The Radical Circle of Ex-Inscribed Circles of a Triangle

In this article, we prove several theorems about the radical center and the radical circle of ex-inscribed circles of a triangle and calculate the radius of the circle from vectorial considerations.

1st Theorem.

The radical center of the ex-inscribed circles of the triangle $ABC$ is the Spiecker’s point of the triangle (the center of the circle inscribed in the median triangle of the triangle $ABC$).

Proof.

We refer in the following to the notation in Figure 1. Let $I_a, I_b, I_c$ be the centers of the ex-inscribed circles of a triangle (the intersections of two external bisectors with the internal bisector of the other angle). Using tangents property taken from a point to a circle to be congruent, we calculate and find that:

$$AF_a = AE_a = BD_b = BF_b = CD_c = CE_c = p,$$

$$BD_c = BF_c = CD_b = CE_b = p - a,$$
$$CE_a = CD_a = AF_c = AE_c = p - b,$$
$$AF_b = AE_b = BF_c = BD_c = p - c.$$ 

If $A_1$ is the middle of segment $D_cD_b$, it follows that $A_1$ has equal powers to the ex-inscribed circles $(I_b)$ and $(I_c)$. Of the previously set, we obtain that $A_1$ is the middle of the side $BC$.

![Diagram](image)

Figure 1.

Also, the middles of the segments $E_bE_c$ and $F_bF_c$, which we denote $U$ and $V$, have equal powers to the circles $(I_b)$ and $(I_c)$.

The radical axis of the circles $(I_b)$, $(I_c)$ will include the points $A_1, U, V$.

Because $AE_b = AF_b$ and $AE_c = AF_c$, it follows that $AU = AY$ and we find that $\angle AUV = \frac{1}{2}\angle A$, therefore the
radical axis of the ex-inscribed circles \((F_b)\) and \((I_c)\) is the parallel taken through the middle \(A_1\) of the side \(BC\) to the bisector of the angle \(BAC\).

Denoting \(B_1\) and \(C_1\) the middles of the sides \(AC\), \(AB\), respectively, we find that the radical center of the ex-inscribed circles is the center of the circle inscribed in the median triangle \(A_1B_1C_1\) of the triangle \(ABC\).

This point, denoted \(S\), is the Spiecker’s point of the triangle \(ABC\).

2\textsuperscript{nd} Theorem.

The radical center of the inscribed circle \((I)\) and of the \(B\) – ex-inscribed and \(C\) – ex-inscribed circles of the triangle \(ABC\) is the center of the \(A_1\) – ex-inscribed circle of the median triangle \(A_1B_1C_1\), corresponding to the triangle \(ABC\).

Proof.

If \(E\) is the contact of the inscribed circle with \(AC\) and \(E_b\) is the contact of the \(B\) – ex-inscribed circle with \(AC\), it is known that these points are isotomic, therefore the middle of the segment \(EE_b\) is the middle of the side \(AC\), which is \(B_1\).

This point has equal powers to the inscribed circle \((I)\) and to the \(B\) – ex-inscribed circle \((I_b)\), so it belongs to their radical axis.
Analogously, $C_1$ is on the radical axis of the circles $(I)$ and $(I_c)$.

The radical axis of the circles $(I)$, $(I_b)$ is the perpendicular taken from $B_1$ to the bisector $II_b$.

This bisector is parallel with the internal bisector of the angle $A_1B_1C_1$, therefore the perpendicular in $B_1$ on $II_b$ is the external bisector of the angle $A_1B_1C_1$ from the median triangle.

Analogously, it follows that the radical axis of the circles $(I)$, $(I_c)$ is the external bisector of the angle $A_1C_1B_1$ from the median triangle.

Because the bisectors intersect in the center of the circle $A_1$-ex-inscribed to the median triangle $A_1B_1C_1$, this point $S_a$ is the center of the radical center of the circles $(I)$, $(I_b)$, $(I_c)$.

**Remark.**

The theorem for the circles $(I)$, $(I_a)$, $(I_b)$ and $(I)$, $(I_a)$, $(I_c)$ can be proved analogously, obtaining the points $S_c$ and $S_b$.

**3rd Theorem.**

The radical circle’s radius of the circles ex-inscribed to the triangle $ABC$ is given by the formula: 
$$\frac{1}{2}\sqrt{r^2 + p^2},$$
where $r$ is the radius of the inscribed circle.
Proof.

The position vector of the circle \( I \) of the inscribed circle in the triangle ABC is:

\[
\overrightarrow{PI} = \frac{1}{2p} (a\overrightarrow{PA} + b\overrightarrow{PB} + c\overrightarrow{PC}).
\]

Spiecker’s point \( S \) is the center of radical circle of ex-inscribed circle and is the center of the inscribed circle in the median triangle \( A_1B_1C_1 \), therefore:

\[
\overrightarrow{PS} = \frac{1}{p} \left( \frac{1}{2} a\overrightarrow{P_{A_1}} + \frac{1}{2} b\overrightarrow{P_{B_1}} + \frac{1}{2} c\overrightarrow{P_{C_1}} \right).
\]

Figure 2.

We denote by \( T \) the contact point with the \( A \)-ex-inscribed circle of the tangent taken from \( S \) to this circle (see Figure 2).
The radical circle's radius is given by:

\[ ST = \sqrt{SI_a^2 - l_a^2} \]

\[ l_a S = \frac{1}{2p} (a l_{aA_1} + b l_{aB_1} + c l_{aC_1}). \]

We evaluate the product of the scales \( l_a S \cdot l_a S \); we have:

\[ l_a S^2 = \frac{1}{4p^2} (a^2 l_{aA_1}^2 + b^2 l_{aB_1}^2 + c^2 l_{aC_1}^2 + 2ab l_{aA_1} \cdot l_{aB_1} + 2bc l_{aB_1} \cdot l_{aC_1} + 2ac l_{aA_1} \cdot l_{aC_1}). \]

From the law of cosines applied in the triangle \( l_a A_1 B_1 \), we find that:

\[ 2l_{aA_1} \cdot l_{aB_1} = l_a A_1^2 + l_a B_1^2 - \frac{1}{4} c^2, \]

therefore:

\[ 2ab l_{aA_1} \cdot l_{aB_1} = ab(l_a A_1^2 + l_a B_1^2 - \frac{1}{4} abc^2). \]

Analogously, we obtain:

\[ 2bc l_{aB_1} \cdot l_{aC_1} = bc(l_a B_1^2 + l_a C_1^2 - \frac{1}{4} a^2 bc), \]

\[ 2ac l_{aA_1} \cdot l_{aC_1} = ac(l_a A_1^2 + l_a C_1^2 - \frac{1}{4} ab^2 c). \]

\[ l_a S^2 = \frac{1}{4p^2} \left[ (a^2 + ab + ac)l_{aA_1}^2 + (b^2 + ab + bc)l_{aB_1}^2 + (c^2 + bc + ac)l_{aC_1}^2 - \frac{abc}{4} (a + b + c) \right], \]

\[ l_a S^2 = \frac{1}{4p^2} \left[ 2p(a l_{aA_1}^2 + b l_{aB_1}^2 + c l_{aC_1}^2) - 2RS_p \right], \]

\[ l_a S^2 = \frac{1}{2p} (a l_{aA_1}^2 + b l_{aB_1}^2 + c l_{aC_1}^2) - \frac{1}{2} Rr. \]

From the right triangle \( l_a D_a A_1 \), we have that:

\[ l_a A_1^2 = r_a^2 + A_1 D_a^2 = r_a^2 + \left[ \frac{a}{2} - (p - c) \right]^2 = r_a^2 + \frac{(c-b)^2}{4}. \]

From the right triangles \( l_a E_a B_1 \) și \( l_a F_a C_1 \), we find:
\[ I_a B_1^2 = r_a^2 + B_1 E_a^2 = r_a^2 + \left[ \frac{b}{2} - (p - b) \right]^2 = r_a^2 + \frac{1}{4} (a + c)^2, \]
\[ I_a C_1^2 = r_a^2 + \frac{1}{4} (a + b)^2. \]
Evaluating \( al_a A_1^2 + bl_a B_1^2 + cl_a C_1^2 \), we obtain:
\[ al_a A_1^2 + bl_a B_1^2 + cl_a C_1^2 = 2pr_a^2 + \frac{1}{2} p(ab + ac + bc) - \frac{1}{4} abc. \]
But:
\[ ab + ac + bc = r^2 + p^2 + 4Rr. \]
It follows that:
\[ \frac{1}{2p} [al_a A_1^2 + bl_a B_1^2 + cl_a C_1^2] = r_a^2 + \frac{1}{4} (r^2 + p^2) + \frac{1}{2} Rr \]
and
\[ I_a S^2 = r_a^2 + \frac{1}{4} (r^2 + p^2). \]
Then, we obtain:
\[ ST = \frac{1}{2} \sqrt{r^2 + p^2}. \]
References.
