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P-Q Relationships and Sequences

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Let $A = \{a_n\}, n \geq 1$ be a sequence of numbers and $q, p$ integers $\geq 1$.

We say that the terms $a_{k+1}, a_{k+2}, \ldots, a_{k+p}, a_{k+p+1}, a_{k+p+2}, \ldots, a_{k+p+q}$ satisfy a $p-q$ relationship if

$$a_{k+1} \circ a_{k+2} \circ \ldots \circ a_{k+p} = a_{k+p+1} \circ a_{k+p+2} \circ \ldots \circ a_{k+p+q}$$

where $\circ$ may be any arithmetic operation, although it is generally a binary relation on $A$. If this relationship is satisfied for any $k \geq 1$, then $\{a_n\}, n \geq 1$ is said to be a $p-q-\circ$ sequence.

For operations such as addition, where $\circ = +$, the sequence is called a $p-q$-additive sequence.

As a specific case, we can easily see that the Fibonacci/Lucas sequence ($a_n + a_{n+1} = a_{n+2}$, for $n \geq 1$), is a $3-1$-additive sequence.

**Definition.** Given any integer $n \geq 1$, the value of the Smarandache function $S(n)$ is the smallest integer $m$ such that $n$ divides $m$.

If we consider the sequence of numbers that are the values of the Smarandache function for the integers $n \geq 1$,

$$1, 2, 3, 4, 5, 3, 7, 4, 6, 5, 11, 4, 13, 7, 5, 6, 17, \ldots$$

they can be incorporated into questions involving the $p-q-\circ$ relationships.
a) How many ordered quadruples are there of the form \((S(n), S(n+1), S(n+2), S(n+3))\) such that \(S(n+1) + S(n+2) = S(n+3) + S(n+4)\) which is a 2 - 2-additive relationship?

The three quadruples
\[
\begin{align*}
S(6) + S(7) &= S(8) + S(9), \quad 3 + 7 = 4 + 6; \\
S(7) + S(8) &= S(9) + S(10), \quad 7 + 4 = 6 + 5; \\
S(28) + S(29) &= S(30) + S(31), \quad 7 + 29 = 5 + 31.
\end{align*}
\]
are known. Are there any others? At this time, these are the only known solutions.

b) How many quadruples satisfy the 2 - 2-subtrac relationship \(S(n+1) - S(n+2) = S(n+3) - S(n+4)\)?

The three quadruples
\[
\begin{align*}
S(1) - S(2) &= S(3) - S(4), \quad 1 - 2 = 3 - 4; \\
S(2) - S(3) &= S(4) - S(5), \quad 2 - 3 = 4 - 5; \\
S(49) - S(50) &= S(51) - S(52), \quad 14 - 10 = 17 - 13
\end{align*}
\]
are known. Are there any others?

c) How many 6-tuples satisfy the 2 - 3-additive relationship \(S(n+1) + S(n+2) + S(n+3) = S(n+4) + S(n+5) + S(n+6)\)?

The only known solution is
\[
S(5) + S(6) + S(7) = S(8) + S(9) + S(10), \quad 5 + 3 + 7 = 4 + 6 + 5.
\]

Charles Ashbacher has a computer program that calculates the values of the Smarandache function. Therefore, he may be able to find additional solutions to these problems.

More general, if \(f_p\) is a \(p\)-ary relation and \(g_q\) a \(q\)-ary relation, both defined on the set \(\{a_1, a_2, a_3, \ldots\}\), then \(a_{i_1}, a_{i_2}, \ldots, a_{i_p}, a_{j_1}, a_{j_2}, \ldots, a_{j_q}\) satisfies a \(f_p - g_q\) relationship if
\[
f(a_{i_1}, a_{i_2}, \ldots, a_{i_p}) = g(a_{j_1}, a_{j_2}, \ldots, a_{j_q}).
\]

If this relationship holds for all terms of the sequence, then \(\{a_n\}, n \geq 1\) is called a \(f_p - g_q\) sequence.

Study some \(f_p - g_q\) relationship for well-known sequences, such as the perfect numbers, Ulam numbers, abundant numbers, Catalan numbers and Cullen numbers. For example, a 2 - 2-additive, subtractive or multiplicative relationship.