Solution of the LLP Limiting case of the Problem of Apollonius via Geometric Algebra, Using Reflections and Rotations

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Abstract

This document adds to the collection of solved problems presented in [1]-[4]. After reviewing, briefly, how reflections and rotations can be expressed and manipulated via GA, it solves the LLP limiting case of the Problem of Apollonius in two ways.

Geometric-Algebra Formulas for Plane (2D) Geometry

The Geometric Product, and Relations Derived from It

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,
$\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{a}$
$\boldsymbol{b}\wedge\boldsymbol{a}=-\boldsymbol{a}\wedge\boldsymbol{b}$
$\boldsymbol{ab}=\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\wedge\boldsymbol{b}$
$\boldsymbol{ba}=\boldsymbol{b}\cdot\boldsymbol{a}+\boldsymbol{b}\wedge\boldsymbol{a}=\boldsymbol{a}\cdot\boldsymbol{b}-\boldsymbol{a}\wedge\boldsymbol{b}$

$\boldsymbol{ab}+\boldsymbol{ba}=2\boldsymbol{a}\cdot\boldsymbol{b}$
$\boldsymbol{ab}-\boldsymbol{ba}=2\boldsymbol{a}\wedge\boldsymbol{b}$
$\boldsymbol{ab}=2\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{ba}$
$\boldsymbol{ab}=2\boldsymbol{a}\wedge\boldsymbol{b}-\boldsymbol{ba}$

Definitions of Inner and Outer Products (Macdonald A. 2010 p. 101.)

The inner product
The inner product of a j-vector A and a k-vector B is
$A\cdot B=\langle AB\rangle_{k-j}$. Note that if $j>k$, then the inner product doesn't exist.
However, in such a case $B\cdot A=\langle BA\rangle_{j-k}$ does exist.

The outer product
The outer product of a j-vector A and a k-vector B is
$A\wedge B=\langle AB\rangle_{k+j}$.

Relations Involving the Outer Product and the Unit Bivector, i.

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,
$\boldsymbol{ia}=-\boldsymbol{ai}$
$\boldsymbol{a}\wedge\boldsymbol{b}=[(\boldsymbol{ai})\cdot\boldsymbol{b}]\,\boldsymbol{i}=-[\boldsymbol{a}\cdot(\boldsymbol{bi})]\,\boldsymbol{i}=-\boldsymbol{b}\wedge\boldsymbol{a}$

Equality of Multivectors

For any two multivectors $\mathcal{M}$ and $\mathcal{N}$,
$\mathcal{M}=\mathcal{N}$ if and only if for all k, $\langle\mathcal{M}\rangle_k=\langle\mathcal{N}\rangle_k$.

Formulas Derived from Projections of Vectors and Equality of Multivectors

Any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written in the form of "Fourier expansions" with respect to a third vector, $\boldsymbol{v}$:

$\boldsymbol{a}=(\boldsymbol{a}\cdot\hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}}+[\boldsymbol{a}\cdot(\hat{\boldsymbol{v}}\boldsymbol{i})]\,\hat{\boldsymbol{v}}\boldsymbol{i}$ and $\boldsymbol{b}=(\boldsymbol{b}\cdot\hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}}+[\boldsymbol{b}\cdot(\hat{\boldsymbol{v}}\boldsymbol{i})]\,\hat{\boldsymbol{v}}\boldsymbol{i}$.

Using these expansions,

$$\boldsymbol{ab}=\{(\boldsymbol{a}\cdot\hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}}+[\boldsymbol{a}\cdot(\hat{\boldsymbol{v}}\boldsymbol{i})]\,\hat{\boldsymbol{v}}\boldsymbol{i}\}\,\{(\boldsymbol{b}\cdot\hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}}+[\boldsymbol{b}\cdot(\hat{\boldsymbol{v}}\boldsymbol{i})]\,\hat{\boldsymbol{v}}\boldsymbol{i}\}$$

Equating the scalar parts of both sides of that equation,

$\boldsymbol{a} \cdot \boldsymbol{b} = [\boldsymbol{a} \cdot \hat{\boldsymbol{v}}] [\boldsymbol{b} \cdot \hat{\boldsymbol{v}}] + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})] [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]$, and
$\boldsymbol{a} \wedge \boldsymbol{b} = \{[\boldsymbol{a} \cdot \hat{\boldsymbol{v}}] [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})] - [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})] [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\} \boldsymbol{i}$.

Also, $a^2 = [\boldsymbol{a} \cdot \hat{\boldsymbol{v}}]^2 + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]^2$, and $b^2 = [\boldsymbol{b} \cdot \hat{\boldsymbol{v}}]^2 + [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]^2$.

Reflections of Vectors, Geometric Products, and Rotation operators
For any vector $\boldsymbol{a}$, the product $\hat{\boldsymbol{v}}\boldsymbol{a}\hat{\boldsymbol{v}}$ is the reflection of $\boldsymbol{a}$ with respect to the direction $\hat{\boldsymbol{v}}$.

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\hat{\boldsymbol{v}}\boldsymbol{a}\boldsymbol{b}\hat{\boldsymbol{v}} = \boldsymbol{b}\boldsymbol{a}$, and $\boldsymbol{v}\boldsymbol{a}\boldsymbol{b}\boldsymbol{v} = v^2\boldsymbol{b}\boldsymbol{a}$. Therefore, $\hat{\boldsymbol{v}}e^{\theta\boldsymbol{i}}\hat{\boldsymbol{v}} = e^{-\theta\boldsymbol{i}}$, and $\boldsymbol{v}e^{\theta\boldsymbol{i}}\boldsymbol{v} = v^2e^{-\theta\boldsymbol{i}}$.

A useful relationship that is valid only in plane geometry: $\boldsymbol{abc} = \boldsymbol{cba}$.
Here is a brief proof:

$$\begin{aligned}
\boldsymbol{abc} &= \{\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}\}\, \boldsymbol{c} \\
&= \{\boldsymbol{a} \cdot \boldsymbol{b} + [(\boldsymbol{a}\boldsymbol{i}) \cdot \boldsymbol{b}]\, \boldsymbol{i}\}\, \boldsymbol{c} \\
&= (\boldsymbol{a} \cdot \boldsymbol{b})\, \boldsymbol{c} + [(\boldsymbol{a}\boldsymbol{i}) \cdot \boldsymbol{b}]\, \boldsymbol{i}\boldsymbol{c} \\
&= \boldsymbol{c}\, (\boldsymbol{a} \cdot \boldsymbol{b}) - \boldsymbol{c}\, [(\boldsymbol{a}\boldsymbol{i}) \cdot \boldsymbol{b}]\, \boldsymbol{i} \\
&= \boldsymbol{c}\, (\boldsymbol{a} \cdot \boldsymbol{b}) + \boldsymbol{c}\, [\boldsymbol{a} \cdot (\boldsymbol{b}\boldsymbol{i})]\, \boldsymbol{i} \\
&= \boldsymbol{c}\, (\boldsymbol{b} \cdot \boldsymbol{a}) + \boldsymbol{c}\, [(\boldsymbol{b}\boldsymbol{i}) \cdot \boldsymbol{a}]\, \boldsymbol{i} \\
&= \boldsymbol{c}\, \{\boldsymbol{b} \cdot \boldsymbol{a} + [(\boldsymbol{b}\boldsymbol{i}) \cdot \boldsymbol{a}]\, \boldsymbol{i}\} \\
&= \boldsymbol{c}\, \{\boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{a}\} \\
&= \boldsymbol{cba}.
\end{aligned}$$

1 Introduction

This document adds to the collection of solved problems presented in [1]-[4]. After reviewing, briefly, how reflections and rotations can be expressed and manipulated via GA, it solves the LLP limiting case of the Problem of Apollonius in two ways.

2 A brief review of reflections and rotations in 2D GA

2.1 Review of reflections

2.1.1 Reflections of a single vector

For any two vectors $\hat{\boldsymbol{u}}$ and $\boldsymbol{v}$, the product $\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}$ is

$$\begin{aligned} \hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}} &= \{2\hat{\boldsymbol{u}} \wedge \boldsymbol{v} + \boldsymbol{v}\hat{\boldsymbol{u}}\}\,\hat{\boldsymbol{u}} & (2.1)\\ &= \boldsymbol{v} + 2\left[(\hat{\boldsymbol{u}}\boldsymbol{i}) \cdot \boldsymbol{v}\right] \boldsymbol{i}\hat{\boldsymbol{u}} & (2.2)\\ &= \boldsymbol{v} - 2\left[\boldsymbol{v} \cdot (\hat{\boldsymbol{u}}\boldsymbol{i})\right] \hat{\boldsymbol{u}}\boldsymbol{i}, & (2.3) \end{aligned}$$

which evaluates to the reflection of the reflection of $\boldsymbol{v}$ with respect to $\hat{\boldsymbol{u}}$ (Fig. 2.1).

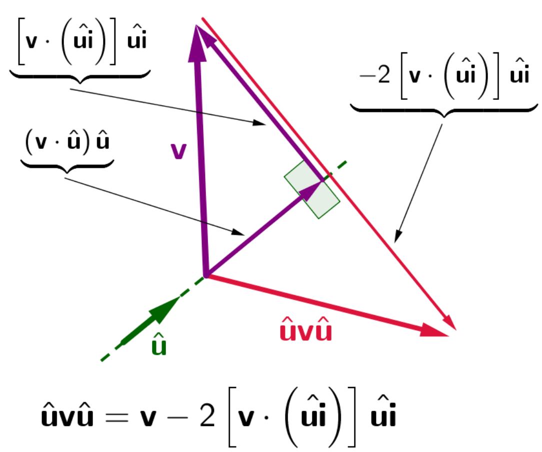

Figure 2.1: Geometric interpretation of $\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}$, showing why it evaluates to the reflection of $\boldsymbol{v}$ with respect to $\hat{\boldsymbol{u}}$.

We also note that because $\boldsymbol{u} = |\boldsymbol{u}|\,\hat{\boldsymbol{u}}$,

$$\boldsymbol{u}\boldsymbol{v}\boldsymbol{u} = u^2\,(\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}) = \boldsymbol{u}^2\boldsymbol{v} - 2\left[\boldsymbol{v} \cdot (\boldsymbol{u}\boldsymbol{i})\right] \boldsymbol{u}\boldsymbol{i}. \quad (2.4)$$

2.1.2 Reflections of a bivector, and of a geometric product of two vectors

The product $\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}}$ is

$$\begin{aligned}
\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}} &= \hat{\boldsymbol{u}}\left(\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{v}\wedge\boldsymbol{w}\right)\hat{\boldsymbol{u}} \\
&= \hat{\boldsymbol{u}}\left(\boldsymbol{v}\cdot\boldsymbol{w}\right)\hat{\boldsymbol{u}}+\hat{\boldsymbol{u}}\left(\boldsymbol{v}\wedge\boldsymbol{w}\right)\hat{\boldsymbol{u}} \\
&= \hat{\boldsymbol{u}}^2\left(\boldsymbol{v}\cdot\boldsymbol{w}\right)+\hat{\boldsymbol{u}}\left[\left(\boldsymbol{v}\boldsymbol{i}\right)\cdot\boldsymbol{w}\right]\boldsymbol{i}\hat{\boldsymbol{u}} \\
&= \boldsymbol{v}\cdot\boldsymbol{w}+\hat{\boldsymbol{u}}\left[-\boldsymbol{v}\cdot\left(\boldsymbol{w}\boldsymbol{i}\right)\right]\left(-\hat{\boldsymbol{u}}\boldsymbol{i}\right) \\
&= \boldsymbol{v}\cdot\boldsymbol{w}+\hat{\boldsymbol{u}}^2\left[\left(\boldsymbol{w}\boldsymbol{i}\right)\cdot\boldsymbol{v}\right]\boldsymbol{i} \\
&= \boldsymbol{w}\cdot\boldsymbol{v}+\boldsymbol{w}\wedge\boldsymbol{v} \\
&= \boldsymbol{w}\boldsymbol{v}.
\end{aligned}$$

In other words, the reflection of the geometric product $\boldsymbol{v}\boldsymbol{w}$ is $\boldsymbol{w}\boldsymbol{v}$, and does not depend on the direction of the vector with respect to which it is reflected. We saw that the scalar part of $\boldsymbol{v}\boldsymbol{w}$ was unaffected by the reflection, but the bivector part was reversed.

Further to that point, the reflection of geometric product of $\boldsymbol{v}$ and $\boldsymbol{w}$ is equal to the geometric product of the two vectors' reflections:

$$\begin{aligned}
\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}} &= \hat{\boldsymbol{u}}\boldsymbol{v}\left(\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}\right)\boldsymbol{w}\hat{\boldsymbol{u}} \\
&= \left(\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}\right)\left(\hat{\boldsymbol{u}}\boldsymbol{w}\hat{\boldsymbol{u}}\right).
\end{aligned}$$

That observation provides a geometric interpretation (Fig. 2.2) of why reflecting a bivector changes its sign: the direction of the turn from $\boldsymbol{v}$ to $\boldsymbol{w}$ reverses.

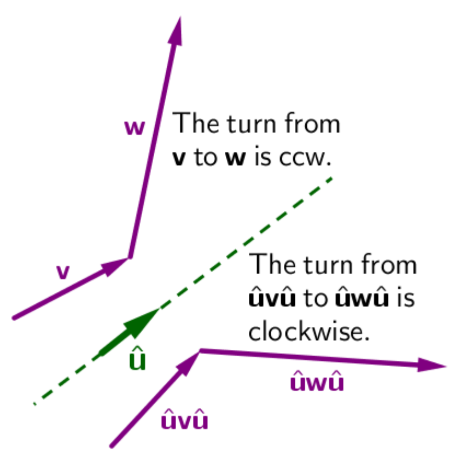

Figure 2.2: Geometric interpretation of $\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}}$, showing why it evaluates to the reflection of $\boldsymbol{v}$ with respect to $\hat{\boldsymbol{u}}$. Note that $\hat{\boldsymbol{u}}\boldsymbol{v}\boldsymbol{w}\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}\boldsymbol{v}\left(\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}\right)\boldsymbol{w}\hat{\boldsymbol{u}} = \left(\hat{\boldsymbol{u}}\boldsymbol{v}\hat{\boldsymbol{u}}\right)\left(\hat{\boldsymbol{u}}\boldsymbol{w}\hat{\boldsymbol{u}}\right)$.

2.2 Review of rotations

One of the most-important rotations—for our purposes—is the one that is produced when a vector is multiplied by the unit bivector, $\boldsymbol{i}$, for the plane: $\boldsymbol{vi}$ is $\boldsymbol{v}$'s 90-degree counter-clockwise rotation, while $\boldsymbol{iv}$ is $\boldsymbol{v}$'s 90-degree clockwise rotation.

Every geometric product $\boldsymbol{bc}$ is a rotation operator, whether we use it as such or not:

$$\boldsymbol{bc} = \|\boldsymbol{b}\|\|\boldsymbol{c}\|e^{\theta \boldsymbol{i}}.$$

where θ is the angle of rotation from $\boldsymbol{b}$ to $\boldsymbol{c}$. From that equation, we obtain the identity

$$e^{\theta \boldsymbol{i}} = \frac{\boldsymbol{bc}}{\|\boldsymbol{b}\|\|\boldsymbol{c}\|} = \left[\frac{\boldsymbol{b}}{\|\hat{\boldsymbol{b}}\|}\right]\left[\frac{\boldsymbol{c}}{\|\hat{\boldsymbol{c}}\|}\right] = \hat{\boldsymbol{b}}\hat{\boldsymbol{c}}.$$

Note that $\boldsymbol{a}\left[\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\right]$ evaluates to the rotation of $\boldsymbol{a}$ by the angle from $\hat{\boldsymbol{u}}$ to $\hat{\boldsymbol{v}}$, while $\left[\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\right]\boldsymbol{a}$ evaluates to the rotation of $\boldsymbol{a}$ by the angle from $\hat{\boldsymbol{v}}$ to $\hat{\boldsymbol{u}}$.

A useful corollary of the foregoing is that any product of an odd number of vectors evaluates to a vector, while the product of an odd number of vectors evaluates to the sum of a scalar and a bivector.

An interesting example of a rotation is the product $\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\boldsymbol{ab}\hat{\boldsymbol{v}}\hat{\boldsymbol{u}}$. Writing that product as $(\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\boldsymbol{a})(\boldsymbol{b}\hat{\boldsymbol{v}}\hat{\boldsymbol{u}})$, and recalling that any geometric product of two vectors acts as a rotation operator in 2D (2.2), we can see why $\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\boldsymbol{ab}\hat{\boldsymbol{v}}\hat{\boldsymbol{u}} = \boldsymbol{ab}$ (Fig. 2.3).

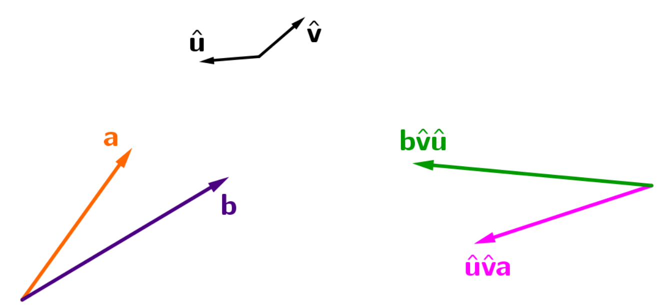

Figure 2.3: Left-multiplying $\boldsymbol{a}$ by the geometric product $\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}$ rotates $\boldsymbol{a}$ by an angle equal (in sign as well as magnitude) to that from $\hat{\boldsymbol{v}}$ to $\hat{\boldsymbol{u}}$. Right-multiplying $\boldsymbol{b}$ by the geometric product $\hat{\boldsymbol{v}}\hat{\boldsymbol{u}}$ rotates $\boldsymbol{b}$ by that same angle. The orientations of $\boldsymbol{a}$ and $\boldsymbol{b}$ with respect to each other are the same after the rotation, and the magnitudes of $\boldsymbol{a}$ and $\boldsymbol{b}$ are unaffected. Therefore, the geometric products $\boldsymbol{ab}$ and $(\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\boldsymbol{a})(\boldsymbol{b}\hat{\boldsymbol{v}}\hat{\boldsymbol{u}})$ are equal.

We can also write $\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\boldsymbol{ab}\hat{\boldsymbol{v}}\hat{\boldsymbol{u}}$ as $\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\left[\boldsymbol{ab}\right]\hat{\boldsymbol{v}}\hat{\boldsymbol{u}}$, giving it the form of a rotation of the geometric product $\boldsymbol{ab}$. Considered in this way, the result $\hat{\boldsymbol{u}}\hat{\boldsymbol{v}}\boldsymbol{ab}\hat{\boldsymbol{v}}\hat{\boldsymbol{u}} = \boldsymbol{ab}$ is a special case of an important theorem: rotations preserve geometric products [5].

3 Use of reflections and rotations to solve the LLP limiting case of the Problem of Apollonius

The Problem of Apollonius is described in detail in [3]. Its LLP limiting case, which we will solve here, is

> *"Given a point $\mathcal{P}$ between two intersecting lines, construct the circles that are tangent to both of the lines, and pass through $\mathcal{P}$"* (Fig. 3.1).

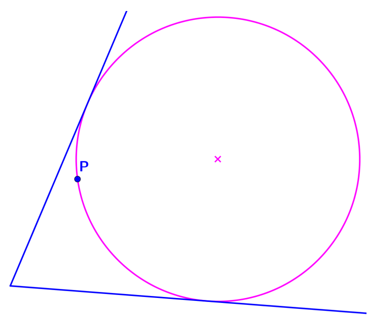

Figure 3.1: Statement of the problem: *"Given a point $\mathcal{P}$ between two intersecting lines, construct the circles that are tangent to both of the lines, and pass through $\mathcal{P}$."*

Because [1]-[4] deal at length with the "translation" of such problems into GA terms, we will go directly to the solution process here. We'll present two solution methods, in both of which we will identify the location of the center of the solution circle.

3.1 First solution method

We'll begin by identifying key elements that can be expressed in GA terms (Fig. 3.2):

The directions of the given lines are captured as $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$. As our origin, we've chosen the point of intersection of those lines. With respect to that origin, the center ($\mathcal{C}$) of the solution circle lies along the direction $\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}}$. For our convenience, we've defined a unit vector ($\hat{\boldsymbol{u}}$) in that direction. Therefore, we can write the location of $\mathcal{C}$ as a scalar multiple of $\hat{\boldsymbol{u}}$:

$$\boldsymbol{c} = \lambda \hat{\boldsymbol{u}}.$$

We then use that result to express the location of the point of tangency T:

$$\boldsymbol{t} = \lambda \left(\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{a}} \right) \hat{\boldsymbol{a}}.$$

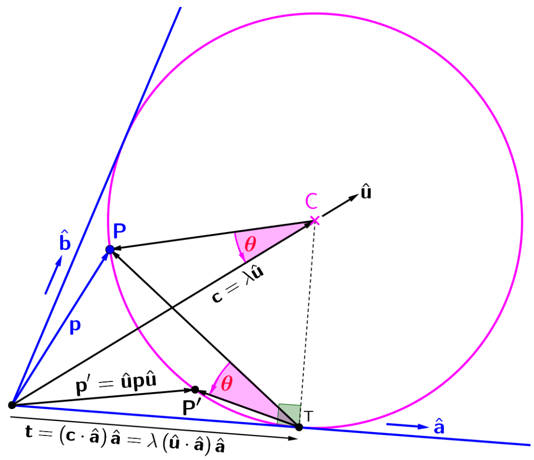

Figure 3.2: One of several possible ways to express key elements of the problem in terms of reflections and rotations, via GA. See the text for explanation.

The point $\mathcal{P}'$ is $\mathcal{P}$'s reflection with respect to $\hat{\boldsymbol{u}}$. The vector from the origin to $\mathcal{P}'$ is $\hat{\boldsymbol{u}}\boldsymbol{p}\hat{\boldsymbol{u}}$ (Section 2.1.1). Because $\mathcal{P}'$ is $\mathcal{P}$'s reflection with respect to a line through $\mathcal{C}$, $\mathcal{P}'$ belongs to the solution circle.

As discussed in [1], the angles θ are equal. Therefore, we can write (see 2.2)

$$\left[\frac{\boldsymbol{p}-\boldsymbol{t}}{\|\boldsymbol{p}-\boldsymbol{t}\|}\right]\left[\frac{\boldsymbol{p}'-\boldsymbol{t}}{\|\boldsymbol{p}'-\boldsymbol{t}\|}\right]=\left[\frac{\boldsymbol{p}-\boldsymbol{c}}{\|\boldsymbol{p}-\boldsymbol{c}\|}\right](-\hat{\boldsymbol{u}}). \tag{3.1}$$

Using ideas presented in [1]-[4], we can transform that result into

$$\hat{\boldsymbol{u}}\left[\boldsymbol{p}-\boldsymbol{c}\right]\left[\boldsymbol{p}-\boldsymbol{t}\right]\left[\boldsymbol{p}'-\boldsymbol{t}\right]=-\|\boldsymbol{p}-\boldsymbol{t}\|\|\boldsymbol{p}'-\boldsymbol{t}\|\|\boldsymbol{p}-\boldsymbol{c}\|,$$

the right side of which is a scalar. Therefore, the according to the postulate for the Equality of Multivectors,

$$\langle\hat{\boldsymbol{u}}\left[\boldsymbol{p}-\boldsymbol{c}\right]\left[\boldsymbol{p}-\boldsymbol{t}\right]\left[\boldsymbol{p}'-\boldsymbol{t}\right]\rangle_2=\langle-\|\boldsymbol{p}-\boldsymbol{t}\|\|\boldsymbol{p}'-\boldsymbol{t}\|\|\boldsymbol{p}-\boldsymbol{c}\|\rangle_2=0.$$

Now, we make the substitutions $\boldsymbol{c}=\lambda\hat{\boldsymbol{u}}$, $\boldsymbol{p}'=\hat{\boldsymbol{u}}\boldsymbol{p}\hat{\boldsymbol{u}}$, and $\boldsymbol{t}=\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}$ to obtain

$$\langle\hat{\boldsymbol{u}}\left[\boldsymbol{p}-\lambda\hat{\boldsymbol{u}}\right]\left[\boldsymbol{p}-\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}\right]\left[\hat{\boldsymbol{u}}\boldsymbol{p}\hat{\boldsymbol{u}}-\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}\right]\rangle_2=0. \tag{3.2}$$

Expanding,

$$\begin{aligned}\langle\hat{\boldsymbol{u}}p^2\hat{\boldsymbol{u}}\boldsymbol{p}\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}p^2\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}-\hat{\boldsymbol{u}}\boldsymbol{p}\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}\hat{\boldsymbol{u}}\boldsymbol{p}\hat{\boldsymbol{u}}\\+\hat{\boldsymbol{u}}\boldsymbol{p}\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}-\hat{\boldsymbol{u}}\lambda\hat{\boldsymbol{u}}\boldsymbol{p}\hat{\boldsymbol{u}}\boldsymbol{p}\hat{\boldsymbol{u}}+\hat{\boldsymbol{u}}\lambda\hat{\boldsymbol{u}}\boldsymbol{p}\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}\\+\hat{\boldsymbol{u}}\lambda\hat{\boldsymbol{u}}\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}\hat{\boldsymbol{u}}\boldsymbol{p}\hat{\boldsymbol{u}}-\hat{\boldsymbol{u}}\lambda\hat{\boldsymbol{u}}\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}\rangle_2=0.\end{aligned}$$

To simplify that equation, we note (see 2.3)that

$$\begin{aligned}\hat{\boldsymbol{u}}\boldsymbol{p}\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}\hat{\boldsymbol{u}}\boldsymbol{p}\hat{\boldsymbol{u}}&=\hat{\boldsymbol{u}}\boldsymbol{p}\left[\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}\hat{\boldsymbol{u}}\right]\boldsymbol{p}\hat{\boldsymbol{u}}\\&=\lambda p^2\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}\hat{\boldsymbol{u}}.\end{aligned}$$

We also note that

$$\hat{\boldsymbol{u}}\lambda\hat{\boldsymbol{u}}\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}}\lambda\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)\hat{\boldsymbol{a}} = \lambda^3\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)^2,$$

which is a scalar. Therefore, its bivector part is zero.

Making use of such observations to find the bivector part of the left-hand side, we obtain (after simplifying) the following quadratic equation for λ:

$$\lambda^2\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)^2\left(\hat{\boldsymbol{u}}\wedge\boldsymbol{p}\right) - \lambda[2\left(\hat{\boldsymbol{u}}\cdot\boldsymbol{p}\right)\left(\hat{\boldsymbol{u}}\wedge\boldsymbol{p}\right)] + p^2\left(\hat{\boldsymbol{u}}\wedge\boldsymbol{p}\right) = 0.$$

The product $\hat{\boldsymbol{u}}\wedge\boldsymbol{p}$ is zero only if $\boldsymbol{p}$ is a scalar multiple of $\hat{\boldsymbol{u}}$; that is, if the given point $\mathcal{P}$ lies along the centerline of the angle formed by the two given lines. We'll omit that possibility from consideration, and multiply both sides of the preceding equation by $\left(\hat{\boldsymbol{u}}\wedge\boldsymbol{p}\right)^{-1}$ to arrive at

$$\lambda^2\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)^2 - \lambda[2\left(\hat{\boldsymbol{u}}\cdot\boldsymbol{p}\right)] + p^2 = 0. \tag{3.3}$$

The two solutions to that quadratic are

$$\begin{aligned}
\lambda &= \frac{\hat{\boldsymbol{u}}\cdot\boldsymbol{p} \pm \sqrt{\left(\hat{\boldsymbol{u}}\cdot\boldsymbol{p}\right)^2 - p^2\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)^2}}{\left(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}}\right)^2} \\
&= 2\left\{\frac{\left(\hat{\boldsymbol{a}}+\hat{\boldsymbol{b}}\right)\cdot\boldsymbol{p} \pm \sqrt{\left[\left(\hat{\boldsymbol{a}}+\hat{\boldsymbol{b}}\right)\cdot\boldsymbol{p}\right]^2 - p^2\left(1+\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}}\right)^2}}{\|\hat{\boldsymbol{a}}+\hat{\boldsymbol{b}}\|\left(1+\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}}\right)}\right\} \\
&= \sqrt{2}\left[\frac{\left(\hat{\boldsymbol{a}}+\hat{\boldsymbol{b}}\right)\cdot\boldsymbol{p} \pm \left\{\left[\left(\hat{\boldsymbol{a}}+\hat{\boldsymbol{b}}\right)\cdot\boldsymbol{p}\right]^2 - p^2\left(1+\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}}\right)^2\right\}^{\frac{1}{2}}}{\left(1+\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}}\right)^{\frac{3}{2}}}\right],
\end{aligned} \tag{3.4}$$

where we've made use of

$$\hat{\boldsymbol{u}} = \frac{\hat{\boldsymbol{a}}+\hat{\boldsymbol{b}}}{\|\hat{\boldsymbol{a}}+\hat{\boldsymbol{b}}\|^2} = \frac{\hat{\boldsymbol{a}}+\hat{\boldsymbol{b}}}{2\left(1+\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}}\right)}.$$

The two solutions are shown in Fig. 3.3:

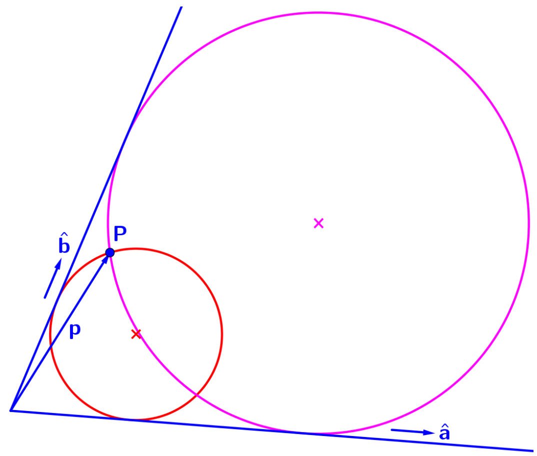

Figure 3.3: The two solution circles.

3.2 Second solution method

This second solution uses only rotations, instead of a combination of rotations and reflections. It is similar to many solutions given in [1]. Again, we'll begin by identifying key elements that can be expressed in GA terms (Fig. 3.4):

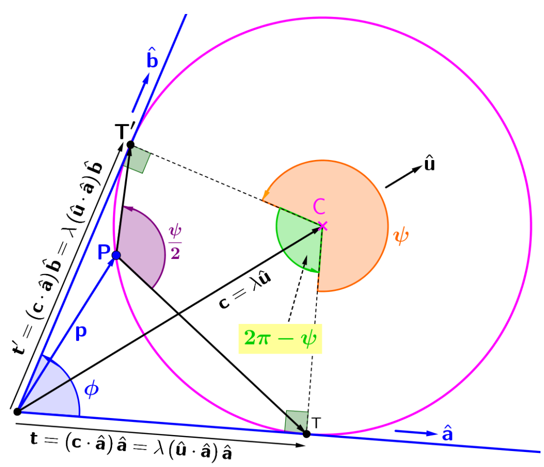

Figure 3.4: A second way to express key elements of the problem in terms of reflections and rotations, via GA. See the text for explanation.

We've made use of the fact that the two tangents drawn two a circle from any external point are equal in length, to write that $\boldsymbol{t'} = \lambda\,(\boldsymbol{\hat{u}} \cdot \boldsymbol{\hat{a}})\,\boldsymbol{\hat{b}}$. Now, we use theorems about angles and circles to write

$$\phi = \frac{1}{2}\left[\psi - (2\pi - \psi)\right]; \quad \therefore\ \frac{\psi}{2} - \frac{\phi}{2} = \frac{\pi}{2}.$$

Next, we use that relationship between ψ and ϕ to obtain an equation for λ:

$$e^{\left(\frac{\psi}{2}-\frac{\phi}{2}\right)\boldsymbol{i}} = e^{\frac{\pi}{2}\boldsymbol{i}}$$

$$\left\{e^{\frac{\psi}{2}\boldsymbol{i}}\right\}\underbrace{\left[e^{-\frac{\phi}{2}\boldsymbol{i}}\right]}_{=\hat{\boldsymbol{u}}\hat{\boldsymbol{a}}} = \boldsymbol{i}$$

$$\langle\left\{e^{\frac{\psi}{2}\boldsymbol{i}}\right\}\hat{\boldsymbol{u}}\hat{\boldsymbol{a}}\rangle_0 = \langle\boldsymbol{i}\rangle_0$$

$$\langle\left\{\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{t}'-\boldsymbol{p}}{\|\boldsymbol{t}'-\boldsymbol{p}\|}\right]\right\}\hat{\boldsymbol{u}}\hat{\boldsymbol{a}}\rangle_0 = 0$$

$$\langle[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{t}'-\boldsymbol{p}]\,\hat{\boldsymbol{u}}\hat{\boldsymbol{a}}\rangle_0 = 0$$

$$\langle[\lambda\,(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}})\,\hat{\boldsymbol{a}}-\boldsymbol{p}]\left[\lambda\,(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}})\,\hat{\boldsymbol{b}}-\boldsymbol{p}\right]\hat{\boldsymbol{u}}\hat{\boldsymbol{a}}\rangle_0 = 0. \tag{3.5}$$

From here, we would expand the left-hand side, thereby arriving at a quadratic equation in λ. We'll omit the details, but if we keep in mind—while simplifying (3.5)—that $\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{b}} = \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{a}} = 1+\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}}$, and $\hat{\boldsymbol{a}}\boldsymbol{p}+\boldsymbol{p}\hat{\boldsymbol{a}} = 2\hat{\boldsymbol{a}}\cdot\boldsymbol{p}$, then this method is arguably easier than the first.

References

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