Theorems with Parallels Taken through a Triangle’s Vertices and Constructions Performed only with the Ruler

In this article, we solve problems of geometric constructions only with the ruler, using known theorems.

1\textsuperscript{st} Problem.

Being given a triangle $ABC$, its circumscribed circle (its center known) and a point $M$ fixed on the circle, construct, using only the ruler, a transversal line $A_1, B_1, C_1$, with $A_1 \in BC, B_1 \in CA, C_1 \in AB$, such that $\angle MA_1 C \equiv \angle MB_1 C \equiv \angle MC_1 A$ (the lines taken though $M$ to generate congruent angles with the sides $BC$, $CA$ and $AB$, respectively).

2\textsuperscript{nd} Problem.

Being given a triangle $ABC$, its circumscribed circle (its center known) and $A_1, B_1, C_1$, such that $A_1 \in$
$BC, B_1 \in CA, C_1 \in AB$ and $A_1, B_1, C_1$ collinear, construct, using only the ruler, a point $M$ on the circle circumscribing the triangle, such that the lines $MA_1, MB_1, MC_1$ to generate congruent angles with $BC$, $CA$ and $AB$, respectively.

3rd Problem.

Being given a triangle $ABC$ inscribed in a circle of given center and $AA'$ a given cevian, $A'$ a point on the circle, construct, using only the ruler, the isogonal cevian $AA_1$ to the cevian $AA'$.

To solve these problems and to prove the theorems for problems solving, we need the following Lemma:

1st Lemma.
(Generalized Simpson's Line)

If $M$ is a point on the circle circumscribed to the triangle $ABC$ and we take the lines $MA_1, MB_1, MC_1$ which generate congruent angles ($A_1 \in BC, B_1 \in CA, C_1 \in AB$) with $BC, CA$ and $AB$ respectively, then the points $A_1, B_1, C_1$ are collinear.
Proof.

Let $M$ on the circle circumscribed to the triangle $ABC$ (see Figure 1), such that:

\[ \angle MA_1C \equiv \angle MB_1C \equiv \angle MC_1A = \varphi. \]  

(1)

From the relation (1), we obtain that the quadrilateral $MB_1A_1C$ is inscriptible and, therefore:

\[ \angle A_1BC \equiv \angle A_1MC. \]  

(2)

Also from (1), we have that $MB_1AC_1$ is inscriptible, and so

\[ \angle AB_1C_1 \equiv \angle AMC_1. \]  

(3)
The quadrilateral MABC is inscribed, hence:
\[ \measuredangle MAC_1 \equiv \measuredangle BCM. \] (4)

On the other hand,
\[ \measuredangle A_1MC = 180^0 - (\overline{BCM} + \varphi), \]
\[ \measuredangle AMC_1 = 180^0 - (\overline{MAC_1} + \varphi). \]

The relation (4) drives us, together with the above relations, to:
\[ \measuredangle A_1MC \equiv \measuredangle AMC_1. \] (5)

Finally, using the relations (5), (2) and (3), we conclude that: \[ \measuredangle A_1B_1C \equiv AB_1C_1, \] which justifies the collinearity of the points \( A_1, B_1, C_1. \)

**Remark.**

The Simson’s Line is obtained in the case when \( \varphi = 90^0. \)

**2nd Lemma.**

If \( M \) is a point on the circle circumscribed to the triangle \( ABC \) and \( A_1, B_1, C_1 \) are points on \( BC, CA \) and \( AB \), respectively, such that \( \measuredangle MA_1C = \measuredangle MB_1C = \measuredangle MC_1A = \varphi \), and \( MA_1 \) intersects the circle a second time in \( A' \), then \( AA' \parallel A_1B_1. \)

**Proof.**

The quadrilateral \( MB_1A_1C \) is inscriptible (see Figure 1); it follows that:
\( \angle CMA' \equiv \angle A_1B_1C. \) \hspace{1cm} (6)

On the other hand, the quadrilateral \( MAA'C \) is also inscriptible, hence:
\( \angle CMA' \equiv \angle A'AC. \) \hspace{1cm} (7)

The relations (6) and (7) imply: \( \angle A'MC \equiv \angle A'AC, \) which gives \( AA' \parallel A_1B_1. \)

3rd Lemma.

(The construction of a parallel with a given diameter using a ruler)

In a circle of given center, construct, using only the ruler, a parallel taken through a point of the circle at a given diameter.

Solution.

In the given circle \( \mathcal{C}(O,R) \), let be a diameter \((AB)]\) and let \( M \in \mathcal{C}(O,R) \). We construct the line \( BM \) (see Figure 2). We consider on this line the point \( D \) (\( M \) between \( D \) and \( B \)). We join \( D \) with \( O \), \( A \) with \( M \) and denote \( DO \cap AM = \{P\} \).

We take \( BP \) and let \( \{N\} = DA \cap BP \). The line \( MN \) is parallel to \( AB \).

Construction’s Proof.

In the triangle \( DAB \), the cevians \( DO \), \( AM \) and \( BN \) are concurrent.

Ceva’s Theorem provides:
\[
\frac{OA}{OB} \cdot \frac{MB}{MD} \cdot \frac{ND}{NA} = 1. \tag{8}
\]

But \(DO\) is a median, \(DO = BO = R\).

From (8), we get \(\frac{MB}{MD} = \frac{NA}{ND}\), which, by Thales reciprocal, gives \(MN \parallel AB\).

**Remark.**

If we have a circle with given center and a certain line \(d\), we can construct though a given point \(M\) a parallel to that line in such way: we take two diameters \([RS]\) and \([UV]\) through the center of the given circle (see Figure 3).
We denote \( RS \cap d = \{P\} \); because \([RO] \equiv [SO]\), we can construct, applying the 3\textsuperscript{rd} Lemma, the parallels through \( U \) and \( V \) to \( RS \) which intersect \( d \) in \( K \) and \( L \), respectively. Since we have on the line \( d \) the points \( K, P, L \), such that \([KP] \equiv [PL]\), we can construct the parallel through \( M \) to \( d \) based on the construction from 3\textsuperscript{rd} Lemma.

1\textsuperscript{st} Theorem.
(P. Aubert – 1899)

If, through the vertices of the triangle \( ABC \), we take three lines parallel to each other, which intersect the circumscribed circle in \( A', B' \) and \( C' \), and \( M \) is a
point on the circumscribed circle, as well \( MA' \cap BC = \{A_1\} \), \( MB' \cap CA = \{B_1\} \), \( MC' \cap AB = \{C_1\} \), then \( A_1, B_1, C_1 \) are collinear and their line is parallel to \( AA' \).

**Proof.**

The point of the proof is to show that \( MA_1, MB_1, MC_1 \) generate congruent angles with \( BC, CA \) and \( AB \), respectively.

\[
m(\overrightarrow{MA_1C}) = \frac{1}{2} [m(\overrightarrow{MC}) + m(\overrightarrow{BA'})] \tag{9}
\]

\[
m(\overrightarrow{MB_1C}) = \frac{1}{2} [m(\overrightarrow{MC}) + m(\overrightarrow{AB'})] \tag{10}
\]

But \( AA' \parallel BB' \) implies \( m(\overrightarrow{BA'}) = m(\overrightarrow{AB'}) \), hence, from (9) and (10), it follows that:

\[
\angle MA_1C \equiv \angle MB_1C, \tag{11}
\]

\[
m(\overrightarrow{MC_1A}) = \frac{1}{2} [m(\overrightarrow{BM}) - m(\overrightarrow{AC'})]. \tag{12}
\]

But \( AA' \parallel CC' \) implies that \( m(\overrightarrow{AC'}) = m(\overrightarrow{AC}) \); by returning to (12), we have that:

\[
m(\overrightarrow{MC_1A}) = \frac{1}{2} [m(\overrightarrow{BM}) - m(\overrightarrow{AC'})] =
\]

\[
= \frac{1}{2} [m(\overrightarrow{BA'}) + m(\overrightarrow{MC})]. \tag{13}
\]

The relations (9) and (13) show that:

\[
\angle MA_1C \equiv \angle MC_1A. \tag{14}
\]

From (11) and (14), we obtain: \( \angle MA_1C \equiv \angle MB_1C \equiv \angle MC_1A \), which, by 1st *Lemma*, verifies the collinearity of points \( A_1, B_1, C_1 \). Now, applying the 2nd *Lemma*, we obtain the parallelism of lines \( AA' \) and \( A_1B_1 \).
2nd Theorem.

(M’Kensie – 1887)

If $A_1B_1C_1$ is a transversal line in the triangle $ABC$ ($A_1 \in BC, B_1 \in CA, C_1 \in AB$), and through the triangle’s vertices we take the chords $AA', BB', CC'$ of a circle circumscribed to the triangle, parallels with the transversal line, then the lines $AA', BB', CC'$ are concurrent on the circumscribed circle.
Proof.

We denote by $M$ the intersection of the line $A_1A'$ with the circumscribed circle (see Figure 5) and with $B_1'$, respectively $C_1'$ the intersection of the line $MB'$ with $AC$ and of the line $MC'$ with $AB$.

![Figure 5.](image)

According to the P. Aubert’s theorem, we have that the points $A_1$, $B_1'$, $C_1'$ are collinear and that the line $A_1B_1'$ is parallel to $AA'$.

From hypothesis, we have that $A_1B_1 \parallel AA'$; from the uniqueness of the parallel taken through $A_1$ to $AA'$, it follows that $A_1B_1 \equiv A_1B_1'$, therefore $B_1' = B_1$, and analogously $C_1' = C_1$. 
Remark.

We have that: $MA_1, MB_1, MC_1$ generate congruent angles with $BC$, $CA$ and $AB$, respectively.

3rd Theorem.

(Beltrami – 1862)

If three parallels are taken through the three vertices of a given triangle, then their isogonals intersect each other on the circle circumscribed to the triangle, and vice versa.

Proof.

Let $AA', BB', CC'$ the three parallel lines with a certain direction (see Figure 6).
To construct the isogonal of the cevian $AA'$, we take $A'M \parallel BC$, $M$ belonging to the circle circumscribed to the triangle, having $\overline{BA'} \equiv \overline{CM}$, it follows that $AM$ will be the isogonal of the cevian $AA'$. (Indeed, from $\overline{BA'} \equiv \overline{CM}$ it follows that $\angle BAA' \equiv \angle CAM$.)

On the other hand, $BB' \parallel A'$ implies $\overline{BA'} \equiv \overline{AB'}$, and since $\overline{BA'} \equiv \overline{CM}$, we have that $\overline{AB'} \equiv \overline{CM}$, which shows that the isogonal of the parallel $BB'$ is $BM$. From $CC' \parallel AA'$, it follows that $A'C \equiv A'C'$, having $\angle B'CM \equiv \angle ACC'$, therefore the isogonal of the parallel $CC'$ is $CM'$.

Reciprocally.

If $AM, BM, CM$ are concurrent cevians in $M$, the point on the circle circumscribed to the triangle $ABC$, let us prove that their isogonals are parallel lines. To construct an isogonal of $AM$, we take $MA' \parallel BC$, $A'$ belonging to the circumscribed circle. We have $\overline{MC} \equiv \overline{BA'}$. Constructing the isogonal $BB'$ of $BM$, with $B'$ on the circumscribed circle, we will have $\overline{CM} \equiv \overline{AB'}$, it follows that $\overline{BA'} \equiv \overline{AB'}$ and, consequently, $\angle BAA' \equiv \angle BAA'$, which shows that $AA' \parallel BB'$. Analogously, we show that $CC' \parallel AA'$.

We are now able to solve the proposed problems.
Solution to the 1\textsuperscript{st} problem.

Using the 3\textsuperscript{rd} Lemma, we construct the parallels $AA', BB', CC'$ with a certain directions of a diameter of the circle circumscribed to the given triangle.

We join $M$ with $A'$, $B'$, $C'$ and denote the intersection between $MA'$ and $BC$, $A_1$; $MB' \cap CA = \{B_1\}$ and $MA' \cap AV = \{C_1\}$.

According to the Aubert’s Theorem, the points $A_1, B_1, C_1$ will be collinear, and $MA'$, $MB'$, $MC'$ generate congruent angles with $BC$, $CA$ and $AB$, respectively.

Solution to the 2\textsuperscript{nd} problem.

Using the 3\textsuperscript{rd} Lemma and the remark that follows it, we construct through $A, B, C$ the parallels to $A_1B_1$; we denote by $A', B', C'$ their intersections with the circle circumscribed to the triangle $ABC$. (It is enough to build a single parallel to the transversal line $A_1B_1C_1$, for example $AA'$).

We join $A'$ with $A_1$ and denote by $M$ the intersection with the circle. The point $M$ will be the point we searched for. The construction’s proof follows from the M’Kensie Theorem.
Solution to the 3rd problem.

We suppose that $A'$ belongs to the little arc determined by the chord $\overline{BC}$ in the circle circumscribed to the triangle $ABC$.

In this case, in order to find the isogonal $AA_1$, we construct (by help of the 3rd Lemma and of the remark that follows it) the parallel $A'A_1$ to $BC$, $A_1$ being on the circumscribed circle, it is obvious that $AA'$ and $AA_1$ will be isogonal cevians.

We suppose that $A'$ belongs to the high arc determined by the chord $\overline{BC}$; we consider $A' \in \overline{AB}$ (the arc $\overline{AB}$ does not contain the point $C$). In this situation, we firstly construct the parallel $BP$ to $AA'$, $P$ belongs to the circumscribed circle, and then through $P$ we construct the parallel $PA_1$ to $AC$, $A_1$ belongs to the circumscribed circle. The isogonal of the line $AA'$ will be $AA_1$. The construction’s proof follows from 3rd Lemma and from the proof of Beltrami’s Theorem.

References.


Apollonius’s Circles
of $k^{th}$ Rank

The purpose of this article is to introduce the notion of Apollonius’s circle of $k^{th}$ rank.

1\textsuperscript{st} Definition.

It is called an internal cevian of $k^{th}$ rank the line $AA_k$ where $A_k \in (BC)$, such that $\frac{BA}{A_kC} = \left(\frac{AB}{AC}\right)^k$ ($k \in \mathbb{R}$).

If $A'_k$ is the harmonic conjugate of the point $A_k$ in relation to $B$ and $C$, we call the line $AA'_k$ an external cevian of $k^{th}$ rank.

2\textsuperscript{nd} Definition.

We call Apollonius’s circle of $k^{th}$ rank with respect to the side $BC$ of $ABC$ triangle the circle which has as diameter the segment line $A_kA'_k$.

1\textsuperscript{st} Theorem.

Apollonius’s circle of $k^{th}$ rank is the locus of points $M$ from $ABC$ triangle's plan, satisfying the relation: $\frac{MB}{MC} = \left(\frac{AB}{AC}\right)^k$.  

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Proof.

Let $O_{Ak}$ the center of the Apollonius's circle of $k^{th}$ rank relative to the side $BC$ of $ABC$ triangle (see Figure 1) and $U, V$ the points of intersection of this circle with the circle circumscribed to the triangle $ABC$. We denote by $D$ the middle of arc $BC$, and we extend $DA_k$ to intersect the circle circumscribed in $U'$. 

In $BU'C$ triangle, $U'D$ is bisector; it follows that

$$\frac{BA_k}{A_kC} = \frac{U'B}{U'C} = \left(\frac{AB}{AC}\right)^k,$$

so $U'$ belongs to the locus.

The perpendicular in $U'$ on $U'A_k$ intersects $BC$ on $A'_k$, which is the foot of the $BUC$ triangle's outer bisector, so the harmonic conjugate of $A_k$ in relation to $B$ and $C$, thus $A''_k = A'_k$.

Therefore, $U'$ is on the Apollonius's circle of rank $k$ relative to the side $BC$, hence $U' = U$.

Figure 3
Let $M$ a point that satisfies the relation from the statement; thus $\frac{MB}{MC} = \frac{BA_k}{A_kC}$; it follows – by using the reciprocal of bisector's theorem – that $MA_k$ is the internal bisector of angle $BMC$. Now let us proceed as before, taking the external bisector; it follows that $M$ belongs to the Apollonius's circle of center $O_{A_k}$. We consider now a point $M$ on this circle, and we construct $C'$ such that $\angle BNA_k \equiv \angle A_kNC'$ (thus $(NA_k$ is the internal bisector of the angle $BNC'$). Because $A_k'N \perp NA_k$, it follows that $A_k$ and $A_k'$ are harmonically conjugated with respect to $B$ and $C'$. On the other hand, the same points are harmonically conjugated with respect to $B$ and $C$; from here, it follows that $C' = C$, and we have $\frac{NB}{NC} = \frac{BA_k}{A_kC} = \left(\frac{AB}{AC}\right)^k$.

3\textsuperscript{rd} Definition.

It is called a complete quadrilateral the geometric figure obtained from a convex quadrilateral by extending the opposite sides until they intersect. A complete quadrilateral has 6 vertices, 4 sides and 3 diagonals.

2\textsuperscript{nd} Theorem.

In a complete quadrilateral, the three diagonals' middles are collinear (Gauss - 1810).
Proof.

Let $ABCDEF$ a given complete quadrilateral (see Figure 2). We denote by $H_1, H_2, H_3, H_4$ respectively the orthocenters of $ABF, ADE, CBE, CDF$ triangles, and let $A_1, B_1, F_1$ the feet of the heights of $ABF$ triangle.

As previously shown, the following relations occur: $H_1A.H_1A_1 - H_1B.H_1B_1 = H_1F.H_1F_1$; they express that the point $H_1$ has equal powers to the circles of diameters $AC, BD, EF$, because those circles contain respectively the points $A_1, B_1, F_1$, and $H_1$ is an internal point.

It is shown analogously that the points $H_2, H_3, H_4$ have equal powers to the same circles, so those points are situated on the radical axis (common to the circles), therefore the circles are part of a fascicle, as
such their centers – which are the middles of the complete quadrilateral's diagonals – are collinear.

The line formed by the middles of a complete quadrilateral's diagonals is called Gauss’s line or Gauss-Newton’s line.

**3rd Theorem.**

The Apollonius’s circle of $k^{th}$ rank of a triangle are part of a fascicle.

**Proof.**

Let $AA_k, BB_k, CC_k$ be concurrent cevians of $k^{th}$ rank and $AA'_k, BB'_k, CC'_k$ be the external cevians of $k^{th}$ rank (see Figure 3). The figure $B'_kC_kB_kC'_kA_kA'_k$ is a complete quadrilateral and 2$^{nd}$ theorem is applied.
4th Theorem.

The Apollonius’s circle of \( k^{th} \) rank of a triangle are the orthogonals of the circle circumscribed to the triangle.

Proof.

We unite \( O \) to \( D \) and \( U \) (see Figure 1), \( OD \perp BC \) and \( m\left( A_k U A_k' \right) = 90^0 \), it follows that \( U A_k' \overline{A_k} = \overline{ODA_k} = \overline{OUA_k} \).

The congruence \( U A_k' \overline{A_k} \equiv \overline{OUA_k} \) shows that \( OU \) is tangent to the Apollonius’s circle of center \( O_{A_k} \).

Analogously, it can be demonstrated for the other Apollonius’s Circle.

1st Remark.

The previous theorem indicates that the radical axis of Apollonius’s circle of \( k^{th} \) rank is the perpendicular taken from \( O \) to the line \( O_{A_k} O_{B_k} \).

5th Theorem.

The centers of Apollonius’s Circle of \( k^{th} \) rank of a triangle are situated on the trilinear polar associated to the intersection point of the cevians of \( 2k^{th} \) rank.
Proof.

From the previous theorem, it results that $OU \perp UO_{Ak}$, so $UO_{Ak}$ is an external cevian of rank 2 for $BCU$ triangle, thus an external symmedian. Henceforth, $\frac{O_{Ak}B}{O_{Ak}C} = \left(\frac{BU}{CU}\right)^2 = \left(\frac{AB}{AC}\right)^{2k}$ (the last equality occurs because $U$ belong to the Apollonius’s circle of rank $k$ associated to the vertex $A$).

6th Theorem.

The Apollonius’s circle of $k^{th}$ rank of a triangle intersects the circle circumscribed to the triangle in two points that belong to the internal and external cevians of $k+1^{th}$ rank.

Proof.

Let $U$ and $V$ points of intersection of the Apollonius’s circle of center $O_{Ak}$ with the circle circumscribed to the $ABC$ (see Figure 1). We take from $U$ and $V$ the perpendiculars $UU_1$, $UU_2$ and $VV_1$, $VV_2$ on $AB$ and $AC$ respectively. The quadrilaterals $ABVC$, $ABCU$ are inscribed, it follows the similarity of triangles $BVV_1$, $CVV_2$ and $BUU_1$, $CUU_2$, from where we get the relations:

$$\frac{BV}{CV} = \frac{VV_1}{VV_2}, \quad \frac{UB}{UC} = \frac{UU_1}{UU_2}.$$
But \( \frac{BV}{CV} = \left( \frac{AB}{AC} \right)^k \), \( \frac{UB}{UC} = \left( \frac{AB}{AC} \right)^k \), \( \frac{VV_1}{VV_2} = \left( \frac{AB}{AC} \right)^k \) and \( \frac{UU_1}{UU_2} = \left( \frac{AB}{AC} \right)^k \), relations that show that \( V \) and \( U \) belong respectively to the internal cevian and the external cevian of rank \( k + 1 \).

**4th Definition.**

If the Apollonius’s circle of \( k^{th} \) rank associated with a triangle has two common points, then we call these points isodynamic points of \( k^{th} \) rank (and we denote them \( W_k, W'_k \)).

**1st Property.**

If \( W_k, W'_k \) are isodynamic centers of \( k^{th} \) rank, then:
\[
W_k A. B C^k = W_k B. A C^k = W_k C. A B^k ;
\]
\[
W'_k A. B C^k = W'_k B. A C^k = W'_k C. A B^k .
\]
The proof of this property follows immediately from **1st Theorem**.

**2nd Remark.**

The Apollonius’s circle of \( 1^{st} \) rank is the investigated Apollonius’s circle (the bisectors are cevians of \( 1^{st} \) rank). If \( k = 2 \), the internal cevians of \( 2^{nd} \) rank are the symmedians, and the external cevians of \( 2^{nd} \) rank are the external symmedians, i.e. the tangents
in triangle’s vertices to the circumscribed circle. In this case, for the Apollonius’s circle of 2\textsuperscript{nd} rank, the 3\textit{rd} Theorem becomes:

\textbf{7\textsuperscript{th} Theorem.}

The Apollonius’s circle of 2\textsuperscript{nd} rank intersects the circumscribed circle to the triangle in two points belonging respectively to the antibisector's isogonal and to the cevian outside of it.

\textit{Proof.}

It follows from the proof of the 6\textsuperscript{th} theorem. We mention that the antibisector is isotomic to the bisector, and a cevian of 3\textsuperscript{rd} rank is isogonic to the antibisector.
References.


