On A Diophantine Equation $x^2 = 2y^4 - 1$


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Abstract: In this note we present a method of solving this Diophantine equation, method which is different from Ljunggren’s, Mordell’s, and R.K. Guy’s.

In his book of unsolved problems Guy shows that the equation $x^2 = 2y^4 - 1$ has, in the set of positive integers, the only solutions $(1, 1)$ and $(239, 13)$; (Ljunggren has proved it in a complicated way). But Mordell gave an easier proof.

We’ll note $t = y^2$. The general integer solution for $x^2 - 2t^2 + 1 = 0$ is

$$
\begin{cases}
    x_{n+1} = 3x_n + 4t_n \\
    t_{n+1} = 2x_n + 3t_n
\end{cases}
$$

for all $n \in \mathbb{N}$, where $(x_0, y_0) = (1, \varepsilon)$, with $\varepsilon = \pm 1$ (see [6]) or

$$
\begin{pmatrix}
    x_n \\
    t_n
\end{pmatrix}
= \begin{pmatrix}
    3 & 4 \\
    2 & 3
\end{pmatrix}^n
\begin{pmatrix}
    1 \\
    \varepsilon
\end{pmatrix},
$$

for all $n \in \mathbb{N}$, where a matrix to the power zero is equal to the unit matrix $I$.

Let’s consider $A = \begin{pmatrix}
    3 & 4 \\
    2 & 3
\end{pmatrix}$, and $\lambda \in \mathbb{R}$. Then $\det(A - \lambda \cdot I) = 0$ implies $\lambda_{1,2} = 3 \pm \sqrt{2}$, whence if $v$ is a vector of dimension two, then: $Av = \lambda_{1,2} \cdot v$.

Let’s consider $P = \begin{pmatrix}
    2 & 2 \\
    \sqrt{2} & -\sqrt{2}
\end{pmatrix}$ and $D = \begin{pmatrix}
    3 + 2\sqrt{2} & 0 \\
    0 & 3 - 2\sqrt{2}
\end{pmatrix}$. We have $P^{-1} \cdot A \cdot P = D$, or

$$
A^n = P \cdot D^n \cdot P^{-1} = \begin{pmatrix}
    \frac{1}{2}(a + b) & \frac{\sqrt{2}}{2}(a - b) \\
    \frac{\sqrt{2}}{4}(a - b) & \frac{1}{2}(a + b)
\end{pmatrix},
$$

where $a = (3 + 2\sqrt{2})^n$ and $b = (3 - 2\sqrt{2})^n$.

Hence, we find:

$$
\begin{pmatrix}
    x_n \\
    t_n
\end{pmatrix}
= \begin{pmatrix}
    \frac{1 + \varepsilon\sqrt{2}}{2} (3 + 2\sqrt{2})^n + \frac{1 - \varepsilon\sqrt{2}}{2} (3 - 2\sqrt{2})^n \\
    \frac{2\varepsilon + \sqrt{2}}{4} (3 + 2\sqrt{2})^n + \frac{2\varepsilon - \sqrt{2}}{4} (3 - 2\sqrt{2})^n
\end{pmatrix},
$$

$n \in \mathbb{N}$.

Or $y_n^2 = \frac{2\varepsilon + \sqrt{2}}{4} (3 + 2\sqrt{2})^n + \frac{2\varepsilon - \sqrt{2}}{4} (3 - 2\sqrt{2})^n$, $n \in \mathbb{N}$.

For $n = 0$, $\varepsilon = 1$ we obtain $y_0^2 = 1$ (whence $x_0^2 = 1$), and for $n = 3, \varepsilon = 1$ we obtain $y_3^2 = 169$ (whence $x_3 = 239$).
We still must prove that \( y_n^2 \) is a perfect square if and only if \( n = 0, 3 \).

We can use a similar method for the Diophantine equation \( x^2 = Dy^4 \pm 1 \), or more generally: \( C \cdot X^2 = D Y^b + E \), with \( a, b \in \mathbb{N}^* \) and \( C, D, E \in \mathbb{Z}^* \); denoting \( X^a = U \), \( Y^b = V \), and applying the results from F.S. [6], the relation (1) becomes very complicated.

REFERENCES
