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The Radical Circle of Ex-Inscribed Circles of a Triangle

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In this article, we prove several theorems about the **radical center** and the **radical circle of ex-inscribed circles of a triangle** and calculate the **radius of the circle from vectorial considerations**.

1st Theorem.

The radical center of the ex-inscribed circles of the triangle ABC is the Spiecker's point of the triangle (the center of the circle inscribed in the median triangle of the triangle ABC).

Proof.

We refer in the following to the notation in *Figure 1*. Let I_a, I_b, I_c be the centers of the ex-inscribed circles of a triangle (the intersections of two external bisectors with the internal bisector of the other angle). Using tangents property taken from a point to a circle to be congruent, we calculate and find that:

$$AF_a = AE_a = BD_b = BF_b = CD_c = CE_c = p,$$
$$BD_c = BF_c = CD_b = CE_b = p - a,$$

$$CE_a = CD_a = AF_c = AE_c = p - b,$$
$$AF_b = AE_b = BF_c = BD_c = p - c.$$

If A_1 is the middle of segment D_cD_b, it follows that A_1 has equal powers to the ex-inscribed circles (I_b) and (I_c). Of the previously set, we obtain that A_1 is the middle of the side BC.

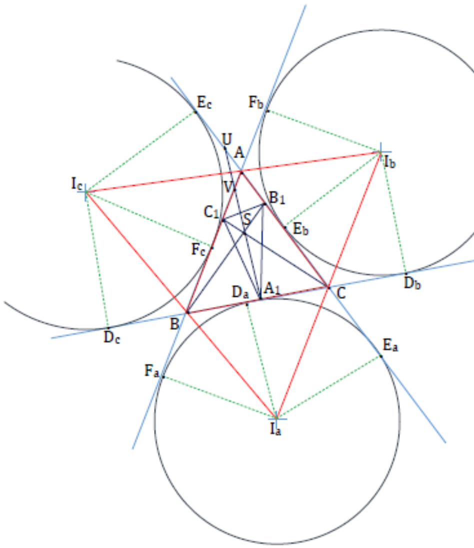

Figure 1.

Also, the middles of the segments E_bE_c and F_bF_c, which we denote U and V, have equal powers to the circles (I_b) and (I_c).

The radical axis of the circles (I_b), (I_c) will include the points A_1, U, V.

Because $AE_b = AF_b$ and $AE_c = AF_c$, it follows that $AU = AY$ and we find that $\sphericalangle AUV = \frac{1}{2}\sphericalangle A$, therefore the radical axis of the ex-inscribed circles (F_b) and (I_c) is the parallel taken through the middle A_1 of the side BC to the bisector of the angle BAC.

Denoting B_1 and C_1 the middles of the sides AC, AB, respectively, we find that the radical center of the ex-inscribed circles is the center of the circle inscribed in the median triangle $A_1B_1C_1$ of the triangle ABC.

This point, denoted S, is the Spiecker's point of the triangle ABC.

2nd Theorem.

The radical center of the inscribed circle (I) and of the B −ex-inscribed and C −ex-inscribed circles of the triangle ABC is the center of the A_1 − ex-inscribed circle of the median triangle $A_1B_1C_1$, corresponding to the triangle ABC).

Proof.

If E is the contact of the inscribed circle with AC and E_b is the contact of the B −ex-inscribed circle with AC , it is known that these points are isotomic, therefore the middle of the segment EE_b is the middle of the side AC, which is B_1.

This point has equal powers to the inscribed circle (I) and to the B −ex-inscribed circle (I_b), so it belongs to their radical axis.

Analogously, C_1 is on the radical axis of the circles (I) and (I_c).

The radical axis of the circles (I), (I_b) is the perpendicular taken from B_1 to the bisector II_b.

This bisector is parallel with the internal bisector of the angle $A_1B_1C_1$, therefore the perpendicular in B_1 on II_b is the external bisector of the angle $A_1B_1C_1$ from the median triangle.

Analogously, it follows that the radical axis of the circles (I), (I_c) is the external bisector of the angle $A_1C_1B_1$ from the median triangle.

Because the bisectors intersect in the center of the circle A_1-ex-inscribed to the median triangle $A_1B_1C_1$, this point S_a is the center of the radical center of the circles (I), (I_b), (I_c).

Remark.

The theorem for the circles (I), (I_a), (I_b) and (I), (I_a), (I_c) can be proved analogously, obtaining the points S_c and S_b.

3rd Theorem.

The radical circle's radius of the circles ex-inscribed to the triangle ABC is given by the formula: $\frac{1}{2}\sqrt{r^2+p^2}$, where r is the radius of the inscribed circle.

Proof.

The position vector of the circle I of the inscribed circle in the triangle ABC is:

$$\overrightarrow{PI} = \frac{1}{2p}\left(a\overrightarrow{PA} + b\overrightarrow{PB} + c\overrightarrow{PC}\right).$$

Spiecker's point S is the center of radical circle of ex-inscribed circle and is the center of the inscribed circle in the median triangle $A_1B_1C_1$, therefore:

$$\overrightarrow{PS} = \frac{1}{p}\left(\frac{1}{2}a\overrightarrow{PA_1} + \frac{1}{2}b\overrightarrow{PB_1} + \frac{1}{2}c\overrightarrow{PC_1}\right).$$

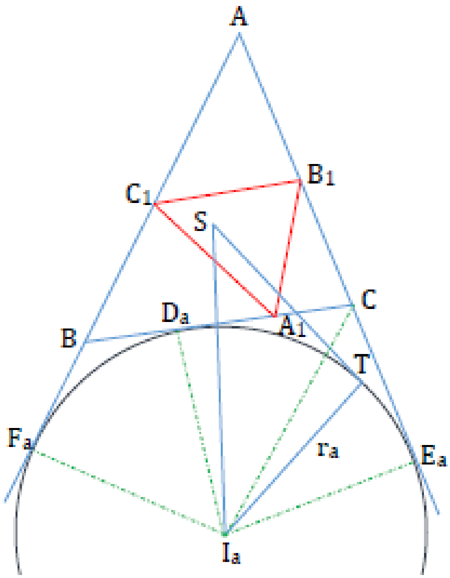

Figure 2.

We denote by T the contact point with the A-ex-inscribed circle of the tangent taken from S to this circle (see *Figure 2*).

The radical circle's radius is given by:

$ST = \sqrt{SI_a^2 - I_a^2}$

$\overrightarrow{I_aS} = \frac{1}{2p}\left(a\overrightarrow{I_aA_1} + b\overrightarrow{I_aB_1} + c\overrightarrow{I_aC_1}\right).$

We evaluate the product of the scales $\overrightarrow{I_aS} \cdot \overrightarrow{I_aS}$; we have:

$I_aS^2 = \frac{1}{4p^2}\left(a^2 I_aA_1^2 + b^2 I_aB_1^2 + c^2 I_aC_1^2 + 2ab\overrightarrow{I_aA_1} \cdot \overrightarrow{I_aB_1} + 2bc\overrightarrow{I_aB_1} \cdot \overrightarrow{I_aC_1} + 2ac\overrightarrow{I_aA_1} \cdot \overrightarrow{I_aC_1}\right).$

From the law of cosines applied in the triangle $I_aA_1B_1$, we find that:

$2\overrightarrow{I_aA_1} \cdot \overrightarrow{I_aB_1} = I_aA_1^2 + I_aB_1^2 - \frac{1}{4}c^2$, therefore:

$2ab\overrightarrow{I_aA_1} \cdot \overrightarrow{I_aB_1} = ab(I_aA_1^2 + I_aB_1^2 - \frac{1}{4}abc^2.$

Analogously, we obtain:

$2bc\overrightarrow{I_aB_1} \cdot \overrightarrow{I_aC_1} = bc(I_aB_1^2 + I_aC_1^2 - \frac{1}{4}a^2bc,$

$2ac\overrightarrow{I_aA_1} \cdot \overrightarrow{I_aC_1} = ac(I_aA_1^2 + I_aC_1^2 - \frac{1}{4}ab^2c.$

$I_aS^2 = \frac{1}{4p^2}\Big[(a^2 + ab + ac)I_aA_1^2 + (b^2 + ab + bc)I_aB_1^2 + (c^2 + bc + ac)I_aC_1^2 - \frac{abc}{4}(a + b + c)\Big],$

$I_aS^2 = \frac{1}{4p^2}\left[2p(aI_aA_1^2 + bI_aB_1^2 + cI_aC_1^2) - 2RS_p\right],$

$I_aS^2 = \frac{1}{2p}(aI_aA_1^2 + bI_aB_1^2 + cI_aC_1^2) - \frac{1}{2}Rr.$

From the right triangle $I_aD_aA_1$, we have that:

$$I_aA_1^2 = r_a^2 + A_1D_a^2 = r_a^2 + \left[\frac{a}{2} - (p - c)\right]^2 =$$

$$= r_a^2 + \frac{(c-b)^2}{4}.$$

From the right triangles $I_aE_aB_1$ și $I_aF_aC_1$, we find:

$$I_aB_1^2 = r_a^2 + B_1E_a^2 = r_a^2 + \left[\frac{b}{2} - (p-b)\right]^2 =$$

$$= r_a^2 + \frac{1}{4}(a+c)^2,$$

$$I_aC_1^2 = r_a^2 + \frac{1}{4}(a+b)^2.$$

Evaluating $aI_aA_1^2 + bI_aB_1^2 + cI_aC_1^2$, we obtain:

$$aI_aA_1^2 + bI_aB_1^2 + cI_aC_1^2 =$$

$$= 2pr_a^2 + \frac{1}{2}p(ab+ac+bc) - \frac{1}{4}abc.$$

But:

$$ab + ac + bc = r^2 + p^2 + 4Rr.$$

It follows that:

$$\frac{1}{2p}\left[aI_aA_1^2 + bI_aB_1^2 + cI_aC_1^2\right] = r_a^2 + \frac{1}{4}(r^2+p^2) + \frac{1}{2}Rr$$

and

$$I_aS^2 = r_a^2 + \frac{1}{4}(r^2+p^2).$$

Then, we obtain:

$$ST = \frac{1}{2}\sqrt{r^2+p^2}.$$

References.

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[2] I. Patrascu, F. Smarandache: *Variance on topics of plane geometry*. Columbus: The Educational Publisher, Ohio, USA, 2013.