A Generalization of the Inequality of Hölder

One generalizes the inequality of Hölder thanks to a reasoning by recurrence. As particular cases, one obtains a generalization of the inequality of Cauchy-Buniakowski-Schwartz, and some interesting applications.

**Theorem:** If $a_i^{(k)} \in \mathbb{R}_+$ and $p_k \in ]1, +\infty[ , i \in \{1, 2, \ldots, n\} , k \in \{1, 2, \ldots, m\}$, such that:

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} = 1 ,$$

then:

$$\sum_{i=1}^{n} \prod_{k=1}^{m} a_i^{(k)} \leq \left( \sum_{i=1}^{n} \left( a_i^{(k)} \right)^{p_k} \right)^{\frac{1}{p_k}} \quad \text{with} \quad m \geq 2 .$$

**Proof:**

For $m = 2$ one obtains exactly the inequality of Hölder, which is true. One supposes that the inequality is true for the values which are strictly smaller than a certain $m$.

Then,:

$$\sum_{i=1}^{n} \prod_{k=1}^{m} a_i^{(k)} = \sum_{i=1}^{n} \left( \prod_{k=1}^{m} a_i^k \right) \left( a_i^{(m-1)} \cdot a_i^{(m)} \right) \leq \prod_{k=1}^{m-2} \left( \sum_{i=1}^{n} \left( a_i^{(k)} \right)^{p_k} \right)^{\frac{1}{p_k}} \cdot \left( \sum_{i=1}^{n} \left( a_i^{(m-1)} \cdot a_i^{(m)} \right)^{p} \right)^{\frac{1}{p}}$$

where $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_{m-2}} + \frac{1}{p} = 1$ and $p_h > 1, 1 \leq h \leq m - 2 , \ p > 1$;

but

$$\sum_{i=1}^{n} \left( a_i^{(m-1)} \right)^p \cdot \left( a_i^{(m)} \right)^p \leq \left( \sum_{i=1}^{n} \left( a_i^{(m-1)} \right)^{p_{t_1}} \right)^{\frac{1}{p_{t_1}}} \cdot \left( \sum_{i=1}^{n} \left( a_i^{(m)} \right)^{p_{t_2}} \right)^{\frac{1}{p_{t_2}}}$$

where $\frac{1}{t_1} + \frac{1}{t_2} = 1$ and $t_1 > 1 , \ t_2 > 2$.

From it results that:

$$\sum_{i=1}^{n} \left( a_i^{(m-1)} \right)^p \cdot \left( a_i^{(m)} \right)^p \leq \left( \sum_{i=1}^{n} \left( a_i^{(m-1)} \right)^{p_{t_1}} \right)^{\frac{1}{p_{t_1}}} \cdot \left( \sum_{i=1}^{n} \left( a_i^{(m)} \right)^{p_{t_2}} \right)^{\frac{1}{p_{t_2}}}$$

with $\frac{1}{p t_1} + \frac{1}{p t_2} = \frac{1}{p}$.
Let us note $p_t = p_{m-1}$ and $p_t = p_m$. Then $\frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_m} = 1$ is true and one has $p_j > 1$ for $1 \leq j \leq m$ and it results the inequality from the theorem.

Note: If one poses $p_j = m$ for $1 \leq j \leq m$ and if one raises to the power $m$ this inequality, one obtains a generalization of the inequality of Cauchy-Buniakowski-Schwartz:

$$\left( \sum_{i=1}^{n} \prod_{k=1}^{m} a_{i}^{(k)} \right)^{m} \leq \prod_{k=1}^{m} \sum_{i=1}^{n} \left( a_{i}^{(k)} \right)^{m}.$$ 

Application:
Let $a_1, a_2, b_1, b_2, c_1, c_2$ be positive real numbers.
Show that:

$$(a_1 b_1 c_1 + a_2 b_2 c_2)^6 \leq 8(a_1^6 + a_2^6)(b_1^6 + b_2^6)(c_1^6 + c_2^6)$$

Solution:
We will use the previous theorem. Let us choose $p_1 = 2$, $p_2 = 3$, $p_3 = 6$; we will obtain the following:

$$a_1 b_1 c_1 + a_2 b_2 c_2 \leq (a_1^2 + a_2^2)^{\frac{1}{2}} (b_1^3 + b_2^3)^{\frac{1}{3}} (c_1^6 + c_2^6)^{\frac{1}{6}},$$

or more:

$$(a_1 b_1 c_1 + a_2 b_2 c_2)^6 \leq (a_1^2 + a_2^2)^{3} (b_1^3 + b_2^3)^{2} (c_1^6 + c_2^6),$$

and knowing that

$$(b_1^3 + b_2^3)^{2} \leq 2(b_1^6 + b_2^6)$$

and that

$$(a_1^2 + a_2^2)^{3} = a_1^6 + a_2^6 + 3(a_1^4 a_2^2 + a_1^2 a_2^4) \leq 4(a_1^6 + a_2^6)$$

since

$$a_1^4 a_2^2 + a_1^2 a_2^4 \leq a_1^6 + a_2^6$$ (because: $-(a_2^2 - a_1^2)^2 (a_1^2 + a_2^2) \leq 0$)

it results the exercise which was proposed.