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Radical Axis of Lemoine's Circles

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In this article, we emphasize the **radical axis of the Lemoine's circles**.

For the start, let us remind:

1st Theorem.

The parallels taken through the simmedian center K of a triangle to the sides of the triangle determine on them six concyclic points (the first Lemoine's circle).

2nd Theorem.

The antiparallels taken through the triangle's simmedian center to the sides of a triangle determine six concyclic points (the second Lemoine's circle).

1st Remark.

If ABC is a scalene triangle and K is its simmedian center, then L, the center of the first Lemoine's circle, is the middle of the segment $[OK]$, where O is the center of the circumscribed circle, and the center of the second Lemoine's circle is K. It follows that the radical axis of Lemoine's circles is perpendicular on the line of the centers LK, therefore on the line OK.

1st Proposition.

The radical axis of Lemoine's circles is perpendicular on the line OK raised in the simmedian center K.

Proof.

Let A_1A_2 be the antiparallel to BC taken through K, then KA_1 is the radius R_{L_2} of the second Lemoine's circle; we have:

$$R_{L_2} = \frac{abc}{a^2+b^2+c^2}.$$

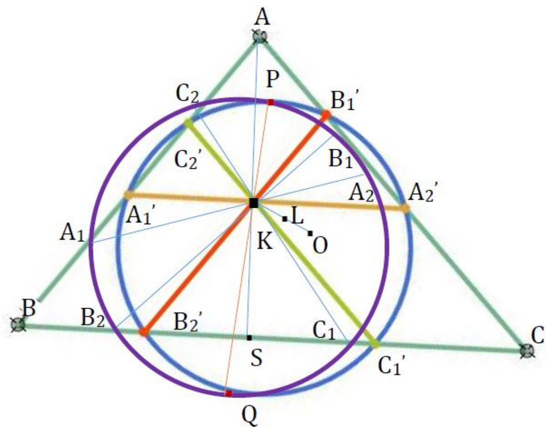

Figure 1

Let $A_1'A_2'$ be the Lemoine's parallel taken to BC; we evaluate the power of K towards the first Lemoine's circle. We have:

$$\overrightarrow{KA_1'} \cdot \overrightarrow{KA_2'} = LK^2 - R_{L_1}^2. \qquad (1)$$

Let S be the simmedian leg from A; it follows that:

$$\frac{KA_1'}{BS} = \frac{AK}{AS} - \frac{KA_2'}{SC} .$$

We obtain:

$$KA_1' = BS \cdot \frac{AK}{AS} \text{ and } KA_2' = SC \cdot \frac{AK}{AS} ,$$

but $\frac{BS}{SC} = \frac{c^2}{b^2}$ and $\frac{AK}{AS} = \frac{b^2+c^2}{a^2+b^2+c^2}$.

Therefore:

$$\overrightarrow{KA_1'} \cdot \overrightarrow{KA_2'} = -BS \cdot SC \cdot \left(\frac{AK}{AS}\right)^2 = \frac{-a^2b^2c^2}{(b^2+c^2)^2};$$

$$\frac{\left(b^2+c^2\right)^2}{(a^2+b^2+c^2)^2} = -R_{L_2}^2 . \qquad (2)$$

We draw the perpendicular in K on the line LK and denote by P and Q its intersection to the first Lemoine's circle.

We have $\overrightarrow{KP} \cdot \overrightarrow{KQ} = -R_{L_2}^2$; by the other hand, $KP = KQ$ (PQ is a chord which is perpendicular to the diameter passing through K).

It follows that $KP = KQ = R_{L_2}$, so P and Q are situated on the second Lemoine's circle.

Because PQ is a chord which is common to the Lemoine's circles, it obviously follows that PQ is the radical axis.

Comment.

Equalizing (1) and (2), or by the Pythagorean theorem in the triangle PKL, we can calculate R_{L_1}.

It is known that: $OK^2 = R^2 - \frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2}$, and since $LK = \frac{1}{2}OK$, we find that:

$$R_{L_1}^2 = \frac{1}{4} \cdot \left[R^2 + \frac{a^2b^2c^2}{(a^2+b^2+c^2)^2}\right].$$

2nd Remark.

The 1st Proposition, ref. the radical axis of the Lemoine's circles, is a particular case of the following *Proposition*, which we leave to the reader to prove.

2nd Proposition.

If $\mathcal{C}(O_1, R_1)$ şi $\mathcal{C}(O_2, R_2)$ are two circles such as the power of center O_1 towards $\mathcal{C}(O_2, R_2)$ is $-R_1^2$, then the radical axis of the circles is the perpendicular in O_1 on the line of centers O_1O_2.

References.

[1] F. Smarandache, Ion Patrascu: *The Geometry of Homological Triangles*, Education Publisher, Ohio, USA, 2012.

[2] Ion Patrascu, F. Smarandache: *Variance on Topics of Plane Geometry*, Education Publisher, Ohio, USA, 2013.

Generating Lemoine's circles

In this paper, we generalize the theorem relative to the first Lemoine's circle and thereby highlight **a method to build Lemoine's circles**.

Firstly, we review some notions and results.

1st Definition.

It is called a simedian of a triangle the symmetric of a median of the triangle with respect to the internal bisector of the triangle that has in common with the median the peak of the triangle.

1st Proposition.

In the triangle ABC, the cevian AS, $S \in (BC)$, is a simedian if and only if $\frac{SB}{SC} = \left(\frac{AB}{AC}\right)^2$. For *Proof*, see [2].

2nd Definition.

It is called a simedian center of a triangle (or Lemoine's point) the intersection of triangle's simedians.

1st Theorem.

The parallels to the sides of a triangle taken through the simedian center intersect the triangle's sides in six concyclic points (the first Lemoine's circle - 1873).

A *Proof* of this theorem can be found in [2].

3rd Definition.

We assert that in a scalene triangle ABC the line MN, where $M \in AB$ and $N \in AC$, is an antiparallel to BC if $\sphericalangle MNA \equiv \sphericalangle ABC$.

1st Lemma.

In the triangle ABC, let AS be a simedian, $S \in (BC)$. If P is the middle of the segment (MN), having $M \in (AB)$ and $N \in (AC)$, belonging to the simedian AS, then MN and BC are antiparallels.

Proof.

We draw through M and N, $MT \parallel AC$ and $NR \parallel AB$, $R, T \in (BC)$, see *Figure 1*. Let $\{Q\} = MT \cap NR$; since $MP = PN$ and $AMQN$ is a parallelogram, it follows that $Q \in AS$.

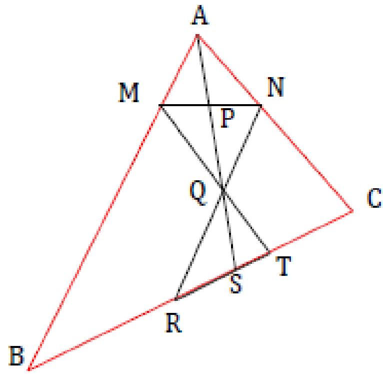

Figure 1.

Thales's Theorem provides the relations:

$$\frac{AN}{AC} = \frac{BR}{BC}; \quad (1)$$

$$\frac{AB}{AM} = \frac{BC}{CT}. \quad (2)$$

From (1) and (2), by multiplication, we obtain:

$$\frac{AN}{AM} \cdot \frac{AB}{AC} = \frac{BR}{TC}. \quad (3)$$

Using again Thales's Theorem, we obtain:

$$\frac{BR}{BS} = \frac{AQ}{AS}, \quad (4)$$

$$\frac{TC}{SC} = \frac{AQ}{AS}. \quad (5)$$

From these relations, we get

$$\frac{BR}{BS} = \frac{TC}{SC}, \quad (6)$$

or

$$\frac{BS}{SC} = \frac{BR}{TC}. \quad (7)$$

In view of *Proposition 1*, the relations (7) and (3) lead to $\frac{AN}{AB} = \frac{AB}{AC}$, which shows that $\Delta AMN \sim \Delta ACB$, so $\sphericalangle AMN \equiv \sphericalangle ABC$.

Therefore, MN and BC are antiparallels in relation to AB and AC.

Remark.

1. The reciprocal of *Lemma 1* is also valid, meaning that if P is the middle of the antiparallel MN to BC, then P belongs to the simedian from A.

2nd Theorem.

(Generalization of the *1st Theorem*)

Let ABC be a scalene triangle and K its simedian center. We take $M \in AK$ and draw $MN \parallel AB, MP \parallel AC$, where $N \in BK$, $P \in CK$. Then:

i. $NP \parallel BC$;

ii. MN, NP and MP intersect the sides of triangle ABC in six concyclic points.

Proof.

In triangle ABC, let AA_1, BB_1, CC_1 the simedians concurrent in K (see *Figure 2*).

We have from Thales' Theorem that:

$$\frac{AM}{MK} = \frac{BN}{NK}; \quad (1)$$

$$\frac{AM}{MK} = \frac{CP}{PK}. \quad (2)$$

From relations (1) and (2), it follows that

$$\frac{BN}{NK} = \frac{CP}{PK}, \quad (3)$$

which shows that $NP \parallel BC$.

Let R, S, V, W, U, T be the intersection points of the parallels MN, MP, NP of the sides of the triangles to the other sides.

Obviously, by construction, the quadrilaterals $ASMW$; $CUPV$; $BRNT$ are parallelograms.

The middle of the diagonal WS falls on AM, so on the simedian AK, and from *1st Lemma* we get that WS is an antiparallel to BC.

Since $TU \parallel BC$, it follows that WS and TU are antiparallels, therefore the points W, S, U, T are concyclic (4).

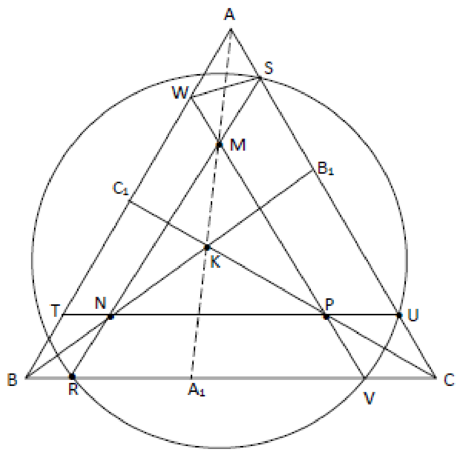

Figure 2.

Analogously, we show that the points U, V, R, S are concyclic (5). From WS and BC antiparallels, UV and AB antiparallels, we have that $\sphericalangle WSA \equiv \sphericalangle ABC$ and $\sphericalangle VUC \equiv \sphericalangle ABC$, therefore: $\sphericalangle WSA \equiv \sphericalangle VUC$, and since

$VW \parallel AC$, it follows that the trapeze $WSUV$ is isosceles, therefore the points W, S, U, V are concyclic (6).

The relations (4), (5), (6) drive to the concyclicality of the points R, U, V, S, W, T, and the theorem is proved.

Further Remarks.

2. For any point M found on the simedian AA_1, by performing the constructions from hypothesis, we get a circumscribed circle of the 6 points of intersection of the parallels taken to the sides of triangle.

3. The *2nd Theorem* generalizes the *1st Theorem* because we get the second in the case the parallels are taken to the sides through the simedian center k.

4. We get a circle built as in *2nd Theorem* from the first Lemoine's circle by homothety of pole k and of ratio $\lambda \in \mathbb{R}$.

5. The centers of Lemoine's circles built as above belong to the line OK, where O is the center of the circle circumscribed to the triangle ABC.

References.

[1] *Exercices de Géométrie*, par F.G.M., Huitième édition, Paris VIe, Librairie Générale, 77, Rue Le Vaugirard.

[2] Ion Patrascu, Florentin Smarandache: *Variance on topics of Plane Geometry*, Educational Publishing, Columbus, Ohio, 2013.