $T_A N_1$ is the radical axis of the B-Lucas’s inner circle and the C-Lucas’s inner circle, therefore it is perpendicular on the line of the centers $L_b L_c$. Analogously, $T_B N_2$ is perpendicular on $L_c L_a$, because the radical axes of the Lucas’s circles are concurrent in $L$, which is the radical center of the Lucas’s circles; it follows that $T_A T_B T_C$ and $L_a L_b L_c$ are orthological and $L$ is the center of orthology. The other center of orthology is $O$ the center of the circle circumscribed to $ABC$.

References.

