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Solving Problems by Using a Function
in The Number Theory

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SOME LINEAR EQUATIONS INVOLVING A FUNCTION IN THE NUMBER THEORY

We have constructed a function \( \eta \) which associates to each non-null integer \( m \) the smallest positive \( n \) such that \( n! \) is a multiple of \( m \).

(a) Solve the equation \( \eta(x) = n \), where \( n \in \mathbb{N} \).

(b) Solve the equation \( \eta(mx) = x \), where \( m \in \mathbb{Z} \).

Discussion.

(c) Let \( \eta^{(i)} \) denote \( \eta \circ \eta \circ \ldots \circ \eta \) of \( i \) times. Prove that there is a \( k \) for which
\[ \eta^{(k)}(m) = \eta^{(k+1)}(m) = \eta(m) \] for all \( m \in \mathbb{Z}^* \setminus \{1\} \).

"Find \( n_m \) and the smallest \( k \) with this property.

Solution

(a) The cases \( n = 0, 1 \) are trivial.

We note the increasing sequence of primes less or equal than \( n \) by \( p_1, p_2, \ldots, p_k \),

\[ \beta_t = \sum_{i=1}^{s} \left[ \frac{n}{p_i^k} \right], \quad t = 1, 2, \ldots, k, \]

where \( [y] \) is greatest integer less or equal than \( y \).

Let \( n = p_1^{a_1} \cdots p_s^{a_s} \), where all \( p_i \) are distinct primes and all \( a_i \) are from \( \mathbb{N} \).

Of course we have \( n \leq x \leq n! \)

Thus \( x = p_1^{\sigma_1} \cdots p_s^{\sigma_s} \) where \( 0 \leq \sigma_t \leq \beta_t \) for all \( t = 1, 2, \ldots, k \) and there exists at least a \( j \in \{1, 2, \ldots, s\} \) for which
\[ \sigma_j = \beta_j, \quad \{\beta_1^t, \ldots, \beta_s^t - \alpha_t + 1}\). \[ \]

Clearly \( n! \) is a multiple of \( x \), and is the smallest one.

(b) See [1] too. We consider \( m \in \mathbb{N}^* \).

Lemma 1. \( \eta(m) \leq m \), and \( \eta(m) = m \) if and only if \( m = 4 \) or \( m \) is a prime.

Of course \( m! \) is a multiple of \( m \).

If \( m \neq 4 \) and \( m \) is not a prime, the Lemma is equivalent to there are \( m_1, m_2 \) such that \( m = m_1 \cdot m_2 \) with \( 1 < m_1 \leq m_2 \) and \( (2m_2 < m) \) or \( 2m_1 < m \). Whence \( \eta(m) \leq 2m_2 < m \), respectively \( \eta(m) \leq \max\{m_2, 2m_1\} < m \).

Lemma 2. Let \( p \) be a prime \( \leq 5 \). Then \( \eta(pz) = z \) if and only if \( z \) is a prime \( > p \), or \( z = 2p \).
Proof: \( \eta(p) = p \). Hence \( x > p \).

Analogously: \( x \) is not a prime and \( x \neq 2p \iff x = z_1z_2, 1 < z_1 \leq z_2 \) and \( 2z_1 < x_1, z_2 \neq p_1 \), and \( 2z_1 < x \iff \eta(x) \leq \max\{p, 2x_2\} < x \) respectively \( \eta(px) \leq \max\{p, 2x_1, z_2\} < x \).

Observations

\( \eta(2x) = x \iff x = 4 \) or \( x \) is an odd prime.

\( \eta(3x) = x \iff x = 4, 6, 9 \) or \( x \) is a prime > 3.

Lemma 3. If \( (m, x) = 1 \) then \( x \) is a prime > \( \eta(m) \).

Of course, \( \eta(mx) = \max\{\eta(m), \eta(x)\} = \eta(x) = x \). And \( x \neq \eta(m) \), because if \( x = \eta(m) \) then \( m \cdot \eta(m) \) divides \( \eta(m)! \), that is \( m \) divides \( \eta(m) - 1 \)! whence \( \eta(m) \leq \eta(m) - 1 \).

Lemma 4. If \( x \) is not a prime then \( \eta(m) < x \leq 2\eta(m) \) and \( x = 2\eta(m) \) if and only if \( \eta(m) \) is a prime.

Proof: If \( x > 2\eta(m) \) there are \( z_1, z_2 \) with \( 1 < z_1 \leq z_2, x = z_1z_2 \). For \( z_1 < \eta(m) \) we have \( (z_1 - 1)! \) is a multiple of \( m \). Same proof for other cases.

Let \( x = 2\eta(m) \); if \( \eta(m) \) is not a prime, then \( x = 2ab, 1 < a \leq b \), but the product \( \eta(m) + 1)(\eta(m) + 2)\ldots(2\eta(m) - 1) \) is divided by \( x \).

If \( \eta(m) \) is a prime, \( \eta(m) \) divides \( m \), whence \( m \cdot 2\eta(m) \) is divided by \( \eta(m) \)², it results in \( \eta(m) \cdot 2\eta(m) \geq 2 \cdot \eta(m) \), but \( (\eta(m) + 1)(\eta(m) + 2)\ldots(2\eta(m)) \) is a multiple of \( 2\eta(m) \), that is \( \eta(m) \cdot 2\eta(m) = 2\eta(m) \).

Conclusion.

All \( x \), prime number > \( \eta(m) \), are solutions.

If \( \eta(m) \) is prime, then \( x = 2\eta(m) \) is a solution.

*If \( x \) is not a prime, \( \eta(m) < x < 2\eta(m) \), and \( x \) does not divide \( (x - 1)!/m \) then \( x \) is a solution (semi-open question). If \( m = 3 \) it adds \( x = 9 \) too. (No other solution exists yet.)

(c)

Lemma 5. \( \eta(ab) \leq \eta(a) + \eta(b) \).

Of course, \( \eta(a) = a' \) and \( \eta(b) = b' \) involves \( a' + b' \)! = \( b'!\left(b' + 1\right)\ldots(a' + a') \). Let \( a' \leq b' \).

Then \( \eta(ab) \leq a' + b' \), because the product of \( a' \) consecutive positive integers is a multiple of \( a' \! \).

Clearly, if \( m \) is a prime then \( k = 1 \) and \( n_m = m \).

If \( m \) is not a prime then \( \eta(m) < m \), whence there is a \( k \) for which \( \eta(k)(m) = \eta(k+1)(m) \).

If \( m \neq 1 \) then \( 2 \leq n_m \leq m \).
Lemma 6. $n_m = 4$ or $n_m$ is a prime.

If $n_m = n_1 n_2, 1 < n_1 \leq n_2,$ then $n(n_m) < n_m$. Absurd. $n_m \neq 4.$

(**) This question remains open.

References


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