Induction and Analogy in a Problem of Finite Sums*

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Abstract What is a general expression for

$$\sum_{k=1}^{n} k^m = 1^m + 2^m + 3^m + \cdots + n^m,$$

where m is a positive integer? Answering this question will be the aim of the paper. We assume the reader has some mathematical maturity and is familiar with the results for

$$1 + 2 + 3 + \cdots + n,$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2,$$

and

$$1^3 + 2^3 + 3^3 + \cdots + n^3$$

and wants to discover the expressions for higher powers. However, we will take the unorthodox approach of presenting the material from the point of view of someone who is trying to solve the problem himself.

Keywords analogy, Johann Faulhaber, finite sums, heuristics, inductive reasoning, number theory, George Polya, problem solving, teaching of mathematics

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†Updated on 23 July 2016 to correct typos, turn suggestions into formal exercises, and clarify the discussion in the last section of the Appendix.

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Part 1

1 Introduction

What is a general expression for

$$\sum_{k=1}^{n} k^m = 1^m + 2^m + 3^m + \cdots + n^m,$$

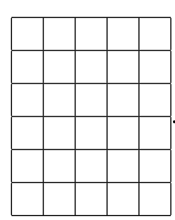
where m is a positive integer? A passage in the Preface to Volume 1 of [2] encapsulates the approach we will take to answer this question:

> Mathematics is regarded as a demonstrative science. Yet, this is only one of its aspects. Finished mathematics presented in a finished form appears as purely demonstrative, consisting of proofs only. Yet mathematics in the making resembles any other human knowledge in the making. You have to guess a mathematical theorem before you prove it; you have to guess the idea of the proof before you carry through the details. You have to combine observations and follow analogies; you have to try and try again. The result of the mathematician's creative work is demonstrative reasoning, a proof; but the proof is discovered by plausible reasoning, by guessing. If the learning of mathematics reflects to any degree the invention of mathematics, it must have a place for guessing, for plausible inference.

We start with the simplest case and a historical example.

2 A Proof from Antiquity

What is the sum of $1 + 2 + 3 + \cdots + n$? Can we write an expression for it in terms of n? What about the special case of $1 + 2 + 3 + 4 + 5$? A proof from antiquity proceeds as follows.[1]

Suppose we draw the following diagram: . There are 5 columns and 6 rows . If we think of each square as having unit area then the total area of the squares is equal to $5 \cdot 6$. Looking at the total area in a different way, we

[1] See the article "Induction and Mathematical Induction" of [1].

may divide it into $1+2+3+4+5$ squares and $1+2+3+4+5$ squares:[2]

X				
X	X			
X	X	X		
X	X	X	X	
X	X	X	X	X

. Therefore we have

$$(1+2+3+4+5)+(1+2+3+4+5)=5\cdot 6,$$

which we may rewrite as

$$1+2+3+4+5=\frac{5\cdot 6}{2}.$$

We generalize that

$$\sum_{k=1}^{n} k = 1+2+3+\cdots+n=\frac{n\,(n+1)}{2}, \tag{1}$$

which is the well-known result.

What about higher powers? What is

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\cdots+n^2\,?$$

What is

$$\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\cdots+n^3\,?$$

Let us postpone looking at these cases and turn to sums of fourth powers.

What is

$$\sum_{k=1}^{n} k^4 = 1^4+2^4+3^4+\cdots+n^4\,?$$

Can we write an expression for it in terms of n? Let us look at the particular case of

$$\sum_{k=1}^{5} k^4 = 1^4+2^4+3^4+4^4+5^4$$

and try to adopt the proof from antiquity.

[2]The diagram is not to scale.

Suppose we write

1^3	2^3	3^3	4^3	5^3
1^3	2^3	3^3	4^3	5^3
1^3	2^3	3^3	4^3	5^3
1^3	2^3	3^3	4^3	5^3
1^3	2^3	3^3	4^3	5^3
1^3	2^3	3^3	4^3	5^3

. Notice that

$$1^4 + 2^4 + 3^4 + 4^4 + 5^4 = 1 \cdot 1^3 + 2 \cdot 2^3 + 3 \cdot 3^3 + 4 \cdot 4^3 + 5 \cdot 5^3.$$

On the top part of the table we have written those terms :

1^3	2^3	3^3	4^3	5^3
	2^3	3^3	4^3	5^3
		3^3	4^3	5^3
			4^3	5^3
				5^3

. On the bottom part of the table we have written

1^3				
1^3	2^3			
1^3	2^3	3^3		
1^3	2^3	3^3	4^3	
1^3	2^3	3^3	4^3	5^3

,

which is

$$1^3 + \left(1^3 + 2^3\right) + \left(1^3 + 2^3 + 3^3\right) + \left(1^3 + 2^3 + 3^3 + 4^3\right) + \left(1^3 + 2^3 + 3^3 + 4^3 + 5^3\right).$$

We may rewrite it as

$$\sum_{k=1}^{5} \sum_{l=1}^{k} l^3.$$

Therefore the sum of the entire table is equal to

$$\sum_{k=1}^{5} k^4 + \sum_{k=1}^{5} \sum_{l=1}^{k} l^3.$$

For the next step, remember that in the original proof we wrote the sum of the area in two different ways. In this proof we can write the sum of the terms in the diagram in a second way too:

$$6 \cdot \sum_{k=1}^{5} k^3.$$

If we set the sums equal to one another then we get

$$\sum_{k=1}^{5} k^4 + \sum_{k=1}^{5} \sum_{l=1}^{k} l^3 = 6 \cdot \sum_{k=1}^{5} k^3,$$

which we may rewrite as

$$\sum_{k=1}^{5} k^{3+1} = (5+1) \cdot \sum_{k=1}^{5} k^3 - \sum_{k=1}^{5} \sum_{l=1}^{k} l^3.$$

We generalize that

$$\sum_{k=1}^{n} k^{3+1} = (n+1) \cdot \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} \sum_{l=1}^{k} l^3. \tag{2}$$

We have reduced a sum of terms each raised to the fourth power to a sum of terms each raised to the third power. How do we simplify such expressions?

3 The First Catch

Before we try to simplify expression 2 and other ones like it, let us pause to prove it in the general case. It will be easier to write it as

$$\sum_{k=1}^{n} k^{m+1} + \sum_{k=1}^{n} \sum_{l=1}^{k} l^m = (n+1) \cdot \sum_{k=1}^{n} k^m, \tag{3}$$

where m is a fixed, positive integer.

We proceed by mathematical induction. Previously we established the result for $\sum_{k=1}^{5} k^4$. Let us assume the result is true for some $n \geq 5$ and a fixed, positive integer m. Then we may write

$$\begin{aligned}
\sum_{k=1}^{n+1} k^{m+1} + \sum_{k=1}^{n+1} \sum_{l=1}^{k} l^m &= \sum_{k=1}^{n} k^{m+1} + (n+1)^{m+1} + \sum_{k=1}^{n} \sum_{l=1}^{k} l^m + \sum_{l=1}^{n+1} l^m \\
&= \sum_{k=1}^{n} k^{m+1} + \sum_{k=1}^{n} \sum_{l=1}^{k} l^m + (n+1)^{m+1} + \sum_{l=1}^{n+1} l^m \\
&= (n+1) \cdot \sum_{k=1}^{n} k^m + (n+1)^{m+1} + \sum_{l=1}^{n+1} l^m \\
&= (n+1) \cdot \left(\sum_{k=1}^{n} k^m + (n+1)^m \right) + \sum_{l=1}^{n+1} l^m \\
&= (n+1) \cdot \sum_{k=1}^{n+1} k^m + \sum_{l=1}^{n+1} l^m.
\end{aligned}$$

Notice that $\sum_{k=1}^{n+1} k^m = \sum_{l=1}^{n+1} l^m$. The same sum is expressed in two different notations. Therefore we may write

$$\sum_{k=1}^{n+1} k^{m+1} + \sum_{k=1}^{n+1} \sum_{l=1}^{k} l^m = (n+2) \cdot \sum_{k=1}^{n+1} k^m.$$

We have proved the desired expression.

4 Back to the Hunt

Earlier, we were interested in rewriting expression 2,

$$\sum_{k=1}^{n} k^{3+1} = (n+1) \cdot \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} \sum_{l=1}^{k} l^3,$$

into something simpler, something more along the lines of expression 1:

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n\,(n+1)}{2}.$$

Along the way we mentioned the sums of squares and cubes, but did not try to find expressions for them. Let us do that now.

What is

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2\,?$$

Expression 3, which we proved in Section 3, tells us

$$\sum_{k=1}^{n} k^2 = (n+1) \cdot \sum_{k=1}^{n} k - \sum_{k=1}^{n} \sum_{l=1}^{k} l. \tag{4}$$

We know also that

$$\sum_{k=1}^{n} k = \frac{n\,(n+1)}{2} = \frac{n+n^2}{2}.$$

Suppose we ***assume*** that

$$\sum_{k=1}^{n} \sum_{l=1}^{k} l = \frac{\sum_{k=1}^{n} k + \sum_{k=1}^{n} k^2}{2}.$$

Then we may rewrite expression 4 as

$$\sum_{k=1}^{n} k^2 = (n+1) \cdot \sum_{k=1}^{n} k - \frac{\sum_{k=1}^{n} k + \sum_{k=1}^{n} k^2}{2}$$

$$\sum_{k=1}^{n} k^2 = (n+1) \cdot \sum_{k=1}^{n} k - \frac{1}{2} \cdot \sum_{k=1}^{n} k - \frac{1}{2} \cdot \sum_{k=1}^{n} k^2$$

$$\frac{3}{2} \cdot \sum_{k=1}^{n} k^2 = \left(n + \frac{1}{2}\right) \cdot \sum_{k=1}^{n} k,$$

which is

$$\sum_{k=1}^{n} k^2 = \frac{2}{3} \cdot \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k,$$

which is the familiar

$$\sum_{k=1}^{n} k^2 = \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2}. \tag{5}$$

What is

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3\,?$$

Expression 3 tells us

$$\sum_{k=1}^{n} k^3 = (n+1) \cdot \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} \sum_{l=1}^{k} l^2. \tag{6}$$

We know also that

$$\sum_{k=1}^{n} k^2 = \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2} = \frac{n + 3n^2 + 2n^3}{6}.$$

Suppose we ***assume*** that

$$\sum_{k=1}^{n} \sum_{l=1}^{k} l^2 = \frac{\sum_{k=1}^{n} k + 3 \cdot \sum_{k=1}^{n} k^2 + 2 \cdot \sum_{k=1}^{n} k^3}{6}.$$

Then we may rewrite expression 6 as

$$\sum_{k=1}^{n} k^3 = (n+1) \cdot \sum_{k=1}^{n} k^2 - \frac{\sum_{k=1}^{n} k + 3 \cdot \sum_{k=1}^{n} k^2 + 2 \cdot \sum_{k=1}^{n} k^3}{6}$$

$$\sum_{k=1}^{n} k^3 = (n+1) \cdot \sum_{k=1}^{n} k^2 - \frac{1}{6} \cdot \sum_{k=1}^{n} k - \frac{1}{2} \cdot \sum_{k=1}^{n} k^2 - \frac{1}{3} \cdot \sum_{k=1}^{n} k^3$$

$$\frac{4}{3} \cdot \sum_{k=1}^{n} k^3 = \left(n + \frac{1}{2}\right) \cdot \sum_{k=1}^{n} k^2 - \frac{1}{6} \cdot \sum_{k=1}^{n} k$$

$$\sum_{k=1}^{n} k^3 = \frac{3}{4} \cdot \frac{2n+1}{2} \cdot \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2} - \frac{3}{24} \cdot \frac{n\,(n+1)}{2}$$

$$= \frac{(2n+1)^2}{8} \cdot \frac{n(n+1)}{2} - \frac{1}{8} \cdot \frac{n(n+1)}{2}$$

$$= \frac{1}{8} \cdot \left((2n+1)^2 - 1\right) \cdot \frac{n(n+1)}{2},$$

which is

$$\sum_{k=1}^{n} k^3 = \frac{1}{8} \cdot \left(4n^2 + 4n\right) \cdot \frac{n(n+1)}{2},$$

which is the familiar

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2. \tag{7}$$

What's going on?

5 An Explanation

Let us look at the assumptions we made in Section 4. Consider the expression for the sum of squares:

$$\sum_{k=1}^{n} k^2 = \frac{2n+1}{3} \cdot \frac{n(n+1)}{2} = \frac{n + 3n^2 + 2n^3}{6}.$$

Suppose we rewrite $\frac{n+3n^2+2n^3}{6}$ as $\frac{1}{6} \cdot n + \frac{3}{6} \cdot n^2 + \frac{2}{6} \cdot n^3$. In an analogous fashion we may write

$$\sum_{l=1}^{k} l^2 = \frac{1}{6} \cdot k + \frac{3}{6} \cdot k^2 + \frac{2}{6} \cdot k^3,$$

which implies

$$\sum_{k=1}^{n} \sum_{l=1}^{k} l^2 = \sum_{k=1}^{n} \left(\frac{1}{6} \cdot k + \frac{3}{6} \cdot k^2 + \frac{2}{6} \cdot k^3\right)$$

$$= \frac{1}{6} \cdot \sum_{k=1}^{n} k + \frac{3}{6} \cdot \sum_{k=1}^{n} k^2 + \frac{2}{6} \cdot \sum_{k=1}^{n} k^3$$

$$= \frac{\sum_{k=1}^{n} k + 3 \cdot \sum_{k=1}^{n} k^2 + 2 \cdot \sum_{k=1}^{n} k^3}{6},$$

which is analogous to $\sum_{k=1}^{n} k^2 = \frac{n+3n^2+2n^3}{6}$.

We have justified the previous, unproven steps. More important, if we have an expression for $\sum_{l=1}^{k} l^m$ then we can derive an expression for $\sum_{k=1}^{n} \sum_{l=1}^{k} l^m$.

6 Now for the Big Game

Let us put together our previous work to derive expressions we might not have seen before. What is

$$\sum_{k=1}^{n} k^4 = 1^4 + 2^4 + 3^4 + \cdots + n^4\,?$$

Expression 3 tells us

$$\sum_{k=1}^{n} k^4 = (n+1) \cdot \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} \sum_{l=1}^{k} l^3. \tag{8}$$

We know also that

$$\sum_{k=1}^{n} k^3 = \left(\frac{n\,(n+1)}{2} \right)^2 = \frac{n^2 + 2n^3 + n^4}{4}.$$

The discussion in Section 5 tells us

$$\sum_{k=1}^{n} \sum_{l=1}^{k} l^3 = \frac{\sum_{k=1}^{n} k^2 + 2 \cdot \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^4}{4}.$$

Therefore we may rewrite expression 8 as

$$\sum_{k=1}^{n} k^4 = (n+1) \cdot \sum_{k=1}^{n} k^3 - \frac{\sum_{k=1}^{n} k^2 + 2 \cdot \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^4}{4}$$

$$\sum_{k=1}^{n} k^4 = (n+1) \cdot \sum_{k=1}^{n} k^3 - \frac{1}{4} \cdot \sum_{k=1}^{n} k^2 - \frac{1}{2} \cdot \sum_{k=1}^{n} k^3 - \frac{1}{4} \cdot \sum_{k=1}^{n} k^4$$

$$\frac{5}{4} \cdot \sum_{k=1}^{n} k^4 = \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k^3 - \frac{1}{4} \cdot \sum_{k=1}^{n} k^2$$

$$\sum_{k=1}^{n} k^4 = \frac{4}{5} \cdot \frac{2n+1}{2} \cdot \left(\frac{n\,(n+1)}{2} \right)^2 - \frac{4}{5} \cdot \frac{1}{4} \cdot \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2}$$

$$= \frac{3}{3} \cdot \frac{4}{5} \cdot \frac{2n+1}{2} \cdot \left(\frac{n\,(n+1)}{2} \right)^2 - \frac{1}{5} \cdot \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2}$$

$$= \frac{2n+1}{3} \cdot \frac{12}{10} \cdot \left(\frac{n\,(n+1)}{2} \right)^2 - \frac{1}{5} \cdot \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2},$$

which is a less familiar

$$\sum_{k=1}^{n} k^4 = \frac{2n+1}{3} \cdot \left(\frac{6}{5} \cdot \left(\frac{n\,(n+1)}{2} \right)^2 - \frac{1}{5} \cdot \frac{n\,(n+1)}{2} \right)$$

or

$$\sum_{k=1}^{n} k^4 = \frac{1}{5} \cdot \left(6 \cdot \frac{n\,(n+1)}{2} - 1\right) \cdot \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2}$$

or

$$\sum_{k=1}^{n} k^4 = \frac{3n\,(n+1)-1}{5} \cdot \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2}, \tag{9}$$

which simplifies to

$$\sum_{k=1}^{n} k^4 = \frac{-n+10n^3+15n^4+6n^5}{30}.$$

What is

$$\sum_{k=1}^{n} k^5 = 1^5 + 2^5 + 3^5 + \cdots + n^5\,?$$

Expression 3 tells us

$$\sum_{k=1}^{n} k^5 = (n+1) \cdot \sum_{k=1}^{n} k^4 - \sum_{k=1}^{n} \sum_{l=1}^{k} l^4. \tag{10}$$

We just derived

$$\sum_{k=1}^{n} k^4 = \frac{-n+10n^3+15n^4+6n^5}{30}.$$

The discussion in Section 5 tells us

$$\sum_{k=1}^{n} \sum_{l=1}^{k} k^4 = \frac{-\sum_{k=1}^{n} k + 10 \cdot \sum_{k=1}^{n} k^3 + 15 \cdot \sum_{k=1}^{n} k^4 + 6 \cdot \sum_{k=1}^{n} k^5}{30}.$$

Therefore we may rewrite expression 10 as

$$\sum_{k=1}^{n} k^5 = (n+1) \cdot \sum_{k=1}^{n} k^4 - \frac{-\sum_{k=1}^{n} k + 10 \cdot \sum_{k=1}^{n} k^3 + 15 \cdot \sum_{k=1}^{n} k^4 + 6 \cdot \sum_{k=1}^{n} k^5}{30}$$

$$\sum_{k=1}^{n} k^5 = (n+1) \cdot \sum_{k=1}^{n} k^4 + \frac{1}{30} \cdot \sum_{k=1}^{n} k - \frac{1}{3} \cdot \sum_{k=1}^{n} k^3 - \frac{1}{2} \cdot \sum_{k=1}^{n} k^4 - \frac{1}{5} \cdot \sum_{k=1}^{n} k^5$$

$$\frac{6}{5} \cdot \sum_{k=1}^{n} k^5 = \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k^4 + \frac{1}{30} \cdot \sum_{k=1}^{n} k - \frac{1}{3} \cdot \sum_{k=1}^{n} k^3.$$

For $\sum_{k=1}^{n} k^4$ we substitute

$$\frac{2n+1}{3} \cdot \left(\frac{6}{5} \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - \frac{1}{5} \cdot \frac{n\,(n+1)}{2}\right)$$

and rewrite the expression as

$$\begin{aligned}\sum_{k=1}^{n} k^5 &= \frac{5}{6}\cdot\frac{2n+1}{2}\cdot\frac{2n+1}{3}\cdot\left(\frac{6}{5}\cdot\left(\frac{n\,(n+1)}{2}\right)^2-\frac{1}{5}\cdot\frac{n\,(n+1)}{2}\right)\\ &\quad+\frac{5}{6}\cdot\frac{1}{30}\cdot\frac{n\,(n+1)}{2}-\frac{5}{6}\cdot\frac{1}{3}\cdot\left(\frac{n\,(n+1)}{2}\right)^2.\end{aligned}$$

For the case of $\sum_{k=1}^{n} k^3$ we rounded up terms of the form $(2n+1)$. We do the same here:

$$\begin{aligned}\sum_{k=1}^{n} k^5 &= (2n+1)^2\cdot\frac{5}{6}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{6}{5}\cdot\left(\frac{n\,(n+1)}{2}\right)^2-(2n+1)^2\cdot\frac{5}{6}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{5}\cdot\frac{n\,(n+1)}{2}\\ &\quad+\frac{1}{36}\cdot\frac{n\,(n+1)}{2}-\frac{5}{18}\cdot\left(\frac{n\,(n+1)}{2}\right)^2\end{aligned}$$

$$=\frac{(2n+1)^2}{6}\cdot\left(\frac{n\,(n+1)}{2}\right)^2-\frac{(2n+1)^2}{36}\cdot\frac{n\,(n+1)}{2}+\frac{1}{36}\cdot\frac{n\,(n+1)}{2}-\frac{5}{18}\cdot\left(\frac{n\,(n+1)}{2}\right)^2$$

$$=\left(\frac{(2n+1)^2}{6}-\frac{5}{18}\right)\cdot\left(\frac{n\,(n+1)}{2}\right)^2+\left(\frac{1}{36}-\frac{(2n+1)^2}{36}\right)\cdot\frac{n\,(n+1)}{2},$$

which is

$$\sum_{k=1}^{n} k^5=\frac{6n\,(n+1)-1}{9}\cdot\left(\frac{n\,(n+1)}{2}\right)^2-\frac{2n\,(n+1)}{18}\cdot\frac{n\,(n+1)}{2}.$$

We may rewrite it further as

$$\sum_{k=1}^{n} k^5=\frac{6n\,(n+1)-1}{9}\cdot\left(\frac{n\,(n+1)}{2}\right)^2-\frac{2}{9}\cdot\frac{n\,(n+1)}{2}\cdot\frac{n\,(n+1)}{2}$$

$$\sum_{k=1}^{n} k^5=\left(\frac{6n\,(n+1)-1}{9}-\frac{2}{9}\right)\cdot\left(\frac{n\,(n+1)}{2}\right)^2$$

$$\sum_{k=1}^{n} k^5=\frac{1}{3}\cdot\left(4\cdot\frac{n\,(n+1)}{2}-1\right)\cdot\left(\frac{n\,(n+1)}{2}\right)^2$$

or

$$\sum_{k=1}^{n} k^5=\frac{2n\,(n+1)-1}{3}\cdot\left(\frac{n\,(n+1)}{2}\right)^2, \tag{11}$$

which simplifies to

$$\sum_{k=1}^{n} k^5=\frac{-n^2+5n^4+6n^5+2n^6}{12}.$$

7 Summary of Part 1

Using the well-known result of

$$\sum_{k=1}^{n} k = \frac{n\,(n+1)}{2}$$

and an analogy of the method from a proof from antiquity we discovered

$$\sum_{k=1}^{n} k^{m+1} + \sum_{k=1}^{n}\sum_{l=1}^{k} l^{m} = (n+1)\cdot\sum_{k=1}^{n} k^{m},$$

which we proved rigorously. We used the new result to derive

$$\sum_{k=1}^{n} k^{2} = \frac{2n+1}{3}\cdot\frac{n\,(n+1)}{2}$$

$$\sum_{k=1}^{n} k^{3} = \left(\frac{n\,(n+1)}{2}\right)^{2}$$

$$\begin{aligned}\sum_{k=1}^{n} k^{4} &= \frac{3n\,(n+1)-1}{5}\cdot\frac{2n+1}{3}\cdot\frac{n\,(n+1)}{2}\\ &= \frac{1}{5}\cdot\left(6\cdot\frac{n\,(n+1)}{2}-1\right)\cdot\frac{2n+1}{3}\cdot\frac{n\,(n+1)}{2}\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{n} k^{5} &= \frac{2n\,(n+1)-1}{3}\cdot\left(\frac{n\,(n+1)}{2}\right)^{2}\\ &= \frac{1}{3}\cdot\left(4\cdot\frac{n\,(n+1)}{2}-1\right)\cdot\left(\frac{n\,(n+1)}{2}\right)^{2}.\end{aligned}$$

The approach can be continued to derive expressions for $\sum_{k=1}^{n} k^{6}$, $\sum_{k=1}^{n} k^{7}$, and $\sum_{k=1}^{n} k^{223}$. In other words, given an expression for $\sum_{k=1}^{n} k^{m}$, we can derive expressions for

$$\sum_{k=1}^{n} k^{m+1}, \sum_{k=1}^{n} k^{m+2}, \ldots.$$

What remains to do is, for any m, to find a general expression for $\sum_{k=1}^{n} k^{m}$.

Part 2

8 Emerging Patterns

Do we notice any patterns in the expressions we derived? Let us look at them again:

$$\sum_{k=1}^{n} k^2 = \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n\,(n+1)}{2}\right)^2$$

$$\sum_{k=1}^{n} k^4 = \frac{3n\,(n+1)-1}{5} \cdot \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2}$$

$$\sum_{k=1}^{n} k^5 = \frac{2n\,(n+1)-1}{3} \cdot \left(\frac{n\,(n+1)}{2}\right)^2.$$

We see that $\frac{3n(n+1)-1}{5}$ is the coefficient for $\sum_{k=1}^{n} k^4$ and that $\frac{2n(n+1)-1}{3}$ is the coefficient for $\sum_{k=1}^{n} k^5$. We believe that $\sum_{k=1}^{n} k^6$ and $\sum_{k=1}^{n} k^7$ will have analogous coefficients, and that if we place all of the expressions together then we will be able to discern a general pattern.

We said "coefficients." Coefficients for what?[3] Let us rewrite the expressions as follows:

$$\sum_{k=1}^{n} k^2 = \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n\,(n+1)}{2}\right)^2$$

$$\sum_{k=1}^{n} k^4 = \frac{3n\,(n+1)-1}{5} \cdot \sum_{k=1}^{n} k^2$$

$$\sum_{k=1}^{n} k^5 = \frac{2n\,(n+1)-1}{3} \cdot \sum_{k=1}^{n} k^3.$$

We see that $\sum_{k=1}^{n} k^2$ appears in the expression for $\sum_{k=1}^{n} k^4$ and that $\sum_{k=1}^{n} k^3$ appears in the expression for $\sum_{k=1}^{n} k^5$. Is it possible that

$$\sum_{k=1}^{n} k^6 = E_6(n) \cdot \sum_{k=1}^{n} k^2$$

[3] By "coefficient" we mean "leading term," not necessarily a constant factor.

and

$$\sum_{k=1}^{n} k^7 = O_7(n) \cdot \sum_{k=1}^{n} k^3,$$

where $E_6(n)$ and $O_7(n)$ are yet-to-be-determined coefficients for $\sum_{k=1}^{n} k^6$ and $\sum_{k=1}^{n} k^7$? Let us find out.[4]

8.1 $\sum_{k=1}^{n} k^6$

What is

$$\sum_{k=1}^{n} k^6 = 1^6 + 2^6 + 3^6 + \cdots + n^6\,?$$

We know that

$$\sum_{k=1}^{n} k^6 = (n+1) \cdot \sum_{k=1}^{n} k^5 - \sum_{k=1}^{n} \sum_{l=1}^{k} l^5. \tag{12}$$

We know also that

$$\sum_{k=1}^{n} k^4 = \frac{3n\,(n+1)-1}{5} \cdot \sum_{k=1}^{n} k^2$$

and

$$\begin{aligned} \sum_{k=1}^{n} k^5 &= \frac{2n\,(n+1)-1}{3} \cdot \sum_{k=1}^{n} k^3 \\ &= \frac{-n^2 + 5n^4 + 6n^5 + 2n^6}{12}, \end{aligned}$$

which implies

$$\sum_{k=1}^{n} \sum_{l=1}^{k} l^5 = \frac{-\sum_{k=1}^{n} k^2 + 5 \cdot \sum_{k=1}^{n} k^4 + 6 \cdot \sum_{k=1}^{n} k^5 + 2 \cdot \sum_{k=1}^{n} k^6}{12}.$$

Therefore we may rewrite expression 12 as

$$\begin{aligned} \sum_{k=1}^{n} k^6 &= (n+1) \cdot \sum_{k=1}^{n} k^5 - \frac{-\sum_{k=1}^{n} k^2 + 5 \cdot \sum_{k=1}^{n} k^4 + 6 \cdot \sum_{k=1}^{n} k^5 + 2 \cdot \sum_{k=1}^{n} k^6}{12} \\ &= (n+1) \cdot \sum_{k=1}^{n} k^5 + \frac{1}{12} \cdot \sum_{k=1}^{n} k^2 - \frac{5}{12} \cdot \sum_{k=1}^{n} k^4 - \frac{1}{2} \cdot \sum_{k=1}^{n} k^5 - \frac{1}{6} \cdot \sum_{k=1}^{n} k^6 \\ \frac{7}{6} \cdot \sum_{k=1}^{n} k^6 &= \left(n + \frac{1}{2}\right) \cdot \sum_{k=1}^{n} k^5 + \frac{1}{12} \cdot \sum_{k=1}^{n} k^2 - \frac{5}{12} \cdot \sum_{k=1}^{n} k^4 \end{aligned}$$

[4] E stands for "even" and O stands for "odd." We distinguish a pattern between even and odd powers.

$$\sum_{k=1}^{n} k^6 = \frac{6}{7} \cdot \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k^5 + \frac{1}{14} \cdot \sum_{k=1}^{n} k^2 - \frac{5}{14} \cdot \sum_{k=1}^{n} k^4.$$

For $\sum_{k=1}^{n} k^4$ we substitute

$$\frac{3n(n+1)-1}{5} \cdot \sum_{k=1}^{n} k^2$$

and rewrite the expression as

$$\begin{aligned}
\sum_{k=1}^{n} k^6 &= \frac{6}{7} \cdot \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k^5 + \frac{1}{14} \cdot \sum_{k=1}^{n} k^2 - \frac{5}{14} \cdot \frac{3n(n+1)-1}{5} \cdot \sum_{k=1}^{n} k^2 \\
&= \frac{6}{7} \cdot \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k^5 + \left(\frac{1}{14} - \frac{5}{14} \cdot \frac{3n(n+1)-1}{5}\right) \cdot \sum_{k=1}^{n} k^2 \\
&= \frac{6}{7} \cdot \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k^5 + \left(\frac{-3n(n+1)+2}{14}\right) \cdot \sum_{k=1}^{n} k^2.
\end{aligned}$$

For $\sum_{k=1}^{n} k^5$ we substitute

$$\frac{2n(n+1)-1}{3} \cdot \sum_{k=1}^{n} k^3$$

and rewrite the left side of the expression as

$$\begin{aligned}
\frac{6}{7} \cdot \frac{2n+1}{2} \cdot \frac{2n(n+1)-1}{3} \cdot \sum_{k=1}^{n} k^3 &= \frac{6}{7} \cdot \frac{2n+1}{3} \cdot \frac{2n(n+1)-1}{2} \cdot \sum_{k=1}^{n} k^3 \\
&= \frac{6}{7} \cdot \frac{2n+1}{3} \cdot \frac{2n(n+1)-1}{2} \cdot \frac{n(n+1)}{2} \cdot \frac{n(n+1)}{2} \\
&= \frac{6}{7} \cdot \frac{2n(n+1)-1}{2} \cdot \frac{n(n+1)}{2} \cdot \frac{2n+1}{3} \cdot \frac{n(n+1)}{2} \\
&= \frac{6}{7} \cdot \frac{2n(n+1)-1}{2} \cdot \frac{n(n+1)}{2} \cdot \sum_{k=1}^{n} k^2.
\end{aligned}$$

Together we have

$$\begin{aligned}
\sum_{k=1}^{n} k^6 &= \frac{6}{7} \cdot \frac{2n(n+1)-1}{2} \cdot \frac{n(n+1)}{2} \cdot \sum_{k=1}^{n} k^2 + \left(\frac{-3n(n+1)+2}{14}\right) \cdot \sum_{k=1}^{n} k^2 \\
&= \frac{3(2n(n+1)-1) \cdot n(n+1)}{14} \cdot \sum_{k=1}^{n} k^2 + \left(\frac{-3n(n+1)+2}{14}\right) \cdot \sum_{k=1}^{n} k^2
\end{aligned}$$

$$= \left(\frac{3\left(2n\left(n+1\right)-1\right)\cdot n\left(n+1\right)}{14} + \frac{-3n\left(n+1\right)+2}{14}\right)\cdot\sum_{k=1}^{n}k^2$$

$$= \left(\frac{6\left(n\left(n+1\right)\right)^2-3n\left(n+1\right)-3n\left(n+1\right)+2}{14}\right)\cdot\sum_{k=1}^{n}k^2,$$

which is

$$\sum_{k=1}^{n}k^6 = \left(\frac{3\left(n\left(n+1\right)\right)^2-3n\left(n+1\right)+1}{7}\right)\cdot\sum_{k=1}^{n}k^2. \tag{13}$$

Success! We wrote $\sum_{k=1}^{n}k^6$ as

$$\sum_{k=1}^{n}k^6 = E_6(n)\cdot\sum_{k=1}^{n}k^2,$$

where

$$E_6(n) = \left(\frac{3\left(n\left(n+1\right)\right)^2-3n\left(n+1\right)+1}{7}\right).$$

Another way to write expression 13 is

$$\sum_{k=1}^{n}k^6 = \frac{1}{7}\cdot\left(12\cdot\left(\frac{n\left(n+1\right)}{2}\right)^2-6\cdot\frac{n\left(n+1\right)}{2}+1\right)\cdot\sum_{k=1}^{n}k^2. \tag{14}$$

Both expressions simplify to

$$\sum_{k=1}^{n}k^6 = \frac{n-7n^3+21n^5+21n^6+6n^7}{42}.$$

9 Success

Let us look at our recent success. We had a hunch we were going to find an expression like

$$\sum_{k=1}^{n}k^6 = E_6(n)\cdot\sum_{k=1}^{n}k^2,$$

and we actually were able to find

$$\sum_{k=1}^{n}k^6 = \left(\frac{3\left(n\left(n+1\right)\right)^2-3n\left(n+1\right)+1}{7}\right)\cdot\sum_{k=1}^{n}k^2.$$

What about $\sum_{k=1}^{n}k^7$? Will it turn out the same way? The calculation in Section 24.1 of the Appendix tells us

$$\sum_{k=1}^{n}k^7 = O_7(n)\cdot\sum_{k=1}^{n}k^3, \tag{15}$$

where

$$O_7(n) = \left(\frac{3(n(n+1))^2 - 4n(n+1) + 2}{6}\right).$$

The same pattern has appeared again. Do we have enough information to figure out the general case? Let us place all of our results together:

$$\sum_{k=1}^{n} k^2 = \frac{2n+1}{3} \cdot \sum_{k=1}^{n} k$$

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$

$$\sum_{k=1}^{n} k^4 = \frac{3n\,(n+1) - 1}{5} \cdot \sum_{k=1}^{n} k^2$$

$$\sum_{k=1}^{n} k^5 = \frac{2n\,(n+1) - 1}{3} \cdot \sum_{k=1}^{n} k^3$$

$$\sum_{k=1}^{n} k^6 = \left(\frac{3\,(n\,(n+1))^2 - 3n\,(n+1) + 1}{7}\right) \cdot \sum_{k=1}^{n} k^2$$

$$\sum_{k=1}^{n} k^7 = \left(\frac{3(n(n+1))^2 - 4n(n+1) + 2}{6}\right) \cdot \sum_{k=1}^{n} k^3.$$

Can we guess general expressions for $E_{2m}(n)$ and $O_{2m+1}(n)$?

10 Refining Our Approach

A general pattern still remains a mystery. Perhaps it will help to calculate $\sum_{k=1}^{n} k^8$ and $\sum_{k=1}^{n} k^9$. Unfortunately, we have reached the limits of our approach. We need to modify the way we carry out the calculations.

From the work we have done, we know the major stumbling point in the calculations has been starting with an expression like

$$\sum_{k=1}^{n} k^6 = (n+1) \cdot \sum_{k=1}^{n} k^5 - \frac{-\sum_{k=1}^{n} k^2 + 5 \cdot \sum_{k=1}^{n} k^4 + 6 \cdot \sum_{k=1}^{n} k^5 + 2 \cdot \sum_{k=1}^{n} k^6}{12},$$

for example. Even with looking for a pattern in $\sum_{k=1}^{n} k^2$ or $\sum_{k=1}^{n} k^3$, unpacking the terms on the right side has become too unwieldy. The difficulty is, for each term in the numerator we have been introducing its coefficients from the start and then trying to simplify them before moving onto the next term. Let us try something else.

10.1 $\sum_{k=1}^{n} k^8$

What is

$$\sum_{k=1}^{n} k^8 = 1^8 + 2^8 + 3^8 + \cdots + n^8\,?$$

Expression 3, of Section 3, tells us

$$\sum_{k=1}^{n} k^8 = (n+1)\cdot\sum_{k=1}^{n} k^7 - \sum_{k=1}^{n}\sum_{l=1}^{k} l^7. \tag{16}$$

The calculations for $\sum_{k=1}^{n} k^7$ in Section 24.1 of the Appendix tell us

$$\sum_{k=1}^{n} k^7 = \frac{2n^2 - 7n^4 + 14n^6 + 12n^7 + 3n^8}{24},$$

which implies

$$\sum_{k=1}^{n}\sum_{l=1}^{k} l^7 = \frac{2\cdot\sum_{k=1}^{n} k^2 - 7\cdot\sum_{k=1}^{n} k^4 + 14\cdot\sum_{k=1}^{n} k^6 + 12\cdot\sum_{k=1}^{n} k^7 + 3\cdot\sum_{k=1}^{n} k^8}{24}.$$

Therefore we may rewrite expression 16 as

$$\sum_{k=1}^{n} k^8 = (n+1)\cdot\sum_{k=1}^{n} k^7 - \frac{2\cdot\sum_{k=1}^{n} k^2 - 7\cdot\sum_{k=1}^{n} k^4 + 14\cdot\sum_{k=1}^{n} k^6 + 12\cdot\sum_{k=1}^{n} k^7 + 3\cdot\sum_{k=1}^{n} k^8}{24}.$$

At this point we will proceed differently. In the previous calculations we never adjusted the indices of the sums. Therefore we will start instead with

$$\sum k^8 = (n+1)\cdot\sum k^7 - \frac{2\cdot\sum k^2 - 7\cdot\sum k^4 + 14\cdot\sum k^6 + 12\cdot\sum k^7 + 3\cdot\sum k^8}{24}.$$

We rewrite it as

$$\sum k^8 = (n+1)\cdot\sum k^7 - \frac{1}{12}\cdot\sum k^2 + \frac{7}{24}\cdot\sum k^4 - \frac{7}{12}\cdot\sum k^6 - \frac{1}{2}\cdot\sum k^7 - \frac{1}{8}\cdot\sum k^8$$

$$\frac{9}{8}\cdot\sum k^8 = \frac{2n+1}{2}\cdot\sum k^7 - \frac{1}{12}\cdot\sum k^2 + \frac{7}{24}\cdot\sum k^4 - \frac{7}{12}\cdot\sum k^6.$$

In order to simplify the calculations we want to delay introducing the coefficients for the terms. In Section 8 we introduced the notation $E_{2m}(n)$ and $O_{2m+1}(n)$. Let us simplify it to E_{2m} and O_{2m+1}. Then we may rewrite the last expression as

$$\begin{aligned}\frac{9}{8}\cdot\sum k^8 \quad &= \quad \frac{2n+1}{2}\cdot O_7\cdot\sum k^3 \\ &\quad -\frac{1}{12}\cdot\sum k^2 + \frac{7}{24}\cdot E_4\cdot\sum k^2 - \frac{7}{12}\cdot E_6\cdot\sum k^2,\end{aligned}$$

where O_7, E_4, and E_6 stem from the previous expressions for $\sum k^7, \sum k^4$, and $\sum k^6$.

Since we believe the final expression will be of the form $\sum k^8 = E_8 \cdot \sum k^2$, we rewrite the term on the left side as follows:

$$\begin{aligned}
\frac{2n+1}{2} \cdot O_7 \cdot \frac{n\,(n+1)}{2} \cdot \frac{n\,(n+1)}{2} &= \frac{3}{3} \cdot \frac{2n+1}{2} \cdot O_7 \cdot \frac{n\,(n+1)}{2} \cdot \frac{n\,(n+1)}{2} \\
&= \frac{3}{2} \cdot \frac{n\,(n+1)}{2} \cdot O_7 \cdot \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2} \\
&= \frac{3}{2} \cdot \frac{n(n+1)}{2} \cdot O_7 \cdot \sum k^2.
\end{aligned}$$

In total we have

$$\begin{aligned}
\frac{9}{8} \cdot \sum k^8 \quad = \quad & \frac{3}{2} \cdot \frac{n(n+1)}{2} \cdot O_7 \cdot \sum k^2 \\
& -\frac{1}{12} \cdot \sum k^2 + \frac{7}{24} \cdot E_4 \cdot \sum k^2 - \frac{7}{12} \cdot E_6 \cdot \sum k^2. \qquad (17)
\end{aligned}$$

Now we are ready to substitute the true coefficients. Only, we are going to use a different set of coefficients.

If we look back to the summary of Part 1 in Section 7, we will remind ourselves we have been carrying two forms of the expressions. Now it will be advantageous to return to the soft forms:

$$\sum k^4 = \frac{1}{5} \cdot \left(6 \cdot \frac{n\,(n+1)}{2} - 1\right) \cdot \sum k^2$$

$$\sum k^6 = \frac{1}{7} \cdot \left(12 \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - 6 \cdot \frac{n\,(n+1)}{2} + 1\right) \cdot \sum k^2$$

$$\sum k^7 = \frac{1}{3} \cdot \left(6 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 4 \cdot \frac{n(n+1)}{2} + 1\right) \cdot \sum k^3.$$

We substitute the coefficients into expression 17 to get

$$\begin{aligned}
\frac{9}{8} \cdot \sum k^8 \quad = \quad & \frac{3}{2} \cdot \frac{n(n+1)}{2} \cdot \frac{1}{3} \cdot \left(6 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 4 \cdot \frac{n(n+1)}{2} + 1\right) \cdot \sum k^2 \\
& -\frac{1}{12} \cdot \sum k^2 \\
& +\frac{7}{24} \cdot \frac{1}{5} \cdot \left(6 \cdot \frac{n\,(n+1)}{2} - 1\right) \cdot \sum k^2 \\
& -\frac{7}{12} \cdot \frac{1}{7} \cdot \left(12 \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - 6 \cdot \frac{n\,(n+1)}{2} + 1\right) \cdot \sum k^2
\end{aligned}$$

$$
\begin{aligned}
\frac{9}{8}\cdot\sum k^8 \quad = \quad & \left(3\cdot\left(\frac{n(n+1)}{2}\right)^3 - 2\cdot\left(\frac{n(n+1)}{2}\right)^2 + \frac{1}{2}\cdot\frac{n(n+1)}{2}\right)\cdot\sum k^2 \\
& -\frac{1}{12}\cdot\sum k^2 \\
& +\left(\frac{42}{120}\cdot\frac{n\,(n+1)}{2} - \frac{7}{120}\right)\cdot\sum k^2 \\
& +\left(-\left(\frac{n\,(n+1)}{2}\right)^2 + \frac{1}{2}\cdot\frac{n\,(n+1)}{2} - \frac{1}{12}\right)\cdot\sum k^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{9}{8}\cdot\sum k^8 \quad = \quad & 3\cdot\left(\frac{n(n+1)}{2}\right)^3\cdot\sum k^2 \\
& -3\cdot\left(\frac{n(n+1)}{2}\right)^2\cdot\sum k^2 \\
& +\left(\frac{1}{2}+\frac{42}{120}+\frac{1}{2}\right)\cdot\frac{n\,(n+1)}{2}\cdot\sum k^2 \\
& +\left(-\frac{1}{12}-\frac{7}{120}-\frac{1}{12}\right)\cdot\sum k^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{9}{8}\cdot\sum k^8 \quad = \quad & 3\cdot\left(\frac{n(n+1)}{2}\right)^3\cdot\sum k^2 \\
& -3\cdot\left(\frac{n(n+1)}{2}\right)^2\cdot\sum k^2 \\
& +\frac{27}{20}\cdot\frac{n(n+1)}{2}\cdot\sum k^2 \\
& -\frac{9}{40}\cdot\sum k^2,
\end{aligned}
$$

which we rewrite as

$$
\begin{aligned}
\sum k^8 \quad &= \quad \frac{8}{9}\cdot\left(3\cdot\left(\frac{n(n+1)}{2}\right)^3 - 3\cdot\left(\frac{n(n+1)}{2}\right)^2 + \frac{27}{20}\cdot\frac{n(n+1)}{2} - \frac{9}{40}\right)\cdot\sum k^2 \\
&= \left(\frac{8}{3}\cdot\left(\frac{n(n+1)}{2}\right)^3 - \frac{8}{3}\cdot\left(\frac{n(n+1)}{2}\right)^2 + \frac{6}{5}\cdot\frac{n(n+1)}{2} - \frac{1}{5}\right)\cdot\sum k^2.
\end{aligned}
$$

Another way to write it is

$$
\sum k^8 = \frac{1}{9}\cdot\left(24\cdot\left(\frac{n(n+1)}{2}\right)^3 - 24\cdot\left(\frac{n(n+1)}{2}\right)^2 + \frac{54}{5}\cdot\frac{n(n+1)}{2} - \frac{9}{5}\right)\cdot\sum k^2. \tag{18}
$$

We have verified the pattern again.[5] The other form is

$$\sum k^8 = \frac{5(n(n+1))^3 - 10(n(n+1))^2 + 9n(n+1) - 3}{15} \cdot \sum k^2.$$

Both expressions simplify to

$$\sum k^8 = \frac{-3n + 20n^3 - 42n^5 + 60n^7 + 45n^8 + 10n^9}{90}.$$

Are we ready to put all of this together for one, final case?

11 $\sum k^9$

What is

$$\sum k^9 = 1^9 + 2^9 + 3^9 + \cdots + n^9\,?$$

We know that

$$\sum k^9 = (n+1) \cdot \sum k^8 - \sum \sum l^8. \tag{19}$$

We just derived

$$\sum k^8 = \frac{-3n + 20n^3 - 42n^5 + 60n^7 + 45n^8 + 10n^9}{90},$$

which implies

$$\sum \sum l^8 = \frac{-3 \cdot \sum k + 20 \cdot \sum k^3 - 42 \cdot \sum k^5 + 60 \cdot \sum k^7 + 45 \cdot \sum k^8 + 10 \cdot \sum k^9}{90}.$$

Therefore we may rewrite expression 19 as

$$\sum k^9 = (n+1) \cdot \sum k^8 - \frac{-3 \cdot \sum k + 20 \cdot \sum k^3 - 42 \cdot \sum k^5 + 60 \cdot \sum k^7 + 45 \cdot \sum k^8 + 10 \cdot \sum k^9}{90}$$

$$\sum k^9 = (n+1) \cdot \sum k^8 + \frac{1}{30} \cdot \sum k - \frac{2}{9} \cdot \sum k^3 + \frac{7}{15} \cdot \sum k^5 - \frac{2}{3} \cdot \sum k^7 - \frac{1}{2} \cdot \sum k^8 - \frac{1}{9} \cdot \sum k^9$$

$$\frac{10}{9} \cdot \sum k^9 = \frac{2n+1}{2} \cdot \sum k^8 + \frac{1}{30} \cdot \sum k - \frac{2}{9} \cdot \sum k^3 + \frac{7}{15} \cdot \sum k^5 - \frac{2}{3} \cdot \sum k^7$$

[5] Notice that

$$\begin{aligned} \sum k^8 &= \frac{1}{9} \cdot \left(24 \cdot \left(\frac{n(n+1)}{2} \right)^3 - 24 \cdot \left(\frac{n(n+1)}{2} \right)^2 + \frac{54}{5} \cdot \frac{n(n+1)}{2} - \frac{9}{5} \right) \cdot \sum k^2 \\ &= \frac{1}{9} \cdot \left(24 \cdot \left(\frac{n(n+1)}{2} \right)^3 - 24 \cdot \left(\frac{n(n+1)}{2} \right)^2 + 9 \cdot \frac{1}{5} \cdot \left(6 \cdot \frac{n(n+1)}{2} - 1 \right) \right) \cdot \sum k^2, \end{aligned}$$

which is a curious

$$\sum k^8 = \frac{1}{9} \cdot \left(24 \cdot \left(\frac{n(n+1)}{2} \right)^3 - 24 \cdot \left(\frac{n(n+1)}{2} \right)^2 \right) \cdot \sum k^2 + \sum k^4.$$

$$
\begin{aligned}
\frac{10}{9}\cdot\sum k^9 \quad = \quad & \frac{2n+1}{2}\cdot E_8\cdot\sum k^2 \\
& +\frac{1}{30}\cdot\sum k \\
& -\frac{2}{9}\cdot\sum k^3+\frac{7}{15}\cdot O_5\cdot\sum k^3-\frac{2}{3}\cdot O_7\cdot\sum k^3, \qquad (20)
\end{aligned}
$$

where

$$E_8=\frac{8}{3}\cdot\left(\frac{n(n+1)}{2}\right)^3-\frac{8}{3}\cdot\left(\frac{n(n+1)}{2}\right)^2+\frac{6}{5}\cdot\frac{n(n+1)}{2}-\frac{1}{5},$$

$$O_5=\frac{1}{3}\cdot\left(4\cdot\frac{n\,(n+1)}{2}-1\right),$$

and

$$O_7=\frac{1}{3}\cdot\left(6\cdot\left(\frac{n(n+1)}{2}\right)^2-4\cdot\frac{n(n+1)}{2}+1\right).$$

We believe the final expression will be of the form $\sum k^9 = O_9\cdot\sum k^3$. However, analogous to the calculation for $\sum k^7$, we will aim first for $\sum k^9 = O_9\cdot\frac{n(n+1)}{2}\cdot\sum k$. For expression 20 we rewrite the terms on the right side as

$$
\begin{aligned}
& -\frac{2}{9}\cdot\sum k^3+\frac{7}{15}\cdot O_5\cdot\sum k^3-\frac{2}{3}\cdot O_7\cdot\sum k^3 \\
= & -\frac{2}{9}\cdot\frac{n(n+1)}{2}\cdot\sum k+\frac{7}{15}\cdot O_5\cdot\frac{n(n+1)}{2}\cdot\sum k-\frac{2}{3}\cdot\frac{n(n+1)}{2}\cdot O_7\cdot\sum k \\
= \quad & -\frac{2}{9}\cdot\frac{n(n+1)}{2}\cdot\sum k+\frac{7}{15}\cdot\frac{1}{3}\cdot\left(4\cdot\frac{n\,(n+1)}{2}-1\right)\cdot\frac{n(n+1)}{2}\cdot\sum k \\
& -\frac{2}{3}\cdot\frac{1}{3}\cdot\left(6\cdot\left(\frac{n(n+1)}{2}\right)^2-4\cdot\frac{n(n+1)}{2}+1\right)\cdot\frac{n(n+1)}{2}\cdot\sum k \\
= \quad & -\frac{2}{9}\cdot\frac{n(n+1)}{2}\cdot\sum k \\
& +\left(\frac{28}{45}\cdot\frac{n(n+1)}{2}-\frac{7}{45}\right)\cdot\frac{n(n+1)}{2}\cdot\sum k \\
& +\left(-\frac{4}{3}\cdot\left(\frac{n(n+1)}{2}\right)^2+\frac{8}{9}\cdot\frac{n(n+1)}{2}-\frac{2}{9}\right)\cdot\frac{n(n+1)}{2}\cdot\sum k \\
= \quad & -\frac{2}{9}\cdot\frac{n(n+1)}{2}\cdot\sum k \\
& +\left(\frac{28}{45}\cdot\left(\frac{n(n+1)}{2}\right)^2-\frac{7}{45}\cdot\frac{n(n+1)}{2}\right)\cdot\sum k \\
& +\left(-\frac{4}{3}\cdot\left(\frac{n(n+1)}{2}\right)^3+\frac{8}{9}\cdot\left(\frac{n(n+1)}{2}\right)^2-\frac{2}{9}\cdot\frac{n(n+1)}{2}\right)\cdot\sum k
\end{aligned}
$$

$$= \left(-\frac{4}{3} \cdot \left(\frac{n(n+1)}{2} \right)^3 + \left(\frac{28}{45} + \frac{8}{9} \right) \cdot \left(\frac{n(n+1)}{2} \right)^2 + \left(-\frac{2}{9} - \frac{7}{45} - \frac{2}{9} \right) \cdot \left(\frac{n(n+1)}{2} \right) \right) \cdot \sum k.$$

$$= \left(-\frac{4}{3} \cdot \left(\frac{n(n+1)}{2} \right)^3 + \frac{68}{45} \cdot \left(\frac{n(n+1)}{2} \right)^2 - \frac{3}{5} \cdot \left(\frac{n(n+1)}{2} \right) \right) \cdot \sum k. \quad (21)$$

Next, we rewrite the term on the left side of expression 20 as

$$\begin{aligned} \frac{2n+1}{2} \cdot E_8 \cdot \sum k^2 &= \frac{2n+1}{2} \cdot E_8 \cdot \frac{2n+1}{3} \cdot \sum k \\ &= \frac{(2n+1)^2}{6} \cdot E_8 \cdot \sum k \\ &= \left(\frac{4}{3} \cdot \frac{n(n+1)}{2} + \frac{1}{6} \right) \cdot E_8 \cdot \sum k \end{aligned}$$

and then substitute the expression for E_8:

$$\left(\frac{4}{3} \cdot \frac{n(n+1)}{2} + \frac{1}{6} \right) \cdot \left(\frac{8}{3} \cdot \left(\frac{n(n+1)}{2} \right)^3 - \frac{8}{3} \cdot \left(\frac{n(n+1)}{2} \right)^2 + \frac{6}{5} \cdot \frac{n(n+1)}{2} - \frac{1}{5} \right) \cdot \sum k$$

$$\begin{aligned} = & \left(\frac{32}{9} \cdot \left(\frac{n(n+1)}{2} \right)^4 - \frac{32}{9} \cdot \left(\frac{n(n+1)}{2} \right)^3 + \frac{24}{15} \cdot \left(\frac{n(n+1)}{2} \right)^2 - \frac{4}{15} \cdot \frac{n(n+1)}{2} \right) \cdot \sum k \\ & + \left(\frac{8}{18} \cdot \left(\frac{n(n+1)}{2} \right)^3 - \frac{8}{18} \cdot \left(\frac{n(n+1)}{2} \right)^2 + \frac{1}{5} \cdot \frac{n(n+1)}{2} - \frac{1}{30} \right) \cdot \sum k \end{aligned}$$

$$\begin{aligned} = \; & \frac{32}{9} \cdot \left(\frac{n(n+1)}{2} \right)^4 \cdot \sum k \\ & + \left(-\frac{32}{9} + \frac{8}{18} \right) \cdot \left(\frac{n(n+1)}{2} \right)^3 \cdot \sum k \\ & + \left(\frac{24}{15} - \frac{8}{18} \right) \cdot \left(\frac{n(n+1)}{2} \right)^2 \cdot \sum k \\ & + \left(-\frac{4}{15} + \frac{1}{5} \right) \cdot \frac{n(n+1)}{2} \cdot \sum k \\ & - \frac{1}{30} \cdot \sum k \end{aligned}$$

$$= \left(\frac{32}{9} \cdot \left(\frac{n(n+1)}{2} \right)^4 - \frac{28}{9} \cdot \left(\frac{n(n+1)}{2} \right)^3 + \frac{52}{45} \cdot \left(\frac{n(n+1)}{2} \right)^2 - \frac{1}{15} \cdot \frac{n(n+1)}{2} \right) \cdot \sum k - \frac{1}{30} \cdot \sum k. \quad (22)$$

The lone term of $-\frac{1}{30} \cdot \sum k$ should make us happy. It cancels with the $\frac{1}{30} \cdot \sum k$ in expression 20.

Finally, if we put together expressions 20, 21, and 22 then we get

$$\begin{aligned}
\frac{10}{9}\cdot\sum k^9 &= \left(\frac{32}{9}\cdot\left(\frac{n(n+1)}{2}\right)^4-\frac{28}{9}\cdot\left(\frac{n(n+1)}{2}\right)^3\right)\cdot\sum k\\
&+\left(\frac{52}{45}\cdot\left(\frac{n(n+1)}{2}\right)^2-\frac{1}{15}\cdot\frac{n(n+1)}{2}\right)\cdot\sum k\\
&-\frac{1}{30}\cdot\sum k+\frac{1}{30}\cdot\sum k\\
&+\left(-\frac{4}{3}\cdot\left(\frac{n(n+1)}{2}\right)^3+\frac{68}{45}\cdot\left(\frac{n(n+1)}{2}\right)^2-\frac{3}{5}\cdot\left(\frac{n(n+1)}{2}\right)\right)\cdot\sum k
\end{aligned}$$

$$\frac{10}{9}\cdot\sum k^9=\left(\frac{32}{9}\cdot\left(\frac{n(n+1)}{2}\right)^4-\frac{40}{9}\cdot\left(\frac{n(n+1)}{2}\right)^3+\frac{8}{3}\cdot\left(\frac{n(n+1)}{2}\right)^2-\frac{2}{3}\cdot\frac{n(n+1)}{2}\right)\cdot\sum k$$

$$\begin{aligned}
\sum k^9 &= \left(\frac{16}{5}\cdot\left(\frac{n(n+1)}{2}\right)^4-4\cdot\left(\frac{n(n+1)}{2}\right)^3+\frac{12}{5}\cdot\left(\frac{n(n+1)}{2}\right)^2-\frac{3}{5}\cdot\frac{n(n+1)}{2}\right)\cdot\sum k\\
&=\left(\frac{16}{5}\cdot\left(\frac{n(n+1)}{2}\right)^3-4\cdot\left(\frac{n(n+1)}{2}\right)^2+\frac{12}{5}\cdot\frac{n(n+1)}{2}-\frac{3}{5}\right)\cdot\sum k^3\\
&=\frac{1}{5}\cdot\left(16\cdot\left(\frac{n(n+1)}{2}\right)^3-20\cdot\left(\frac{n(n+1)}{2}\right)^2+12\cdot\frac{n(n+1)}{2}-3\right)\cdot\sum k^3. \qquad (23)
\end{aligned}$$

The other form is

$$\sum k^9=\frac{2(n(n+1))^3-5(n(n+1))^2+6n(n+1)-3}{5}\cdot\sum k^3.$$

Both expressions simplify to

$$\sum k^9=\frac{-3n^2+10n^4-14n^6+15n^8+10n^9+2n^{10}}{20}.$$

We got our $\sum k^9=O_9\cdot\sum k^3$.

12 Summary of Part 2

Building upon the methods and results of Part 1, we guessed the sums might follow patterns for even and odd powers:

$$\sum_{k=1}^{n} k^{2m} = E_{2m}(n) \cdot \sum_{k=1}^{n} k^2$$

$$\sum_{k=1}^{n} k^{2m+1} = O_{2m+1}(n) \cdot \sum_{k=1}^{n} k^3,$$

where $E_{2m}(n)$ and $O_{2m+1}(n)$ were rational expressions involving n. We were able to verify them for the next cases of $m = 6, 7$:

$$\begin{aligned}\sum k^6 &= \frac{1}{7} \cdot \left(12 \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - 6 \cdot \frac{n\,(n+1)}{2} + 1\right) \cdot \sum k^2 \\ &= \left(\frac{3\,(n\,(n+1))^2 - 3n\,(n+1) + 1}{7}\right) \cdot \sum k^2\end{aligned}$$

$$\begin{aligned}\sum k^7 &= \frac{1}{3} \cdot \left(6 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 4 \cdot \frac{n(n+1)}{2} + 1\right) \cdot \sum k^3 \\ &= \left(\frac{3(n(n+1))^2 - 4n(n+1) + 2}{6}\right) \cdot \sum k^3.\end{aligned}$$

We suspected the same would be true for $m = 8, 9$, but the calculations became too difficult. We modified our approach and derived

$$\begin{aligned}\sum k^8 &= \frac{1}{9} \cdot \left(24 \cdot \left(\frac{n(n+1)}{2}\right)^3 - 24 \cdot \left(\frac{n(n+1)}{2}\right)^2 + \frac{54}{5} \cdot \frac{n(n+1)}{2} - \frac{9}{5}\right) \cdot \sum k^2 \\ &= \frac{5(n(n+1))^3 - 10(n(n+1))^2 + 9n(n+1) - 3}{15} \cdot \sum k^2\end{aligned}$$

$$\begin{aligned}\sum k^9 &= \frac{1}{5} \cdot \left(16 \cdot \left(\frac{n(n+1)}{2}\right)^3 - 20 \cdot \left(\frac{n(n+1)}{2}\right)^2 + 12 \cdot \frac{n(n+1)}{2} - 3\right) \cdot \sum k^3 \\ &= \frac{2(n(n+1))^3 - 5(n(n+1))^2 + 6n(n+1) - 3}{5} \cdot \sum k^3,\end{aligned}$$

which verified the patterns again. Are we able to guess general expressions for E_{2m} and O_{2m+1}?

Part 3

13 A Long List

We have assembled a long list of results. Let us look at all of them together:

$$
\begin{aligned}
\sum k^2 &= \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2} \\
\sum k^3 &= \left(\frac{n\,(n+1)}{2}\right)^2 \\
\sum k^4 &= \frac{1}{5} \cdot \left(6 \cdot \frac{n\,(n+1)}{2} - 1\right) \cdot \sum k^2 \\
\sum k^5 &= \frac{1}{3} \cdot \left(4 \cdot \frac{n\,(n+1)}{2} - 1\right) \cdot \sum k^3 \\
\sum k^6 &= \frac{1}{7} \cdot \left(12 \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - 6 \cdot \frac{n\,(n+1)}{2} + 1\right) \cdot \sum k^2 \\
\sum k^7 &= \frac{1}{3} \cdot \left(6 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 4 \cdot \frac{n(n+1)}{2} + 1\right) \cdot \sum k^3 \\
\sum k^8 &= \frac{1}{9} \cdot \left(24 \cdot \left(\frac{n(n+1)}{2}\right)^3 - 24 \cdot \left(\frac{n(n+1)}{2}\right)^2 + \frac{54}{5} \cdot \frac{n(n+1)}{2} - \frac{9}{5}\right) \cdot \sum k^2 \\
\sum k^9 &= \frac{1}{5} \cdot \left(16 \cdot \left(\frac{n(n+1)}{2}\right)^3 - 20 \cdot \left(\frac{n(n+1)}{2}\right)^2 + 12 \cdot \frac{n(n+1)}{2} - 3\right) \cdot \sum k^3,
\end{aligned}
$$

and of course, $\sum k = \frac{n(n+1)}{2}$. Let us group them in even and odd powers:

$$
\begin{aligned}
\sum k^2 &= \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2} \\
\sum k^4 &= \frac{1}{5} \cdot \left(6 \cdot \frac{n\,(n+1)}{2} - 1\right) \cdot \sum k^2 \\
\sum k^6 &= \frac{1}{7} \cdot \left(12 \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - 6 \cdot \frac{n\,(n+1)}{2} + 1\right) \cdot \sum k^2 \\
\sum k^8 &= \frac{1}{9} \cdot \left(24 \cdot \left(\frac{n(n+1)}{2}\right)^3 - 24 \cdot \left(\frac{n(n+1)}{2}\right)^2 + \frac{54}{5} \cdot \frac{n(n+1)}{2} - \frac{9}{5}\right) \cdot \sum k^2
\end{aligned}
$$

$$
\begin{aligned}
\sum k^3 &= \left(\frac{n\,(n+1)}{2}\right)^2 \\
\sum k^5 &= \frac{1}{3}\cdot\left(4\cdot\frac{n\,(n+1)}{2}-1\right)\cdot\sum k^3 \\
\sum k^7 &= \frac{1}{3}\cdot\left(6\cdot\left(\frac{n(n+1)}{2}\right)^2-4\cdot\frac{n(n+1)}{2}+1\right)\cdot\sum k^3 \\
\sum k^9 &= \frac{1}{5}\cdot\left(16\cdot\left(\frac{n(n+1)}{2}\right)^3-20\cdot\left(\frac{n(n+1)}{2}\right)^2+12\cdot\frac{n(n+1)}{2}-3\right)\cdot\sum k^3.
\end{aligned}
$$

For the odd powers, something is amiss with the leading fractions. Let us try

$$
\begin{aligned}
\sum k^2 &= \frac{2n+1}{3}\cdot\frac{n\,(n+1)}{2} \\
\sum k^4 &= \frac{1}{5}\cdot\left(6\cdot\frac{n\,(n+1)}{2}-1\right)\cdot\sum k^2 \\
\sum k^6 &= \frac{1}{7}\cdot\left(12\cdot\left(\frac{n\,(n+1)}{2}\right)^2-6\cdot\frac{n\,(n+1)}{2}+1\right)\cdot\sum k^2 \\
\sum k^8 &= \frac{1}{9}\cdot\left(24\cdot\left(\frac{n(n+1)}{2}\right)^3-24\cdot\left(\frac{n(n+1)}{2}\right)^2+\frac{54}{5}\cdot\frac{n(n+1)}{2}-\frac{9}{5}\right)\cdot\sum k^2
\end{aligned}
$$

$$
\begin{aligned}
\sum k^3 &= \left(\frac{n\,(n+1)}{2}\right)^2 \\
\sum k^5 &= \frac{1}{6}\cdot\left(8\cdot\frac{n\,(n+1)}{2}-2\right)\cdot\sum k^3 \\
\sum k^7 &= \frac{1}{8}\cdot\left(\frac{1}{3}\cdot\left(48\cdot\left(\frac{n(n+1)}{2}\right)^2-32\cdot\frac{n(n+1)}{2}+8\right)\right)\cdot\sum k^3 \\
\sum k^9 &= \frac{1}{10}\cdot\left(32\cdot\left(\frac{n(n+1)}{2}\right)^3-40\cdot\left(\frac{n(n+1)}{2}\right)^2+24\cdot\frac{n(n+1)}{2}-6\right)\cdot\sum k^3.
\end{aligned}
$$

Yes, that looks better. Do we notice any patterns?

We observe the following:

1. the leading fractions for the sums seem to be based on the powers of the terms: for example, $\frac{1}{5}$ for $\sum k^4$ and $\frac{1}{10}$ for $\sum k^9$. We expect to find $\frac{1}{11}$ for $\sum k^{10}$ and $\frac{1}{12}$ for $\sum k^{11}$.

2. the terms of $\frac{n(n+1)}{2}$ in the coefficients for the sums are raised to the powers

$$1$$
$$2, 1$$
$$3, 2, 1.$$

We believe the next cases will be

$$4, 3, 2, 1$$
$$5, 4, 3, 2, 1,$$

and so forth.

3. as for the coefficients for the terms of $\frac{n(n+1)}{2}$ themselves, either they do not follow a pattern or they follow one which remains a mystery still. We have

$$6, -1$$
$$12, -6, 1$$
$$24, -24, \tfrac{54}{5}, -\tfrac{9}{5}$$

and

$$8, -2$$
$$16, -\tfrac{32}{3}, \tfrac{8}{3}$$
$$32, -40, 24, -6.$$

Why is a multiple of $\frac{1}{5}$ introduced? of $\frac{1}{3}$? We do not know. If we clear the fractions then we get

$$6, -1$$
$$12, -6, 1$$
$$120, -120, 54, -9$$

and

$$8, -2$$
$$48, -32, 8$$
$$32, -40, 24, -6.$$

It still is hard to discern a pattern. About the only thing we can pinpoint are the alternating signs in front of the coefficients.

Honestly, we are disappointed. After all of this work we expected the general patterns for $\sum k^{2m}$ and $\sum k^{2m+1}$ to fall into our laps. We are a far cry from that. Worse, we remember how lengthy the calculations were for $\sum k^8$ and $\sum k^9$ and are reluctant to attempt them for $\sum k^{10}$ and $\sum k^{11}$. What else can we do?

14 A Different Point of View

Perhaps we are looking at the expressions too directly. Perhaps we would benefit from taking a different point of view. Let us summarize our results as follows:

1. we have observed a pattern for even powers and a pattern for odd powers. The patterns have a close resemblance to one another.

2. after distinguishing the two patterns, the forms of the total sums have become less important and the forms of the coefficients for the sums have become more important.

Let us turn our attention to the coefficients.

For the calculation for $\sum k^8$ in Section 10, in Footnote 5 we remarked we noticed something curious:

$$\begin{aligned}\sum k^8 &= \frac{1}{9}\cdot\left(24\cdot\left(\frac{n(n+1)}{2}\right)^3 - 24\cdot\left(\frac{n(n+1)}{2}\right)^2 + \frac{54}{5}\cdot\frac{n(n+1)}{2} - \frac{9}{5}\right)\cdot\sum k^2 \\ &= \frac{1}{9}\cdot\left(24\cdot\left(\frac{n(n+1)}{2}\right)^3 - 24\cdot\left(\frac{n(n+1)}{2}\right)^2 + 9\cdot\frac{1}{5}\cdot\left(6\cdot\frac{n(n+1)}{2} - 1\right)\right)\cdot\sum k^2 \\ &= \frac{1}{9}\cdot\left(24\cdot\left(\frac{n(n+1)}{2}\right)^3 - 24\cdot\left(\frac{n(n+1)}{2}\right)^2\right)\cdot\sum k^2 + \sum k^4,\end{aligned}$$

where

$$\sum k^4 = \frac{1}{5}\cdot\left(6\cdot\frac{n\,(n+1)}{2} - 1\right)\cdot\sum k^2.$$

The expression for $\sum k^4$ ended up in the expression for $\sum k^8$. How? More curious, if we look at

$$\sum k^6 = \frac{1}{7}\cdot\left(12\cdot\left(\frac{n\,(n+1)}{2}\right)^2 - 6\cdot\frac{n\,(n+1)}{2} + 1\right)\cdot\sum k^2$$

then we notice

$$\begin{aligned}\sum k^6 &= \frac{1}{7}\cdot\left(12\cdot\left(\frac{n\,(n+1)}{2}\right)^2 - 5\cdot\frac{1}{5}\left(6\cdot\frac{n\,(n+1)}{2} - 1\right)\right)\cdot\sum k^2 \\ &= \frac{12}{7}\cdot\left(\frac{n\,(n+1)}{2}\right)^2\cdot\sum k^2 - \frac{5}{7}\cdot\sum k^4.\end{aligned}$$

$\sum k^4$ appeared again. What is the explanation?

14.1 Even

In Section 8 we introduced the notation $E_{2m}(n)$ and $O_{2m+1}(n)$, which refer to the coefficients for $\sum k^{2m}$ and $\sum k^{2m+1}$, respectively. Let us return to it.

For $\sum k^4, \sum k^6$, and $\sum k^8$ we look at the coefficients rather than the full expressions:

$$E_4 = \frac{1}{5} \cdot \left(6 \cdot \frac{n\,(n+1)}{2} - 1\right)$$

$$E_6 = \frac{1}{7} \cdot \left(12 \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - 6 \cdot \frac{n\,(n+1)}{2} + 1\right)$$

$$E_8 = \frac{1}{9} \cdot \left(24 \cdot \left(\frac{n(n+1)}{2}\right)^3 - 24 \cdot \left(\frac{n(n+1)}{2}\right)^2 + \frac{9}{5} \cdot \left(6 \cdot \frac{n(n+1)}{2} - 1\right)\right).$$

The new observations allow us to write

$$E_6 = \frac{12}{7} \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - \frac{5}{7} \cdot E_4$$

$$E_8 = \frac{1}{9} \cdot \left(24 \cdot \left(\frac{n(n+1)}{2}\right)^3 - 24 \cdot \left(\frac{n(n+1)}{2}\right)^2\right) + E_4.$$

What about E_4? In E_6 and E_8 we noticed the appearance of E_4. In

$$E_4 = \frac{1}{5} \cdot \left(6 \cdot \frac{n\,(n+1)}{2} - 1\right)$$

do we notice anything? What is the term of -1?

Ah, we have forgotten something. Remember that

$$\sum k^2 = \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2}.$$

If we write

$$\sum k^2 = \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2} = 1 \cdot \frac{2n+1}{3} \cdot \frac{n\,(n+1)}{2} = 1 \cdot \sum k^2$$

then we get $E_2 = 1$. This allows us to rewrite E_4 as

$$E_4 = \frac{1}{5} \cdot \left(6 \cdot \frac{n\,(n+1)}{2} - E_2\right)$$

$$5E_4 = 6 \cdot \frac{n\,(n+1)}{2} - E_2$$

or

$$E_2 + 5E_4 = 6 \cdot \frac{n\,(n+1)}{2}. \tag{24}$$

In an analogous fashion we may rewrite E_6 as

$$E_6 = \frac{12}{7} \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - \frac{5}{7} \cdot E_4$$

$$7E_6 = 12 \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - 5E_4$$

or

$$5E_4 + 7E_6 = 12 \cdot \left(\frac{n\,(n+1)}{2}\right)^2, \tag{25}$$

and E_8 as

$$E_8 = \frac{1}{9} \cdot \left(24 \cdot \left(\frac{n(n+1)}{2}\right)^3 - 24 \cdot \left(\frac{n(n+1)}{2}\right)^2\right) + E_4$$

$$-E_4 + E_8 = \frac{1}{9} \cdot \left(24 \cdot \left(\frac{n(n+1)}{2}\right)^3 - 24 \cdot \left(\frac{n(n+1)}{2}\right)^2\right)$$

$$-9E_4 + 9E_8 = 24 \cdot \left(\frac{n(n+1)}{2}\right)^3 - 2 \cdot (5E_4 + 7E_6)$$

or

$$E_4 + 14E_6 + 9E_8 = 24 \cdot \left(\frac{n(n+1)}{2}\right)^3. \tag{26}$$

Together we have

$$\begin{aligned} E_2 &= 1 \\ E_2 + 5E_4 &= 6 \cdot \tfrac{n(n+1)}{2} \\ 5E_4 + 7E_6 &= 12 \cdot \left(\tfrac{n(n+1)}{2}\right)^2 \\ E_4 + 14E_6 + 9E_8 &= 24 \cdot \left(\tfrac{n(n+1)}{2}\right)^3. \end{aligned} \tag{27}$$

This is remarkable. On the right side of the expressions we have

$$6 \cdot \frac{n\,(n+1)}{2},\ 12 \cdot \left(\frac{n\,(n+1)}{2}\right)^2,\ 24 \cdot \left(\frac{n\,(n+1)}{2}\right)^3.$$

We see the terms of $\frac{n(n+1)}{2}$ are raised to the powers $1, 2, 3$ and we notice the simple pattern of

$$6 = 2 \cdot 3,\ 12 = 2^2 \cdot 3,\ 24 = 2^3 \cdot 3.$$

On the left side of the expressions we have sums involving E_2, E_4, E_6, and E_8 and coefficients in integers. We might not be able to explain the integers yet, but we are confident enough to conjecture the next expression will be

$$e_4E_4 + e_6E_6 + e_8E_8 + e_{10}E_{10} = 48 \cdot \left(\frac{n(n+1)}{2}\right)^4 \tag{28}$$

for some integers e_4, e_6, e_8, e_{10}. Do the coefficients for sums of odd powers follow an analogous pattern?

14.2 Odd

We remind ourselves that

$$O_3 = 1$$

$$\begin{aligned} O_5 &= \frac{1}{6} \cdot \left(8 \cdot \frac{n\,(n+1)}{2} - 2\right) \\ &= \frac{1}{3} \cdot \left(4 \cdot \frac{n\,(n+1)}{2} - 1\right) \end{aligned}$$

$$\begin{aligned} O_7 &= \frac{1}{8} \cdot \left(\frac{1}{3} \cdot \left(48 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 32 \cdot \frac{n(n+1)}{2} + 8\right)\right) \\ &= \frac{1}{3} \cdot \left(6 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 4 \cdot \frac{n(n+1)}{2} + 1\right) \end{aligned}$$

$$O_9 = \frac{1}{10} \cdot \left(32 \cdot \left(\frac{n(n+1)}{2}\right)^3 - 40 \cdot \left(\frac{n(n+1)}{2}\right)^2 + 24 \cdot \frac{n(n+1)}{2} - 6\right).$$

We rewrite O_5 as

$$O_5 = \frac{1}{3} \cdot \left(4 \cdot \frac{n\,(n+1)}{2} - O_3\right)$$

$$3O_5 = 4 \cdot \frac{n\,(n+1)}{2} - O_3$$

or

$$O_3 + 3O_5 = 4 \cdot \frac{n\,(n+1)}{2}. \tag{29}$$

We rewrite O_7 as

$$O_7 = \frac{1}{3} \cdot \left(6 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 3 \cdot \frac{1}{3} \cdot \left(4 \cdot \frac{n(n+1)}{2} - 1\right)\right)$$

$$3O_7 = 6 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 3O_5$$

$$3O_5 + 3O_7 = 6 \cdot \left(\frac{n(n+1)}{2}\right)^2$$

$$O_5 + O_7 = 2 \cdot \left(\frac{n(n+1)}{2}\right)^2$$

or

$$4O_5 + 4O_7 = 8 \cdot \left(\frac{n(n+1)}{2}\right)^2. \tag{30}$$

We rewrite O_9 as

$$O_9 = \frac{1}{10} \cdot \left(32 \cdot \left(\frac{n(n+1)}{2} \right)^3 - 40 \cdot \left(\frac{n(n+1)}{2} \right)^2 + 18 \cdot \frac{1}{3} \cdot \left(4 \cdot \frac{n(n+1)}{2} - 1 \right) \right)$$

$$10 O_9 = 32 \cdot \left(\frac{n(n+1)}{2} \right)^3 - 5 \cdot (4 O_5 + 4 O_7) + 18 \cdot O_5$$

$$10 O_9 = 32 \cdot \left(\frac{n(n+1)}{2} \right)^3 - 2 O_5 - 20 O_7$$

$$2 O_5 + 20 O_7 + 10 O_9 = 32 \cdot \left(\frac{n(n+1)}{2} \right)^3$$

or

$$O_5 + 10 \cdot O_7 + 5 O_9 = 16 \cdot \left(\frac{n(n+1)}{2} \right)^3 . \tag{31}$$

Together we have

$$\begin{gathered} O_3 = 1 \\ O_3 + 3 O_5 = 4 \cdot \tfrac{n(n+1)}{2} \\ 4 O_5 + 4 O_7 = 8 \cdot \left(\tfrac{n(n+1)}{2} \right)^2 \\ O_5 + 10 \cdot O_7 + 5 O_9 = 16 \cdot \left(\tfrac{n(n+1)}{2} \right)^3 . \end{gathered} \tag{32}$$

Again, this is remarkable. On the right side we have

$$4 \cdot \frac{n\,(n+1)}{2},\ 8 \cdot \left(\frac{n\,(n+1)}{2} \right)^2,\ 16 \cdot \left(\frac{n\,(n+1)}{2} \right)^3 .$$

We see the terms of $\frac{n(n+1)}{2}$ are raised to the powers $1, 2, 3$ and we notice the simple pattern of

$$4 = 2^2,\ 8 = 2^3,\ 16 = 2^4.$$

On the left side we have sums involving O_3, O_5, O_7, and O_9 and coefficients in integers. We might not be able to explain the integers yet, but we are confident enough to conjecture the next expression will be

$$o_5 O_5 + o_7 O_7 + o_9 O_9 + o_{11} O_{11} = 32 \cdot \left(\frac{n(n+1)}{2} \right)^4 \tag{33}$$

for some integers o_5, o_7, o_9, o_{11}.

15 Back to the Hunt 2

We are so excited by our new conjectures we hardly can wait to test them out. Let us try the one for even powers:

$$e_4 E_4 + e_6 E_6 + e_8 E_8 + e_{10} E_{10} = 2^4 \cdot 3 \cdot \left(\frac{n(n+1)}{2}\right)^4 \tag{34}$$

for some integers e_4, e_6, e_8, e_{10}. Can we use it to find E_{10}?

15.1 E_{10}

From our previous work we suspect E_{10} will have a form like

$$E_{10} = \frac{1}{11} \cdot \left(a_1 \cdot \left(\frac{n(n+1)}{2}\right)^4 - a_2 \cdot \left(\frac{n(n+1)}{2}\right)^3 + a_3 \cdot \left(\frac{n(n+1)}{2}\right)^2 - a_4 \cdot E_4\right), \tag{35}$$

where a_1, a_2, a_3, a_4 are rational numbers and E_4 is the familiar

$$E_4 = \frac{1}{5} \cdot \left(6 \cdot \frac{n(n+1)}{2} - 1\right).$$

The coefficient of $2^4 \cdot 3$ in expression 34 tells us $a_1 = 48$. Looking at expression 35 and the earlier list in (27), we choose $e_{10} = 11$. Therefore we rewrite expression 34 as

$$e_4 E_4 + e_6 E_6 + e_8 E_8 + 11 E_{10} = 48 \cdot \left(\frac{n(n+1)}{2}\right)^4 \tag{36}$$

and expression 35 as

$$E_{10} = \frac{1}{11} \cdot \left(48 \cdot \left(\frac{n(n+1)}{2}\right)^4 - a_2 \cdot \left(\frac{n(n+1)}{2}\right)^3 + a_3 \cdot \left(\frac{n(n+1)}{2}\right)^2 - a_4 \cdot E_4\right). \tag{37}$$

What does expression 36 tell us? On the right side of the expression we see only $48 \cdot \left(\frac{n(n+1)}{2}\right)^4$. By expression 37 we *suspect* E_{10} has terms of the form $\left(\frac{n(n+1)}{2}\right)^4, \left(\frac{n(n+1)}{2}\right)^3, \left(\frac{n(n+1)}{2}\right)^2$, and E_4. We *know* E_4, E_6, and E_8 have terms of the form $\left(\frac{n(n+1)}{2}\right)^3, \left(\frac{n(n+1)}{2}\right)^2$, and E_4. That means all of the other terms must cancel out. In other words, we have the following system of equations:

$$\begin{array}{rlll}
E_{10}: & -a_2 \cdot \left(\frac{n(n+1)}{2}\right)^3 & + a_3 \cdot \left(\frac{n(n+1)}{2}\right)^2 & -a_4 \cdot E_4 \\
E_8: & e_8 \cdot \frac{1}{9} \cdot 24 \cdot \left(\frac{n(n+1)}{2}\right)^3 & - e_8 \cdot \frac{1}{9} \cdot 24 \cdot \left(\frac{n(n+1)}{2}\right)^2 & +e_8 \cdot \frac{1}{9} \cdot 9 \cdot E_4 \\
E_6: & & e_6 \cdot \frac{1}{7} \cdot 12 \cdot \left(\frac{n\,(n+1)}{2}\right)^2 & -e_6 \cdot \frac{1}{7} \cdot 5 \cdot E_4 \\
E_4: & & & e_4 \cdot E_4
\end{array}$$

where

$$-a_2 + e_8 \cdot \tfrac{1}{9} \cdot 24 = 0$$
$$a_3 - e_8 \cdot \tfrac{1}{9} \cdot 24 + e_6 \cdot \tfrac{1}{7} \cdot 12 = 0$$
$$-a_4 + e_8 \cdot \tfrac{1}{9} \cdot 9 - e_6 \cdot \tfrac{1}{7} \cdot 5 + e_4 = 0,$$

which we simplify to

$$\begin{aligned} \frac{8}{3} \cdot e_8 &= a_2 \\ -\frac{12}{7} \cdot e_6 + \frac{8}{3} \cdot e_8 &= a_3 \\ e_4 - \frac{5}{7} \cdot e_6 + e_8 &= a_4. \end{aligned} \tag{38}$$

How do we solve such a system of equations?

First, we need to decide whether we solve for e_i or a_j. That is easy to answer. If we knew the a_j then we would have solved the problem already. Second, that means if we substitute the values for e_i then we will get the values for a_j. What *are* the values for e_4, e_6, and e_8?

If we look back to the previous expressions then we will remind ourselves we *derived*

$$\begin{aligned} E_2 &= 1 \\ E_2 + 5E_4 &= 6 \cdot \tfrac{n(n+1)}{2} \\ 5E_4 + 7E_6 &= 12 \cdot \left(\tfrac{n(n+1)}{2}\right)^2 \\ E_4 + 14E_6 + 9E_8 &= 24 \cdot \left(\tfrac{n(n+1)}{2}\right)^3 \end{aligned} \tag{39}$$

and *guessed*

$$e_4E_4 + e_6E_6 + e_8E_8 + 11E_{10} = 48 \cdot \left(\frac{n(n+1)}{2}\right)^4.$$

We *do not know* the values for e_4, e_6, and e_8. What do we do?

Let us start by looking at the equations in (38). We observe the following:

1. We have $\frac{8}{3} \cdot e_8 = a_2$, which is equivalent to $e_8 = \frac{3a_2}{8}$. Since we suspect e_8 is an integer, if a_2 *also* is an integer then it follows that a_2 is a multiple of 8: $a_2 = 8c$ for some integer c. Therefore we may write $e_8 = \frac{3 \cdot 8c}{8} = 3c$, which implies e_8 is a multiple of 3.[6]

2. We have $-\frac{12}{7} \cdot e_6 + \frac{8}{3} \cdot e_8 = a_3$, which is equivalent to $e_6 = \frac{7}{12} \cdot \left(\frac{8}{3} \cdot e_8 - a_3\right)$. Analogous to the case for e_8, if both e_6 and a_3 are integers then it follows that e_6 is a multiple of 7: $e_6 = 7d$ for some integer d.

[6]For another observation concerning e_8, see Section 24.2 of the Appendix.

3. We have $e_4 - \frac{5}{7} \cdot e_6 + e_8 = a_4$, which is equivalent to $e_4 = \frac{5}{7} \cdot e_6 - e_8 + a_4$. That says little about e_4. All it seems to suggest is that a_4 might be an integer too.

This is little to go on. What do we do?

Let us return to the expressions in (39), which suggested the conjecture originally. However, like the approach of Section 14, let us focus on the coefficients of the expressions rather than the total expressions:

$$\begin{aligned} 1 &= 1 \\ 6 &= 1 + 5 \\ 12 &= 0 + 5 + 7 \\ 24 &= 0 + 1 + 14 + 9. \end{aligned} \tag{40}$$

Remarkable! In each expression the sum of the coefficients on the right side is equal to the coefficient on the left side. Does it follow that

$$e_4 + e_6 + e_8 + 11 = 48, \tag{41}$$

which is equivalent to

$$e_4 + e_6 + e_8 = 37\,? \tag{42}$$

If so, will it allow us to solve the previous system of equations?

Wait, there is more! Suppose we add the terms from expression 41 to the list in (40). Then we get

$$\begin{aligned} 1 &= 1 \\ 6 &= 1 + 5 \\ 12 &= 0 + 5 + 7 \\ 24 &= 0 + 1 + 14 + 9 \\ 48 &= 0 + e_4 + e_6 + e_8 + 11. \end{aligned} \tag{43}$$

Look at the terms on the diagonal containing e_8: 1,5,14. Have we seen them before? Of course! They are

$$1 = \sum_{k=1}^{1} k^2,\ 5 = \sum_{k=1}^{2} k^2,\ 14 = \sum_{k=1}^{3} k^2.$$

Does it follow that

$$e_8 = 30 = \sum_{k=1}^{4} k^2\,?$$

Remembering our previous observations on the list of expressions in (38), we noticed that e_8 is a multiple of 3. 30 is a multiple of 3! If we substitute $e_8 = 30$ into expression 42 then we get $e_4 + e_6 = 7$. We noticed also that e_6 is a multiple of 7. 7 is a multiple of 7! Suppose we choose $e_6 = 7$, which implies $e_4 = 0$. Then we may rewrite the list in (43) as

$$\begin{aligned}1 &= 1\\ 6 &= 1+5\\ 12 &= 0+5+7\\ 24 &= 0+1+14+9\\ 48 &= 0+0+7+30+11\end{aligned} \tag{44}$$

and expression 36 as

$$7E_6 + 30E_8 + 11E_{10} = 48 \cdot \left(\frac{n(n+1)}{2}\right)^4. \tag{45}$$

Did we find the correct values for e_4, e_6, and e_8? If we substitute them into the equations of (38) then we get

$$\begin{aligned}a_2 &= \frac{8}{3} \cdot 30 = 80\\ a_3 &= -\frac{12}{7} \cdot 7 + \frac{8}{3} \cdot 30 = 68\\ a_4 &= 0 - \frac{5}{7} \cdot 7 + 30 = 25.\end{aligned}$$

If we substitute these values for a_2, a_3, and a_4 into expression 37 then we get

$$E_{10} = \frac{1}{11} \cdot \left(48 \cdot \left(\frac{n(n+1)}{2}\right)^4 - 80 \cdot \left(\frac{n(n+1)}{2}\right)^3 + 68 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 25 \cdot E_4\right). \tag{46}$$

Last, if we return to the general form for sums of even powers then we get

$$11 \cdot \frac{\sum k^{10}}{\sum k^2} = 48 \cdot \left(\frac{n(n+1)}{2}\right)^4 - 80 \cdot \left(\frac{n(n+1)}{2}\right)^3 + 68 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 25 \cdot \frac{1}{5} \cdot \left(6 \cdot \frac{n\,(n+1)}{2} - 1\right), \tag{47}$$

which actually gives us a way to test the new result:

$$\sum_{k=1}^{10} k^{10} = 14,914,341,925$$

and

$$\frac{1}{11} \cdot \left(48 \cdot 55^4 - 80 \cdot 55^3 + 68 \cdot 55^2 - 25 \cdot \frac{1}{5} \cdot (6 \cdot 55 - 1)\right) \cdot \sum_{k=1}^{10} k^2$$

$$= \frac{1}{11} \cdot 426124055 \cdot 385 = 14,914,341,925.$$

It's only the single case of $n = 10$, but that's good enough for now.

15.2 O_{11}

Let us try the conjecture for the next odd power, O_{11}:

$$o_5 O_5 + o_7 O_7 + o_9 O_9 + o_{11} O_{11} = 2^5 \cdot \left(\frac{n(n+1)}{2}\right)^4 \tag{48}$$

for some integers o_5, o_7, o_9, o_{11}, where

$$o_5 + o_7 + o_9 + o_{11} = 2^5 = 32. \tag{49}$$

Since this case is a bit different than the previous one for E_{10}, we remind ourselves that

$$O_3 = 1$$

$$O_5 = \frac{1}{3} \cdot \left(4 \cdot \frac{n\,(n+1)}{2} - 1\right)$$

$$O_7 = \frac{1}{3} \cdot \left(6 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 3 \cdot O_5\right)$$

$$O_9 = \frac{1}{10} \cdot \left(32 \cdot \left(\frac{n(n+1)}{2}\right)^3 - 40 \cdot \left(\frac{n(n+1)}{2}\right)^2 + 18 \cdot O_5\right)$$

and

$$\begin{aligned} O_3 &= 1 \\ O_3 + 3O_5 &= 4 \cdot \tfrac{n(n+1)}{2} \\ 4O_5 + 4O_7 &= 8 \cdot \left(\tfrac{n(n+1)}{2}\right)^2 \\ O_5 + 10 \cdot O_7 + 5O_9 &= 16 \cdot \left(\tfrac{n(n+1)}{2}\right)^3. \end{aligned} \tag{50}$$

We suspect O_{11} will have a form like

$$O_{11} = \frac{1}{6} \cdot \left(32 \cdot \left(\frac{n(n+1)}{2}\right)^4 - b_2 \cdot \left(\frac{n(n+1)}{2}\right)^3 + b_3 \cdot \left(\frac{n(n+1)}{2}\right)^2 - b_4 \cdot O_5\right) \tag{51}$$

for some rational numbers b_2, b_3, b_4. Therefore we rewrite expression 48 as

$$o_5 O_5 + o_7 O_7 + o_9 O_9 + 6 \cdot O_{11} = 32 \cdot \left(\frac{n(n+1)}{2}\right)^4 \tag{52}$$

and expression 49 as

$$o_5 + o_7 + o_9 + 6 = 32, \tag{53}$$

which is equivalent to

$$o_5 + o_7 + o_9 = 26. \tag{54}$$

Let us gather more information about o_5, o_7, and o_9.

From the list in (50) we know that

$$\begin{aligned}1 &= 1\\ 4 &= 1+3\\ 8 &= 0+4+4\\ 16 &= 0+1+10+5.\end{aligned}$$

If we add expression 53 to it the we get

$$\begin{aligned}1 &= 1\\ 4 &= 1+3\\ 8 &= 0+4+4\\ 16 &= 0+1+10+5\\ 32 &= 0+o_5+o_7+o_9+6. \qquad (55)\end{aligned}$$

What are the terms along the diagonal for o_9? We suspect

$$4-1=3=1+2=\sum_{k=1}^{2} k$$

$$10-4=6=1+2+3=\sum_{k=1}^{3} k$$

and

$$o_9-10=10=1+2+3+4=\sum_{k=1}^{4} k,$$

which implies $o_9 = 20$. Therefore we rewrite the list in (55) as

$$\begin{aligned}1 &= 1\\ 4 &= 1+3\\ 8 &= 0+4+4\\ 16 &= 0+1+10+5\\ 32 &= 0+o_5+o_7+20+6 \qquad (56)\end{aligned}$$

and expression 52 as

$$o_5O_5+o_7O_7+20\cdot O_9+6\cdot O_{11}=32\cdot\left(\frac{n(n+1)}{2}\right)^4.$$

That still leaves

$$o_5+o_7=6$$

and

$$O_{11}=\frac{1}{6}\cdot\left(32\cdot\left(\frac{n(n+1)}{2}\right)^4-b_2\cdot\left(\frac{n(n+1)}{2}\right)^3+b_3\cdot\left(\frac{n(n+1)}{2}\right)^2-b_4\cdot O_5\right).$$

How do we figure out o_5, o_7 and b_2, b_3, b_4?

In order to determine all of the values for o_i and b_j we need to solve the following system of equations:

$$\begin{array}{rcccc}
O_{11}: & & -b_2 \cdot \left(\frac{n(n+1)}{2}\right)^3 & + b_3 \cdot \left(\frac{n(n+1)}{2}\right)^2 & -b_4 \cdot O_5 \\
O_9: & o_9 \cdot \frac{32}{10} \cdot & \left(\frac{n(n+1)}{2}\right)^3 & - o_9 \cdot \frac{40}{10} \cdot \left(\frac{n(n+1)}{2}\right)^2 & +o_9 \cdot \frac{18}{10} \cdot O_5 \\
O_7: & & & o_7 \cdot \frac{6}{3} \cdot \left(\frac{n(n+1)}{2}\right)^2 & -o_7 \cdot \frac{3}{3} \cdot O_5 \\
O_5: & & & & o_5 \cdot O_5,
\end{array}$$

where

$$\begin{gathered}
-b_2 + o_9 \cdot \tfrac{32}{10} = 0 \\
b_3 - o_9 \cdot \tfrac{40}{10} + o_7 \cdot \tfrac{6}{3} = 0 \\
-b_4 + o_9 \cdot \tfrac{18}{10} - o_7 \cdot \tfrac{3}{3} + o_5 = 0,
\end{gathered}$$

which we simplify to

$$\begin{aligned}
o_9 \cdot \frac{16}{5} &= b_2 \\
o_9 \cdot 4 - o_7 \cdot 2 &= b_3 \\
o_9 \cdot \frac{9}{5} - o_7 + o_5 &= b_4.
\end{aligned} \tag{57}$$

$o_9 = 20$ implies

$$b_2 = 20 \cdot \frac{16}{5} = 64,$$

which allows us to rewrite expression 51 as

$$O_{11} = \frac{1}{6} \cdot \left(32 \cdot \left(\frac{n(n+1)}{2}\right)^4 - 64 \cdot \left(\frac{n(n+1)}{2}\right)^3 + b_3 \cdot \left(\frac{n(n+1)}{2}\right)^2 - b_4 \cdot O_5\right). \tag{58}$$

As for the remaining variables, from the equations in (57) we observe

1. $o_9 \cdot \frac{16}{5} = b_2$, which is equivalent to $o_9 = \frac{5b_2}{16}$. If both o_9 and b_2 are integers then b_2 is a multiple of 16, which implies o_9 is a multiple of 5: $o_9 = 5c$ for some integer c.

2. $o_9 \cdot 4 - o_7 \cdot 2 = b_3$, which is equivalent to $2 \cdot (o_7 - 2 \cdot o_9) = b_3$. If both o_7 and o_9 are integers then b_3 is even.

3. $o_9 \cdot \frac{9}{5} - o_7 + o_5 = b_4$ might imply b_4 is an integer.

Once again, that is little to go on. All it suggests is the guess of $o_9 = 20$ might be correct. What do we do?

Suppose we make the simple choice of $o_7 = 6$, which implies $o_5 = 0$. Then we may rewrite the list in (56) as

$$\begin{aligned} 1 &= 1 \\ 4 &= 1 + 3 \\ 8 &= 0 + 4 + 4 \\ 16 &= 0 + 1 + 10 + 5 \\ 32 &= 0 + 0 + 6 + 20 + 6 \end{aligned} \tag{59}$$

and expression 52 as

$$6 \cdot O_7 + 20 \cdot O_9 + 6 \cdot O_{11} = 32 \cdot \left(\frac{n(n+1)}{2}\right)^4. \tag{60}$$

Did we find the correct values for o_5, o_7 and o_9? If we substitute them into the equations of (57) then we get

$$\begin{aligned} b_3 &= 20 \cdot 4 - 6 \cdot 2 = 68 \\ b_4 &= 20 \cdot \frac{9}{5} - 6 + 0 = 30. \end{aligned}$$

If we substitute these values for b_3 and b_4 into expression 58 then we get

$$O_{11} = \frac{1}{6} \cdot \left(32 \cdot \left(\frac{n(n+1)}{2}\right)^4 - 64 \cdot \left(\frac{n(n+1)}{2}\right)^3 + 68 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 30 \cdot O_5\right). \tag{61}$$

Last, if we return to the general form for sums of odd powers then we get

$$6 \cdot \frac{\sum k^{11}}{\sum k^3} = 32 \cdot \left(\frac{n(n+1)}{2}\right)^4 - 64 \cdot \left(\frac{n(n+1)}{2}\right)^3 + 68 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 30 \cdot \frac{1}{3} \cdot \left(4 \cdot \frac{n\,(n+1)}{2} - 1\right), \tag{62}$$

which actually gives us a way to test the new result:

$$\sum_{k=1}^{9} k^{11} = 42{,}364{,}319{,}625$$

and

$$\frac{1}{6} \cdot \left(32 \cdot 45^4 - 64 \cdot 45^3 + 68 \cdot 45^2 - 30 \cdot \frac{1}{3} \cdot (4 \cdot 45 - 1)\right) \cdot \sum_{k=1}^{9} k^3$$

$$= \frac{1}{6} \cdot 125523910 \cdot 2025 = 42{,}364{,}319{,}625.$$

We'll take it.

16 A Chance to Catch Our Breath

Our new results for even powers, which are only conjectures, are

$$7E_6 + 30E_8 + 11E_{10} = 48 \cdot \left(\frac{n(n+1)}{2}\right)^4$$

$$E_{10} = \frac{1}{11} \cdot \left(48 \cdot \left(\frac{n(n+1)}{2}\right)^4 - 80 \cdot \left(\frac{n(n+1)}{2}\right)^3 + 68 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 25 \cdot E_4\right)$$

$$11 \cdot \frac{\sum k^{10}}{\sum k^2} = 48 \cdot \left(\frac{n(n+1)}{2}\right)^4 - 80 \cdot \left(\frac{n(n+1)}{2}\right)^3 + 68 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 25 \cdot \frac{1}{5} \cdot \left(6 \cdot \frac{n\,(n+1)}{2} - 1\right).$$

They allow us to expand our list to

$$\begin{aligned} E_2 &= 1 \\ E_2 + 5E_4 &= 2 \cdot 3 \cdot \tfrac{n(n+1)}{2} \\ 5E_4 + 7E_6 &= 2^2 \cdot 3 \cdot \left(\tfrac{n(n+1)}{2}\right)^2 \\ E_4 + 14E_6 + 9E_8 &= 2^3 \cdot 3 \cdot \left(\tfrac{n(n+1)}{2}\right)^3 \\ 7E_6 + 30E_8 + 11E_{10} &= 2^4 \cdot 3 \cdot \left(\tfrac{n(n+1)}{2}\right)^4, \end{aligned}$$

which we find more advantageous to express as

$$\begin{aligned} 1 &= 1 \\ 6 &= 1 + 5 \\ 12 &= 0 + 5 + 7 \\ 24 &= 0 + 1 + 14 + 9 \\ 48 &= 0 + 0 + 7 + 30 + 11. \end{aligned}$$

Our new results for odd powers, which also are only conjectures, are

$$6 \cdot O_7 + 20 \cdot O_9 + 6 \cdot O_{11} = 32 \cdot \left(\frac{n(n+1)}{2}\right)^4$$

$$O_{11} = \frac{1}{6} \cdot \left(32 \cdot \left(\frac{n(n+1)}{2}\right)^4 - 64 \cdot \left(\frac{n(n+1)}{2}\right)^3 + 68 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 30 \cdot O_5\right)$$

$$6 \cdot \frac{\sum k^{11}}{\sum k^3} = 32 \cdot \left(\frac{n(n+1)}{2}\right)^4 - 64 \cdot \left(\frac{n(n+1)}{2}\right)^3 + 68 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 30 \cdot \frac{1}{3} \cdot \left(4 \cdot \frac{n\,(n+1)}{2} - 1\right).$$

They allow us to expand our list to

$$\begin{aligned}
O_3 &= 1 \\
O_3 + 3O_5 &= 2^2 \cdot \tfrac{n(n+1)}{2} \\
4O_5 + 4O_7 &= 2^3 \cdot \left(\tfrac{n(n+1)}{2}\right)^2 \\
O_5 + 10 \cdot O_7 + 5O_9 &= 2^4 \cdot \left(\tfrac{n(n+1)}{2}\right)^3 \\
6 \cdot O_7 + 20 \cdot O_9 + 6 \cdot O_{11} &= 2^5 \cdot \left(\tfrac{n(n+1)}{2}\right)^4,
\end{aligned}$$

which we find more advantageous to express as

$$\begin{aligned}
1 &= 1 \\
4 &= 1 + 3 \\
8 &= 0 + 4 + 4 \\
16 &= 0 + 1 + 10 + 5 \\
32 &= 0 + 0 + 6 + 20 + 6.
\end{aligned}$$

The outer terms suggest we recast some previous expressions for O_{2m+1}:

$$\begin{aligned}
O_7 &= \frac{1}{3} \cdot \left(6 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 3 \cdot O_5\right) \\
&= \frac{1}{4} \cdot \left(8 \cdot \left(\frac{n(n+1)}{2}\right)^2 - 4 \cdot O_5\right)
\end{aligned}$$

$$\begin{aligned}
O_9 &= \frac{1}{10} \cdot \left(32 \cdot \left(\frac{n(n+1)}{2}\right)^3 - 40 \cdot \left(\frac{n(n+1)}{2}\right)^2 + 18 \cdot O_5\right) \\
&= \frac{1}{5} \cdot \left(16 \cdot \left(\frac{n(n+1)}{2}\right)^3 - 20 \cdot \left(\frac{n(n+1)}{2}\right)^2 + 9 \cdot O_5\right).
\end{aligned}$$

17 Summary of Part 3

The result of our hard work in Parts 1 and 2 were explicit expressions for

$$\sum k^2, \sum k^3, \ldots, \sum k^8, \sum k^9,$$

which we divided into even and odd powers. Unfortunately, after inspecting the list of sums we were unable to discover any general patterns. Following up on an earlier observation, we came across the idea that in expressing the sums as

$$\sum k^{2m} = E_{2m} \cdot \sum k^2$$

$$\sum k^{2m+1} = O_{2m+1} \cdot \sum k^3,$$

perhaps some insight was to be gained by removing the coefficients E_{2m} and O_{2m+1} and looking at their relationships amongst only themselves.

To our shock we discovered the equations

$$\begin{aligned} E_2 &= 1 \\ E_2 + 5E_4 &= 2 \cdot 3 \cdot \tfrac{n(n+1)}{2} \\ 5E_4 + 7E_6 &= 2^2 \cdot 3 \cdot \left(\tfrac{n(n+1)}{2}\right)^2 \\ E_4 + 14E_6 + 9E_8 &= 2^3 \cdot 3 \cdot \left(\tfrac{n(n+1)}{2}\right)^3 \end{aligned}$$

and

$$\begin{aligned} O_3 &= 1 \\ O_3 + 3O_5 &= 2^2 \cdot \tfrac{n(n+1)}{2} \\ 4O_5 + 4O_7 &= 2^3 \cdot \left(\tfrac{n(n+1)}{2}\right)^2 \\ O_5 + 10 \cdot O_7 + 5O_9 &= 2^4 \cdot \left(\tfrac{n(n+1)}{2}\right)^3, \end{aligned}$$

which led us to conjecture

$$e_4E_4 + e_6E_6 + e_8E_8 + e_{10}E_{10} = 2^4 \cdot 3 \cdot \left(\frac{n(n+1)}{2}\right)^4$$

$$o_5O_5 + o_7O_7 + o_9O_9 + o_{11}O_{11} = 2^5 \cdot \left(\frac{n(n+1)}{2}\right)^4.$$

Through inductive reasoning and a simple system of equations we derived

$$7E_6 + 30E_8 + 11E_{10} = 48 \cdot \left(\frac{n(n+1)}{2}\right)^4$$

$$6 \cdot O_7 + 20 \cdot O_9 + 6 \cdot O_{11} = 32 \cdot \left(\frac{n(n+1)}{2}\right)^4,$$

which yielded

$$E_{10} = \frac{1}{11} \cdot \left(48 \cdot \left(\frac{n(n+1)}{2} \right)^4 - 80 \cdot \left(\frac{n(n+1)}{2} \right)^3 + 68 \cdot \left(\frac{n(n+1)}{2} \right)^2 - 25 \cdot E_4 \right)$$

$$O_{11} = \frac{1}{6} \cdot \left(32 \cdot \left(\frac{n(n+1)}{2} \right)^4 - 64 \cdot \left(\frac{n(n+1)}{2} \right)^3 + 68 \cdot \left(\frac{n(n+1)}{2} \right)^2 - 30 \cdot O_5 \right).$$

Even though the new results were only conjectures, by reinserting them into

$$\sum k^{10} = E_{10} \cdot \sum k^2$$

$$\sum k^{11} = O_{11} \cdot \sum k^3$$

we were able to verify them in special cases. Given how complicated the sums were, we had good reason to believe we found the correct expressions.[7]

Last, analogous to how in the expressions for the sums the coefficients turned out to hold the more important relationships, we believe that in the expressions for the coefficients *their* coefficients will turn out to hold the more important relationships, namely

$$\begin{aligned} 1 &= 1 \\ 6 &= 1 + 5 \\ 12 &= 0 + 5 + 7 \\ 24 &= 0 + 1 + 14 + 9 \\ 48 &= 0 + 0 + 7 + 30 + 11 \end{aligned}$$

and

$$\begin{aligned} 1 &= 1 \\ 4 &= 1 + 3 \\ 8 &= 0 + 4 + 4 \\ 16 &= 0 + 1 + 10 + 5 \\ 32 &= 0 + 0 + 6 + 20 + 6. \end{aligned}$$

[7]For an example of when inductive reasoning goes wrong, see Section 24.3 of the Appendix.

Part 4

18 A Frenchman Comes Aboard

At the close of Part 3 we placed our hopes in new observations about E_{2m} and O_{2m+1}, the expressions

$$\begin{gathered}
E_2 = 1 \\
E_2 + 5E_4 = 2 \cdot 3 \cdot \tfrac{n(n+1)}{2} \\
5E_4 + 7E_6 = 2^2 \cdot 3 \cdot \left(\tfrac{n(n+1)}{2}\right)^2 \\
E_4 + 14E_6 + 9E_8 = 2^3 \cdot 3 \cdot \left(\tfrac{n(n+1)}{2}\right)^3 \\
7E_6 + 30E_8 + 11E_{10} = 2^4 \cdot 3 \cdot \left(\tfrac{n(n+1)}{2}\right)^4
\end{gathered}$$

and

$$\begin{gathered}
O_3 = 1 \\
O_3 + 3O_5 = 2^2 \cdot \tfrac{n(n+1)}{2} \\
4O_5 + 4O_7 = 2^3 \cdot \left(\tfrac{n(n+1)}{2}\right)^2 \\
O_5 + 10 \cdot O_7 + 5O_9 = 2^4 \cdot \left(\tfrac{n(n+1)}{2}\right)^3 \\
6O_7 + 20 \cdot O_9 + 6O_{11} = 2^5 \cdot \left(\tfrac{n(n+1)}{2}\right)^4,
\end{gathered}$$

and the relationships amongst their coefficients,

$$\begin{aligned}
1 &= 1 \\
2 \cdot 3 &= 1 + 5 \\
2^2 \cdot 3 &= 0 + 5 + 7 \\
2^3 \cdot 3 &= 0 + 1 + 14 + 9 \\
2^4 \cdot 3 &= 0 + 0 + 7 + 30 + 11
\end{aligned} \tag{63}$$

and

$$\begin{aligned}
1 &= 1 \\
2^2 &= 1 + 3 \\
2^3 &= 0 + 4 + 4 \\
2^4 &= 0 + 1 + 10 + 5 \\
2^5 &= 0 + 0 + 6 + 20 + 6.
\end{aligned} \tag{64}$$

We believed they held the key to discovering general expressions for E_{2m} and O_{2m+1}. To continue the investigation, from the outward appearances of (63) and (64) we hardly can do any better than to look for relationships in Pascal's Triangle.

To construct Pascal's Triangle we start with a 1 at the top and a 1,2,1 in the second row, place a 1 at the start and end of each subsequent row, and fill in the middle of the triangle by adding adjacent terms:

$$3 = 1 + 2,\ 3 = 2 + 1$$

$$4 = 1 + 3,\ 6 = 3 + 3,\ 4 = 3 + 1$$

$$5 = 1 + 4,\ 10 = 4 + 6,\ 10 = 6 + 4,\ 5 = 4 + 1,$$

and so forth. We get

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
 & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}$$

We may continue the process for as long as we like. To refer to entries in the triangle we use the the notation $\binom{n}{k}$. For example,

$$\binom{1}{1} = 1,\ \binom{4}{0} = 1,\ \binom{4}{1} = 2,\ \binom{4}{4} = 4,\ \binom{7}{2} = 21.$$

Entries outside the table we set equal to zero: for example, $\binom{6}{-1} = 0$ and $\binom{2}{10} = 0$. Does Pascal's Triangle help us to notice any new relationships in the lists in (63) and (64)?

DOES IT EVER!? If we look at the coefficients for O_{2m+1}, it's as if the entire solution has been laid bare before our eyes:

$$\begin{aligned}
1 &= 1\\
2^2 &= \binom{3}{0} + \binom{3}{2}\\
2^3 &= \binom{4}{1} + \binom{4}{3}\\
2^4 &= \binom{5}{0} + \binom{5}{2} + \binom{5}{4}\\
2^5 &= \binom{6}{1} + \binom{6}{3} + \binom{6}{5}.
\end{aligned}$$

More important, from these few cases we believe we can extend the table indefinitely:

$$\begin{aligned}
1 &= 1\\
2^2 &= \binom{3}{0} + \binom{3}{2}\\
2^3 &= \binom{4}{1} + \binom{4}{3}\\
2^4 &= \binom{5}{0} + \binom{5}{2} + \binom{5}{4}\\
2^5 &= \binom{6}{1} + \binom{6}{3} + \binom{6}{5}\\
2^6 &= \binom{7}{0} + \binom{7}{2} + \binom{7}{4} + \binom{7}{6}\\
2^7 &= \binom{8}{1} + \binom{8}{3} + \binom{8}{5} + \binom{8}{7}\\
&\vdots\\
2^m &= \binom{m+1}{0} + \binom{m+1}{2} + \binom{m+1}{4} + \cdots + \binom{m+1}{m}\\
2^{m+1} &= \binom{m+2}{1} + \binom{m+2}{3} + \binom{m+2}{5} + \cdots + \binom{m+2}{m+1}.
\end{aligned}$$

Of course sometimes it will be helpful to write it as

$$\begin{aligned}
1 &= 1\\
4 &= 1+3\\
8 &= 0+4+4\\
16 &= 0+1+10+5\\
32 &= 0+0+6+20+6\\
64 &= 0+0+1+21+35+7\\
128 &= 0+0+0+8+56+56+8.
\end{aligned} \tag{65}$$

What about the coefficients for even powers, E_{2m}? The entries are a bit harder to find, but we believe they are

$$\begin{aligned}
1 &= 1\\
2\cdot 3 &= \left[\binom{2}{-1}+\binom{3}{0}\right]+\left[\binom{2}{1}+\binom{3}{2}\right]\\
2^2\cdot 3 &= \left[\binom{3}{0}+\binom{4}{1}\right]+\left[\binom{3}{2}+\binom{4}{3}\right]\\
2^3\cdot 3 &= \left[\binom{4}{-1}+\binom{5}{0}\right]+\left[\binom{4}{1}+\binom{5}{2}\right]+\left[\binom{4}{3}+\binom{5}{4}\right]\\
2^4\cdot 3 &= \left[\binom{5}{0}+\binom{6}{1}\right]+\left[\binom{5}{2}+\binom{6}{3}\right]+\left[\binom{5}{4}+\binom{6}{5}\right]\\
2^5\cdot 3 &= \left[\binom{6}{-1}+\binom{7}{0}\right]+\left[\binom{6}{1}+\binom{7}{2}\right]+\left[\binom{6}{3}+\binom{7}{4}\right]+\left[\binom{6}{5}+\binom{7}{6}\right]\\
2^6\cdot 3 &= \left[\binom{7}{0}+\binom{8}{1}\right]+\left[\binom{7}{2}+\binom{8}{3}\right]+\left[\binom{7}{4}+\binom{8}{5}\right]+\left[\binom{7}{6}+\binom{8}{7}\right]\\
&\vdots\\
2^{m-1}\cdot 3 &= \left[\binom{m}{-1}+\binom{m+1}{0}\right]+\left[\binom{m}{1}+\binom{m+1}{2}\right]+\left[\binom{m}{3}+\binom{m+1}{4}\right]\\
&\quad+\cdots+\left[\binom{m}{m-1}+\binom{m+1}{m}\right]\\
2^{m}\cdot 3 &= \left[\binom{m+1}{0}+\binom{m+2}{1}\right]+\left[\binom{m+1}{2}+\binom{m+2}{3}\right]+\left[\binom{m+1}{4}+\binom{m+2}{5}\right]\\
&\quad+\cdots+\left[\binom{m+1}{m}+\binom{m+2}{m+1}\right],
\end{aligned}$$

which we still can write as

$$
\begin{aligned}
1 &= 1 \\
6 &= 1 + 5 \\
12 &= 0 + 5 + 7 \\
24 &= 0 + 1 + 14 + 9 \\
48 &= 0 + 0 + 7 + 30 + 11 \\
96 &= 0 + 0 + 1 + 27 + 55 + 13 \\
192 &= 0 + 0 + 0 + 9 + 77 + 91 + 15.
\end{aligned} \tag{66}
$$

How did we deserve such luck?

19 Disbelief

Before we continue, let us look at the new discovery in more detail. We observe the following:

1. put simply, we are at a loss for words about how the introduction of Pascal's Triangle eliminated our troubles in one fell swoop. Is there a deeper connection with the problem? Is there something we missed? Also, we wonder why it took so long to make the connection. Next time, in similar circumstances, will we be able to make such a connection quicker?

2. when recast in terms of entries in Pascal's Triangle, the previous relationships, which are rather ordinary, suggest new, concrete expressions. For example, the sum

$$32 = 0 + 0 + 6 + 20 + 6$$

becomes

$$2^5 = \binom{6}{1} + \binom{6}{3} + \binom{6}{5}.$$

The sum

$$48 = 0 + 0 + 7 + 30 + 11$$

becomes

$$2^4 \cdot 3 = \left[\binom{5}{0} + \binom{6}{1}\right] + \left[\binom{5}{2} + \binom{6}{3}\right] + \left[\binom{5}{4} + \binom{6}{5}\right].$$

Or, the sums $\sum k$ and $\sum k^2$ can be expressed as

$$
\begin{aligned}
\sum_{k=1}^{5} k &= 1 + 2 + 3 + 4 + 5 = 15 \\
&= 1 + \binom{2}{1} + \binom{3}{2} + \binom{4}{3} + \binom{5}{4} = \binom{6}{4}
\end{aligned}
$$

$$\begin{aligned}\sum_{k=1}^{5} k^2 &= 1+4+9+16+25 = 55 \\ &= 20+35 = \binom{6}{3} + \binom{7}{4} = \binom{6}{3} + \binom{7}{3}\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{1} k + \sum_{k=1}^{2} k + \sum_{k=1}^{3} k + \sum_{k=1}^{4} k + \sum_{k=1}^{5} k &= 1+3+6+10+15 = 35 \\ &= 1 + \binom{3}{1} + \binom{4}{2} + \binom{5}{3} + \binom{6}{4} = \binom{7}{4}\end{aligned}$$

$$\sum_{k=1}^{1} k^2 + \sum_{k=1}^{2} k^2 + \sum_{k=1}^{3} k^2 + \sum_{k=1}^{4} k^2 + \sum_{k=1}^{5} k^2 = 1+5+14+30+55 = 105$$

$$\begin{aligned}&= 1 + \left[\binom{3}{0} + \binom{4}{1}\right] + \left[\binom{4}{1} + \binom{5}{2}\right] + \left[\binom{5}{2} + \binom{6}{3}\right] + \left[\binom{6}{3} + \binom{7}{4}\right] \\ &= 35 + 70 = \binom{7}{3} + \binom{8}{4}.\end{aligned}$$

We believe we only have begun to scratch the surface. Unfortunately, we must save such investigations for another time.[8]

20 Back to the Hunt 3

Our new results lead to a substantial improvement in our ability to calculate the desired expressions. Consider E_{12} and O_{13}.

20.1 E_{12}

From the list in (66) we conjecture

$$E_6 + 27E_8 + 55E_{10} + 13E_{12} = 96 \cdot \left(\frac{n(n+1)}{2}\right)^5, \tag{67}$$

where

$$\begin{aligned}96 &= e_6 + e_8 + e_{10} + e_{12} \\ &= 1 + 27 + 55 + 13.\end{aligned} \tag{68}$$

[8]See Section 24.4 of the Appendix.

We suspect E_{12} will have the form

$$13 \cdot E_{12} = 96 \cdot \left(\frac{n(n+1)}{2}\right)^5 - a_2 \cdot \left(\frac{n(n+1)}{2}\right)^4 + a_3 \cdot \left(\frac{n(n+1)}{2}\right)^3 - a_4 \cdot \left(\frac{n(n+1)}{2}\right)^2 + a_5 \cdot E_4. \tag{69}$$

The system of equations is

$$\begin{array}{lllll}
E_{12}: & -a_2 \cdot \left(\frac{n(n+1)}{2}\right)^4 & + a_3 \cdot \left(\frac{n(n+1)}{2}\right)^3 & -a_4 \cdot \left(\frac{n(n+1)}{2}\right)^2 & + a_5 \cdot E_4 \\
E_{10}: & 55 \cdot \frac{48}{11} \cdot \left(\frac{n(n+1)}{2}\right)^4 & - 55 \cdot \frac{80}{11} \cdot \left(\frac{n(n+1)}{2}\right)^3 & +55 \cdot \frac{68}{11} \cdot \left(\frac{n(n+1)}{2}\right)^2 & - 55 \cdot \frac{25}{11} \cdot E_4 \\
E_8: & & 27 \cdot \frac{24}{9} \cdot \left(\frac{n(n+1)}{2}\right)^3 & -27 \cdot \frac{24}{9} \cdot \left(\frac{n(n+1)}{2}\right)^2 & + 27 \cdot \frac{9}{9} \cdot E_4 \\
E_6: & & & 1 \cdot \frac{12}{7} \cdot \left(\frac{n(n+1)}{2}\right)^2 & - 1 \cdot \frac{5}{7} \cdot E_4,
\end{array}$$

where

$$\begin{gathered}
-a_2 + 55 \cdot \tfrac{48}{11} = 0 \\
a_3 - 55 \cdot \tfrac{80}{11} + 27 \cdot \tfrac{24}{9} = 0 \\
-a_4 + 55 \cdot \tfrac{68}{11} - 27 \cdot \tfrac{24}{9} + 1 \cdot \tfrac{12}{7} = 0 \\
a_5 - 55 \cdot \tfrac{25}{11} + 27 \cdot \tfrac{9}{9} - 1 \cdot \tfrac{5}{7} = 0.
\end{gathered}$$

We can solve it immediately:

$$\begin{aligned}
a_2 &= 240 \\
a_3 &= 328 \\
a_4 &= \frac{1888}{7} \\
a_5 &= \frac{691}{7}.
\end{aligned}$$

Therefore we can rewrite expression 69 as

$$13 \cdot E_{12} = 96 \cdot \left(\frac{n(n+1)}{2}\right)^5 - 240 \cdot \left(\frac{n(n+1)}{2}\right)^4 + 328 \cdot \left(\frac{n(n+1)}{2}\right)^3 - \frac{1888}{7} \cdot \left(\frac{n(n+1)}{2}\right)^2 + \frac{691}{7} \cdot E_4. \tag{70}$$

If we insert it into

$$\sum k^{12} = E_{12} \cdot \sum k^2$$

then we can test the result:

$$\sum_{k=1}^{9} k^{12} = 367,428,536,133$$

and

$$\frac{1}{13}\cdot\left(96\cdot 45^5-240\cdot 45^4+328\cdot 45^3-\frac{1888}{7}\cdot 45^2+\frac{691}{7}\cdot\frac{1}{5}\cdot(6\cdot 45-1)\right)\cdot\sum_{k=1}^{9}k^2$$

$$=\frac{1}{13}\cdot\frac{83799490697}{5}\cdot 285=367,428,536,133.$$

20.2 O_{13}

From the list in (65) we conjecture

$$O_7+21\cdot O_9+35\cdot O_{11}+7\cdot O_{13}=64\cdot\left(\frac{n(n+1)}{2}\right)^5, \tag{71}$$

where

$$\begin{aligned}64 &= o_7+o_9+o_{11}+o_{13}\\ &= 1+21+35+7.\end{aligned} \tag{72}$$

We suspect O_{13} will have the form

$$7\cdot O_{13}=64\cdot\left(\frac{n(n+1)}{2}\right)^5-b_2\cdot\left(\frac{n(n+1)}{2}\right)^4+b_3\cdot\left(\frac{n(n+1)}{2}\right)^3-b_4\cdot\left(\frac{n(n+1)}{2}\right)^2+b_5\cdot O_5. \tag{73}$$

The system of equations is

$$\begin{array}{llll}
O_{13}: & -b_2\cdot\left(\frac{n(n+1)}{2}\right)^4+b_3\cdot\left(\frac{n(n+1)}{2}\right)^3 & -b_4\cdot\left(\frac{n(n+1)}{2}\right)^2+b_5\cdot O_5 \\
O_{11}: & 35\cdot\frac{32}{6}\cdot\left(\frac{n(n+1)}{2}\right)^4-35\cdot\frac{64}{6}\cdot\left(\frac{n(n+1)}{2}\right)^3 & +35\cdot\frac{68}{6}\cdot\left(\frac{n(n+1)}{2}\right)^2-35\cdot\frac{30}{6}\cdot O_5 \\
O_9: & 21\cdot\frac{16}{5}\cdot\left(\frac{n(n+1)}{2}\right)^3 & -21\cdot\frac{20}{5}\cdot\left(\frac{n(n+1)}{2}\right)^2+21\cdot\frac{9}{5}\cdot O_5 \\
O_7: & & 1\cdot\frac{8}{4}\cdot\left(\frac{n(n+1)}{2}\right)^2-1\cdot\frac{4}{4}\cdot O_5,
\end{array}$$

where

$$\begin{gathered}-b_2+35\cdot\tfrac{32}{6}=0\\ b_3-35\cdot\tfrac{64}{6}+21\cdot\tfrac{16}{5}=0\\ -b_4+35\cdot\tfrac{68}{6}-21\cdot\tfrac{20}{5}+1\cdot\tfrac{8}{4}=0\\ b_5-35\cdot\tfrac{30}{6}+21\cdot\tfrac{9}{5}-1\cdot\tfrac{4}{4}=0.\end{gathered}$$

We can solve it immediately:

$$\begin{aligned}
b_2 &= \frac{560}{3} \\
b_3 &= \frac{4592}{15} \\
b_4 &= \frac{944}{3} \\
b_5 &= \frac{691}{5}.
\end{aligned}$$

Therefore we can rewrite expression 73 as

$$7 \cdot O_{13} = 64 \cdot \left(\frac{n(n+1)}{2}\right)^5 - \frac{560}{3} \cdot \left(\frac{n(n+1)}{2}\right)^4 + \frac{4592}{15} \cdot \left(\frac{n(n+1)}{2}\right)^3 - \frac{944}{3} \cdot \left(\frac{n(n+1)}{2}\right)^2 + \frac{691}{5} \cdot O_5. \tag{74}$$

If we insert it into

$$\sum k^{13} = O_{13} \cdot \sum k^3$$

then we can test the result:

$$\sum_{k=1}^{9} k^{13} = 3,202,860,761,145$$

and

$$\frac{1}{7} \cdot \left(64 \cdot 45^5 - \frac{560}{3} \cdot 45^4 + \frac{4592}{15} \cdot 45^3 - \frac{944}{3} \cdot 45^2 + \frac{691}{5} \cdot \frac{1}{3} \cdot (4 \cdot 45 - 1)\right) \cdot \sum_{k=1}^{9} k^3$$

$$= \frac{1}{7} \cdot \frac{166074261689}{15} \cdot 2025 = 3,202,860,761,145.$$

21 A Long List 2

One last time, let us place our results for E_{2m} and O_{2m+1} into a list:

$$
\begin{aligned}
E_2 &= 1\\
5\cdot E_4 &= 6\cdot\left(\frac{n(n+1)}{2}\right) - 1\\
7\cdot E_6 &= 12\cdot\left(\frac{n(n+1)}{2}\right)^2 - 5\cdot E_4\\
9\cdot E_8 &= 24\cdot\left(\frac{n(n+1)}{2}\right)^3 - 24\cdot\left(\frac{n(n+1)}{2}\right)^2 + 9\cdot E_4\\
11\cdot E_{10} &= 48\cdot\left(\frac{n(n+1)}{2}\right)^4 - 80\cdot\left(\frac{n(n+1)}{2}\right)^3 + 68\cdot\left(\frac{n(n+1)}{2}\right)^2 - 25\cdot E_4\\
13\cdot E_{12} &= 96\cdot\left(\frac{n(n+1)}{2}\right)^5 - 240\cdot\left(\frac{n(n+1)}{2}\right)^4 + 328\cdot\left(\frac{n(n+1)}{2}\right)^3 - \frac{1888}{7}\cdot\left(\frac{n(n+1)}{2}\right)^2 + \frac{691}{7}\cdot E_4
\end{aligned}
$$

$$
\begin{aligned}
O_3 &= 1\\
3\cdot O_5 &= 4\cdot\left(\frac{n(n+1)}{2}\right) - 1\\
4\cdot O_7 &= 8\cdot\left(\frac{n(n+1)}{2}\right)^2 - 4\cdot O_5\\
5\cdot O_9 &= 16\cdot\left(\frac{n(n+1)}{2}\right)^3 - 20\cdot\left(\frac{n(n+1)}{2}\right)^2 + 9\cdot O_5\\
6\cdot O_{11} &= 32\cdot\left(\frac{n(n+1)}{2}\right)^4 - 64\cdot\left(\frac{n(n+1)}{2}\right)^3 + 68\cdot\left(\frac{n(n+1)}{2}\right)^2 - 30\cdot O_5\\
7\cdot O_{13} &= 64\cdot\left(\frac{n(n+1)}{2}\right)^5 - \frac{560}{3}\cdot\left(\frac{n(n+1)}{2}\right)^4 + \frac{4592}{15}\cdot\left(\frac{n(n+1)}{2}\right)^3 - \frac{944}{3}\cdot\left(\frac{n(n+1)}{2}\right)^2 + \frac{691}{5}\cdot O_5.
\end{aligned}
$$

As a reminder, the expressions for E_{10}, E_{12} and O_{11}, O_{13} are conjectures only. Do we have any observations?

1. we have come a long way. The list is a large improvement over the one in Section 13. The leading fractions, the coefficients for $\frac{n(n+1)}{2}$, the alternating signs: everything is in unison. We truly believe we have found the correct expressions. Of course we doubt we could have guessed coefficients like
$$\frac{1888}{7}, \frac{691}{7}; \frac{560}{3}, \frac{4592}{15}, \frac{944}{3}, \frac{691}{5},$$
and we find their appearance even a bit strange, but now we can explain where they come from.

22 Summary of Part 4

We began Part 4 with our most promising results from Part 3, the lists

$$\begin{aligned} 1 &= 1 \\ 2 \cdot 3 &= 1 + 5 \\ 2^2 \cdot 3 &= 0 + 5 + 7 \\ 2^3 \cdot 3 &= 0 + 1 + 14 + 9 \\ 2^4 \cdot 3 &= 0 + 0 + 7 + 30 + 11 \end{aligned}$$

and

$$\begin{aligned} 1 &= 1 \\ 2^2 &= 1 + 3 \\ 2^3 &= 0 + 4 + 4 \\ 2^4 &= 0 + 1 + 10 + 5 \\ 2^5 &= 0 + 0 + 6 + 20 + 6, \end{aligned}$$

which contained relationships for terms in the expressions for E_{2m} and O_{2m+1}. The outward appearances of the lists suggested we turn to Pascal's Triangle. In little short of a miracle we noticed

$$\begin{aligned} 1 &= 1 \\ 2 \cdot 3 &= \left[\binom{2}{-1} + \binom{3}{0}\right] + \left[\binom{2}{1} + \binom{3}{2}\right] \\ 2^2 \cdot 3 &= \left[\binom{3}{0} + \binom{4}{1}\right] + \left[\binom{3}{2} + \binom{4}{3}\right] \\ 2^3 \cdot 3 &= \left[\binom{4}{-1} + \binom{5}{0}\right] + \left[\binom{4}{1} + \binom{5}{2}\right] + \left[\binom{4}{3} + \binom{5}{4}\right] \\ 2^4 \cdot 3 &= \left[\binom{5}{0} + \binom{6}{1}\right] + \left[\binom{5}{2} + \binom{6}{3}\right] + \left[\binom{5}{4} + \binom{6}{5}\right] \end{aligned}$$

and

$$\begin{aligned} 1 &= 1 \\ 2^2 &= \binom{3}{0} + \binom{3}{2} \\ 2^3 &= \binom{4}{1} + \binom{4}{3} \\ 2^4 &= \binom{5}{0} + \binom{5}{2} + \binom{5}{4} \\ 2^5 &= \binom{6}{1} + \binom{6}{3} + \binom{6}{5}. \end{aligned}$$

All of our previous difficulties seemed to disappear. After guessing expressions for the general terms,

$$\begin{aligned}
2^{m-1}\cdot 3 &= \left[\binom{m}{-1}+\binom{m+1}{0}\right]+\left[\binom{m}{1}+\binom{m+1}{2}\right]+\left[\binom{m}{3}+\binom{m+1}{4}\right] \\
&\quad+\cdots+\left[\binom{m}{m-1}+\binom{m+1}{m}\right] \\
2^{m}\cdot 3 &= \left[\binom{m+1}{0}+\binom{m+2}{1}\right]+\left[\binom{m+1}{2}+\binom{m+2}{3}\right]+\left[\binom{m+1}{4}+\binom{m+2}{5}\right] \\
&\quad+\cdots+\left[\binom{m+1}{m}+\binom{m+2}{m+1}\right]
\end{aligned}$$

and

$$\begin{aligned}
2^{m} &= \binom{m+1}{0}+\binom{m+1}{2}+\binom{m+1}{4}+\cdots+\binom{m+1}{m} \\
2^{m+1} &= \binom{m+2}{1}+\binom{m+2}{3}+\binom{m+2}{5}+\cdots+\binom{m+2}{m+1},
\end{aligned}$$

and recasting some previous relationships in terms of entries in Pascal's Triangle, we used the new results to calculate the next cases of E_{12} and O_{13}. With less effort than before we found

$$13\cdot E_{12} = 96\cdot\left(\frac{n(n+1)}{2}\right)^5 - 240\cdot\left(\frac{n(n+1)}{2}\right)^4 + 328\cdot\left(\frac{n(n+1)}{2}\right)^3 - \frac{1888}{7}\cdot\left(\frac{n(n+1)}{2}\right)^2 + \frac{691}{7}\cdot E_4$$

$$7\cdot O_{13} = 64\cdot\left(\frac{n(n+1)}{2}\right)^5 - \frac{560}{3}\cdot\left(\frac{n(n+1)}{2}\right)^4 + \frac{4592}{15}\cdot\left(\frac{n(n+1)}{2}\right)^3 - \frac{944}{3}\cdot\left(\frac{n(n+1)}{2}\right)^2 + \frac{691}{5}\cdot O_5,$$

which we were able to verify in the special cases of

$$\sum_{k=1}^{9} k^{12} = E_{12}(n)\cdot\sum_{k=1}^{9} k^2$$

$$\sum_{k=1}^{9} k^{13} = O_{13}(n)\cdot\sum_{k=1}^{9} k^3.$$

Finally, we placed all of our expressions for E_{2m} and O_{2m+1} into a long list and made some quick observations.

23 Final Remarks

At this point we believe we have answered the question we posed at the start,

> "What is a general expression for
>
> $$\sum_{k=1}^{n} k^m = 1^m + 2^m + 3^m + \cdots + n^m,$$
>
> where m is a positive integer?"

The answer is,

> "For even powers the expression is
>
> $$\begin{aligned}(2m+1)\cdot\frac{\sum_{k=1}^{n} k^{2m}}{\sum_{k=1}^{n} k^2} &= a_1\cdot\left(\frac{n(n+1)}{2}\right)^{m-1} - a_2\cdot\left(\frac{n(n+1)}{2}\right)^{m-2} + a_3\cdot\left(\frac{n(n+1)}{2}\right)^{m-3} \\ &\mp\cdots\mp a_{m-2}\cdot\left(\frac{n(n+1)}{2}\right)^2 + a_{m-1}\cdot\frac{1}{5}\cdot\left(6\cdot\frac{n(n+1)}{2}-1\right),\end{aligned}$$
>
> and for odd powers,
>
> $$\begin{aligned}(m+1)\cdot\frac{\sum_{k=1}^{n} k^{2m+1}}{\sum_{k=1}^{n} k^3} &= b_1\cdot\left(\frac{n(n+1)}{2}\right)^{m-1} - b_2\cdot\left(\frac{n(n+1)}{2}\right)^{m-2} + b_3\cdot\left(\frac{n(n+1)}{2}\right)^{m-3} \\ &\mp\cdots\mp b_{m-2}\cdot\left(\frac{n(n+1)}{2}\right)^2 + b_{m-1}\cdot\frac{1}{3}\cdot\left(4\cdot\frac{n(n+1)}{2}-1\right),\end{aligned}$$
>
> where a_i and b_j are rational numbers."

Whether or not we were aware of it previously, what our work has produced is an algorithm for finding expressions for $\sum k^m$. Even though it still is a conjecture, it is far preferable to the established result of Section 3,

$$\sum_{k=1}^{n} k^{m+1} + \sum_{k=1}^{n}\sum_{l=1}^{k} l^m = (n+1)\cdot\sum_{k=1}^{n} k^m,$$

which involves wading through lengthy calculations.[9] With the old result we gave up on calculations after $\sum k^9$. A case as far out as $\sum k^{16}$ almost would be unthinkable. It would require taming an expression like

$$\sum\sum l^{15} = \frac{c_1\cdot\sum k + c_2\cdot\sum k^2 + c_3\cdot\sum k^3 + \cdots + c_{16}\cdot\sum k^{16}}{C},$$

which might contain as many as nine terms. For comparison, with the new algorithm we can do the following.

[9]For a discussion of a proof, see Section 24.4 of the Appendix.

We start with

$$\sum k^{16} = E_{16} \cdot \sum k^2.$$

The exact form of E_{16} satisfies

$$e_2E_2+e_4E_4+e_6E_6+e_8E_8+e_{10}E_{10}+e_{12}E_{12}+e_{14}E_{14}+e_{16}E_{16} = 384\cdot\left(\frac{n(n+1)}{2}\right)^7. \tag{75}$$

In order to find the values for $e_2, e_4, e_6, \ldots, e_{16}$ we look at the previous list of

$$\begin{aligned}
1 &= 1\\
6 &= 1+5\\
12 &= 0+5+7\\
24 &= 0+1+14+9\\
48 &= 0+0+7+30+11\\
96 &= 0+0+1+27+55+13\\
192 &= 0+0+0+9+77+91+15.
\end{aligned}$$

It does not contain the values for E_{16}. Not to worry. From the relationships in Pascal's Triangle we notice

$$\begin{aligned}
2^5\cdot 3 &= \left[\binom{6}{-1}+\binom{7}{0}\right]+\left[\binom{6}{1}+\binom{7}{2}\right]+\left[\binom{6}{3}+\binom{7}{4}\right]+\left[\binom{6}{5}+\binom{7}{6}\right]\\
2^6\cdot 3 &= \left[\binom{7}{0}+\binom{8}{1}\right]+\left[\binom{7}{2}+\binom{8}{3}\right]+\left[\binom{7}{4}+\binom{8}{5}\right]+\left[\binom{7}{6}+\binom{8}{7}\right],
\end{aligned}$$

which suggest the next entry will be

$$\begin{aligned}
2^7\cdot 3 \quad &= \quad \left[\binom{8}{-1}+\binom{9}{0}\right]+\left[\binom{8}{1}+\binom{9}{2}\right]+\left[\binom{8}{3}+\binom{9}{4}\right]+\left[\binom{8}{5}+\binom{9}{6}\right]+\left[\binom{8}{7}+\binom{9}{8}\right]\\
&= 1+44+182+140+17.
\end{aligned}$$

We write it as

$$384 = 0+0+0+1+44+182+140+17,$$

which tells us $e_2 = e_4 = e_6 = 0$. We substitute these values into expression 75,

$$1\cdot E_8 + 44\cdot E_{10} + 182\cdot E_{12} + 140\cdot E_{14} + 17\cdot E_{16} = 384\cdot\left(\frac{n(n+1)}{2}\right)^7,$$

and then simplify

$$17\cdot E_{16} = 384\cdot\left(\frac{n(n+1)}{2}\right)^7 - \left(E_8 + 44\cdot E_{10} + 182\cdot E_{12} + 140\cdot E_{14}\right),$$

which, for comparison, contains only four terms. The final expression will be

$$\begin{aligned} 17 \cdot E_{16} &= 384 \cdot \left(\frac{n(n+1)}{2}\right)^7 - a_2 \cdot \left(\frac{n(n+1)}{2}\right)^6 + a_3 \cdot \left(\frac{n(n+1)}{2}\right)^5 \\ &\quad \mp \cdots - a_6 \cdot \left(\frac{n(n+1)}{2}\right)^2 + a_7 \cdot E_4 \end{aligned}$$

for some rational numbers $a_2, a_3, \ldots, a_6, a_7$. Last, we reinsert it into

$$\sum k^{16} = E_{16} \cdot \sum k^2.$$

Unfortunately, the algorithm still is recursive. In order to find the expression for E_{16} we need to know already the expressions for $E_2, E_4, E_6, \ldots, E_{14}$. Can we improve upon this? For the previous expressions for E_{2m} and O_{2m+1} it is tempting to replace the coefficients with the entries from Pascal's Triangle and then to try to rewrite them into explicit expressions. However, to proceed that way seems to place us back into lengthy calculations reminiscent of the old result. Therefore, for the question at the start of the paper we have an answer, but we cannot give the exact values for the coefficients, a_i and b_j. We have to find them.

Historical Note After the author had completed the paper he was informed he had rediscovered a result credited to Johann Faulhaber, 1580-1635. For a discussion of Faulhaber's original work and some developments over the ensuing centuries, the reader can consult the paper "Johann Faulhaber and Sums of Powers" by Donald E. Knuth. Perhaps at a later time the author will address the matter himself.

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On a lighter note, the author thanks Mark Twain for writing "The Adventures of Huckleberry Finn," which he had been reading in his time away from working on this paper and which had provided the inspiration for some of the section titles.

Last, the author is indebted to someone who wishes to remain anonymous, who was kind enough to read a rough version of the paper and suggested ways to improve it.

References

[1] George Polya, *How to Solve It*, Princeton University Press, 1988.

[2] George Polya, *Mathematics and Plausible Reasoning*, Princeton University Press, 1990.

[3] Mark Twain, *The Adventures of Huckleberry Finn*, Washington Square Press, 1973.

24 Appendix

24.1 $\sum_{k=1}^{n} k^7$

What is

$$\sum_{k=1}^{n} k^7 = 1^7 + 2^7 + 3^7 + \cdots + n^7\,?$$

We know that

$$\sum_{k=1}^{n} k^7 = (n+1) \cdot \sum_{k=1}^{n} k^6 - \sum_{k=1}^{n} \sum_{l=1}^{k} l^6. \tag{76}$$

We know also that

$$\sum_{k=1}^{n} k^6 = \frac{n - 7n^3 + 21n^5 + 21n^6 + 6n^7}{42},$$

which implies

$$\sum_{k=1}^{n} \sum_{l=1}^{k} l^6 = \frac{\sum_{k=1}^{n} k - 7 \cdot \sum_{k=1}^{n} k^3 + 21 \cdot \sum_{k=1}^{n} k^5 + 21 \cdot \sum_{k=1}^{n} k^6 + 6 \cdot \sum_{k=1}^{n} k^7}{42}.$$

Therefore we may rewrite expression 76 as

$$\begin{aligned} \sum_{k=1}^{n} k^7 &= (n+1) \cdot \sum_{k=1}^{n} k^6 - \frac{\sum_{k=1}^{n} k - 7 \cdot \sum_{k=1}^{n} k^3 + 21 \cdot \sum_{k=1}^{n} k^5 + 21 \cdot \sum_{k=1}^{n} k^6 + 6 \cdot \sum_{k=1}^{n} k^7}{42} \\ &= (n+1) \cdot \sum_{k=1}^{n} k^6 - \frac{1}{42} \cdot \sum_{k=1}^{n} k + \frac{1}{6} \cdot \sum_{k=1}^{n} k^3 - \frac{1}{2} \cdot \sum_{k=1}^{n} k^5 - \frac{1}{2} \cdot \sum_{k=1}^{n} k^6 - \frac{1}{7} \cdot \sum_{k=1}^{n} k^7 \end{aligned}$$

$$\frac{8}{7} \cdot \sum_{k=1}^{n} k^7 = \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k^6 - \frac{1}{42} \cdot \sum_{k=1}^{n} k + \frac{1}{6} \cdot \sum_{k=1}^{n} k^3 - \frac{1}{2} \cdot \sum_{k=1}^{n} k^5.$$

For $\sum_{k=1}^{n} k^5$ we substitute

$$\frac{2n\,(n+1) - 1}{3} \cdot \sum_{k=1}^{n} k^3$$

and collect the terms which involve $\sum_{k=1}^{n} k^3$:

$$\frac{8}{7} \cdot \sum_{k=1}^{n} k^7 = \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k^6 - \frac{1}{42} \cdot \sum_{k=1}^{n} k + \left(\frac{1}{6} - \frac{1}{2} \cdot \frac{2n\,(n+1) - 1}{3}\right) \cdot \sum_{k=1}^{n} k^3$$

$$\frac{8}{7} \cdot \sum_{k=1}^{n} k^7 = \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k^6 - \frac{1}{42} \cdot \sum_{k=1}^{n} k + \left(\frac{1 - n\,(n+1)}{3}\right) \cdot \sum_{k=1}^{n} k^3.$$

Since

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 = \sum_{k=1}^{n} k \cdot \sum_{k=1}^{n} k,$$

we collect terms of the form $\sum_{k=1}^{n} k$:

$$\frac{8}{7} \cdot \sum_{k=1}^{n} k^7 = \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k^6 - \frac{1}{42} \cdot \sum_{k=1}^{n} k + \left(\frac{1-n(n+1)}{3}\right) \cdot \sum_{k=1}^{n} k \cdot \sum_{k=1}^{n} k$$

$$\frac{8}{7} \cdot \sum_{k=1}^{n} k^7 = \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k^6 - \left(\frac{1}{42} - \left(\frac{1-n(n+1)}{3}\right) \cdot \frac{n(n+1)}{2}\right) \cdot \sum_{k=1}^{n} k$$

$$\frac{8}{7} \cdot \sum_{k=1}^{n} k^7 = \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k^6 - \left(\frac{1}{42} - \frac{n(n+1) - (n(n+1))^2}{6}\right) \cdot \sum_{k=1}^{n} k$$

$$\frac{8}{7} \cdot \sum_{k=1}^{n} k^7 = \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k^6 - \left(\frac{7(n(n+1))^2 - 7n(n+1) + 1}{42}\right) \cdot \sum_{k=1}^{n} k,$$

which allows us to write

$$\sum_{k=1}^{n} k^7 = \frac{7}{8} \cdot \frac{2n+1}{2} \cdot \sum_{k=1}^{n} k^6 - \left(\frac{7(n(n+1))^2 - 7n(n+1) + 1}{48}\right) \cdot \sum_{k=1}^{n} k. \quad (77)$$

For $\sum_{k=1}^{n} k^6$ we substitute

$$\sum_{k=1}^{n} k^6 = \left(\frac{3(n(n+1))^2 - 3n(n+1) + 1}{7}\right) \cdot \sum_{k=1}^{n} k^2$$

and rewrite the left terms of expression 77 as

$$\frac{7}{8} \cdot \frac{2n+1}{2} \cdot \left(\frac{3(n(n+1))^2 - 3n(n+1) + 1}{7}\right) \cdot \sum_{k=1}^{n} k^2$$

$$= \frac{7}{8} \cdot \frac{2n+1}{2} \cdot \left(\frac{3(n(n+1))^2 - 3n(n+1) + 1}{7}\right) \cdot \frac{2n+1}{3} \cdot \frac{n(n+1)}{2}$$

$$= \frac{7}{8} \cdot \frac{2n+1}{2} \cdot \frac{2n+1}{3} \cdot \left(\frac{3(n(n+1))^2 - 3n(n+1) + 1}{7}\right) \cdot \frac{n(n+1)}{2}$$

$$= \frac{7(2n+1)^2}{48} \cdot \left(\frac{3(n(n+1))^2 - 3n(n+1) + 1}{7}\right) \cdot \sum_{k=1}^{n} k$$

$$= (2n+1)^2 \cdot \left(\frac{3(n(n+1))^2 - 3n(n+1) + 1}{48}\right) \cdot \sum_{k=1}^{n} k.$$

We substitute this back into expression 77 to get

$$\begin{aligned}\sum_{k=1}^{n} k^7 &= (2n+1)^2 \cdot \left(\frac{3\,(n\,(n+1))^2 - 3n\,(n+1) + 1}{48}\right) \cdot \sum_{k=1}^{n} k \\ &\quad - \left(\frac{7\,(n\,(n+1))^2 - 7n\,(n+1) + 1}{48}\right) \cdot \sum_{k=1}^{n} k \\ &= \left((2n+1)^2 \cdot \frac{3\,(n\,(n+1))^2 - 3n\,(n+1) + 1}{48} - \frac{7\,(n\,(n+1))^2 - 7n\,(n+1) + 1}{48}\right) \cdot \sum_{k=1}^{n} k \\ &= \frac{1}{8} \cdot \left((2n+1)^2 \cdot \frac{3\,(n\,(n+1))^2 - 3n\,(n+1) + 1}{6} - \frac{7\,(n\,(n+1))^2 - 7n\,(n+1) + 1}{6}\right) \cdot \sum_{k=1}^{n} k. \qquad (78)\end{aligned}$$

Consider expression 78 a first derivation for $\sum_{k=1}^{n} k^7$.

Let us continue. We rewrite the expression inside the parentheses as

$$\begin{aligned}&(2n+1)^2 \cdot 2 \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - (2n+1)^2 \cdot \frac{n\,(n+1)}{2} + (2n+1)^2 \cdot \frac{1}{6} \\ &\quad - \frac{14}{3} \cdot \left(\frac{n\,(n+1)}{2}\right)^2 + \frac{7}{3} \cdot \frac{n\,(n+1)}{2} - \frac{1}{6} \\ &= \left(2 \cdot (2n+1)^2 - \frac{14}{3}\right) \cdot \left(\frac{n\,(n+1)}{2}\right)^2 + \left(-(2n+1)^2 + \frac{7}{3}\right) \cdot \frac{n\,(n+1)}{2} + \left(\frac{(2n+1)^2}{6} - \frac{1}{6}\right) \\ &= \frac{24n^2 + 24n - 8}{3} \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - \frac{12n^2 + 12n - 4}{3} \cdot \frac{n\,(n+1)}{2} + \frac{4}{3} \cdot \frac{n\,(n+1)}{2}.\end{aligned}$$

If we multiply the expression by

$$\sum_{k=1}^{n} k = \frac{n\,(n+1)}{2}$$

then we get

$$\begin{aligned}&\frac{24n^2 + 24n - 8}{3} \cdot \frac{n\,(n+1)}{2} \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - \frac{12n^2 + 12n - 4}{3} \cdot \frac{n\,(n+1)}{2} \cdot \frac{n\,(n+1)}{2} \\ &\quad + \frac{4}{3} \cdot \frac{n\,(n+1)}{2} \cdot \frac{n\,(n+1)}{2} \\ &= \frac{24n^2 + 24n - 8}{3} \cdot \frac{n\,(n+1)}{2} \cdot \sum_{k=1}^{n} k^3 - \frac{12n^2 + 12n - 4}{3} \cdot \sum_{k=1}^{n} k^3 + \frac{4}{3} \cdot \sum_{k=1}^{n} k^3\end{aligned}$$

$$= \frac{8\left(3(n(n+1))^2 - n(n+1)\right)}{6} \cdot \sum_{k=1}^{n} k^3 - \frac{4\,(3n(n+1)-1)}{3} \cdot \sum_{k=1}^{n} k^3 + \frac{4}{3} \cdot \sum_{k=1}^{n} k^3.$$

Remembering the form of expression 78, if we multiply by $\frac{1}{8}$ then we get

$$\sum_{k=1}^{n} k^7 = \frac{1}{8} \cdot \left(\frac{8\left(3(n(n+1))^2 - n(n+1)\right)}{6} \cdot \sum_{k=1}^{n} k^3 - \frac{4\,(3n(n+1)-1)}{3} \cdot \sum_{k=1}^{n} k^3 + \frac{4}{3} \cdot \sum_{k=1}^{n} k^3 \right)$$

$$= \frac{1}{8} \cdot \left(\frac{8\left(3(n(n+1))^2 - n(n+1)\right)}{6} - \frac{4\,(3n(n+1)-1)}{3} + \frac{4}{3} \right) \cdot \sum_{k=1}^{n} k^3$$

$$= \frac{1}{8} \cdot \left(\frac{12(n(n+1))^2 - 16n(n+1) + 8}{3} \right) \cdot \sum_{k=1}^{n} k^3$$

$$= \left(\frac{3(n(n+1))^2 - 4n(n+1) + 2}{6} \right) \cdot \sum_{k=1}^{n} k^3. \tag{79}$$

Another way to write it is

$$\sum_{k=1}^{n} k^7 = \frac{1}{8} \cdot \left(\frac{12(n(n+1))^2 - 16n(n+1) + 8}{3} \right) \cdot \sum_{k=1}^{n} k^3$$

$$= \frac{1}{8} \cdot \left(\frac{48}{3} \cdot \left(\frac{n(n+1)}{2} \right)^2 - \frac{32}{3} \cdot \frac{n(n+1)}{2} + \frac{8}{3} \right) \cdot \sum_{k=1}^{n} k^3$$

$$= \frac{1}{3} \cdot \left(6 \cdot \left(\frac{n(n+1)}{2} \right)^2 - 4 \cdot \frac{n(n+1)}{2} + 1 \right) \cdot \sum_{k=1}^{n} k^3. \tag{80}$$

Both expressions simplify to

$$\sum_{k=1}^{n} k^7 = \frac{2n^2 - 7n^4 + 14n^6 + 12n^7 + 3n^8}{24}.$$

24.2 Does a prime $p = 2m + 1$ divide $\sum_{k=1}^{m} k^2$?

In the calculations for E_{10}, under the assumption that both e_8 and a_2 are integers, the equation

$$\frac{8}{3} \cdot e_8 = a_2$$

tells us that $e_8 = 3c$ for some integer c. If we were to carry out the calculations for E_{12}, the analogous conjecture would be

$$e_4 E_4 + e_6 E_6 + e_8 E_8 + e_{10} E_{10} + 13 E_{12} = 96 \cdot \left(\frac{n(n+1)}{2} \right)^5$$

and the list of coefficients would be

$$\begin{aligned} 1 &= 1 \\ 6 &= 1 + 5 \\ 12 &= 0 + 5 + 7 \\ 24 &= 0 + 1 + 14 + 9 \\ 48 &= 0 + 0 + 7 + 30 + 11 \\ 96 &= 0 + e_4 + e_6 + e_8 + e_{10} + 13. \end{aligned}$$

Looking at the diagonal containing e_{10}, we would suspect $e_{10} = 55 = \sum_{k=1}^{5} k^2$.

Further, under an analogous system of equations, under the assumption that both e_{10} and a_2 were integers, the equation

$$\frac{48}{11} \cdot e_{10} = a_2$$

would tell us $e_{10} = 11d$ for some integer d. Therefore we would have $11d = 55 = \sum_{k=1}^{5} k^2$, which means 11 divides $55 = \sum_{k=1}^{5} k^2$. We write $11 \mid 55 = \sum_{k=1}^{5} k^2$. That happens to be true: $11 \cdot 5 = 55$. If we look again at the list of coefficients then we notice

$$3 \mid 30 = \sum_{k=1}^{4} k^2, \; 7 \mid 14 = \sum_{k=1}^{3} k^2, \; 5 \mid 5 = \sum_{k=1}^{2} k^2.$$

If we ignore the case of $3 \mid 30 = \sum_{k=1}^{4} k^2$ for the moment, we may make the following observation:

Exercise if $p = 2m + 1$ is prime then p divides $\sum_{k=1}^{m} k^2$. Prove or disprove.

24.3 Inductive Reasoning Sometimes Goes Wrong

In Section 3 we proved formally that

$$\sum_{k=1}^{n} k^{m+1} + \sum_{k=1}^{n}\sum_{l=1}^{k} l^m = (n+1)\cdot\sum_{k=1}^{n} k^m$$

for positive integers m. Over the course of Parts 1 and 2 we used it to derive

$$\sum k^3 = O_3 \cdot \sum k^3$$

$$\sum k^5 = O_5 \cdot \sum k^3$$

$$\sum k^7 = O_7 \cdot \sum k^3$$

$$\sum k^9 = O_9 \cdot \sum k^3,$$

where

$$O_3 = 1$$

$$O_5 = \frac{1}{3}\cdot\left(4\cdot\frac{n\,(n+1)}{2} - 1\right)$$

$$O_7 = \frac{1}{4}\cdot\left(8\cdot\left(\frac{n(n+1)}{2}\right)^2 - 4\cdot O_5\right)$$

$$O_9 = \frac{1}{5}\cdot\left(16\cdot\left(\frac{n(n+1)}{2}\right)^3 - 20\cdot\left(\frac{n(n+1)}{2}\right)^2 + 9\cdot O_5\right). \tag{81}$$

The expressions for $\sum k^3, \sum k^5, \sum k^7$, and $\sum k^9$ were true. In Part 3 we used inductive reasoning to arrive at

$$O_{11} = \frac{1}{6}\cdot\left(32\cdot\left(\frac{n(n+1)}{2}\right)^4 - 64\cdot\left(\frac{n(n+1)}{2}\right)^3 + 68\cdot\left(\frac{n(n+1)}{2}\right)^2 - 30\cdot O_5\right), \tag{82}$$

which we believed satisfied

$$\sum k^{11} = O_{11}\cdot\sum k^3.$$

As a check on our inductive reasoning we verified it in the special case of

$$\sum_{k=1}^{9} k^{11} = 42,364,319,625.$$

Given how complicated the sums were, we believed that was sufficient evidence to suggest the result was correct. What if instead we had verified it in the following way?

In the expressions in (81) the coefficients satisfy

$$\begin{aligned}3 &= 4-1\\4 &= 8-4\\5 &= 16-20+9.\end{aligned}$$

In the new result for O_{11} they satisfy

$$6 = 32-64+68-30.$$

Impressive, right? It also bears an analogy with our previous conjectures. Is it enough to justify we found the correct expression? Not quite.

In the derivation for O_{11} we had the conjectures

$$o_5 O_5 + o_7 O_7 + o_9 O_9 + o_{11} O_{11} = 32 \cdot \left(\frac{n(n+1)}{2}\right)^4$$

and

$$o_5 + o_7 + o_9 + o_{11} = 32 \tag{83}$$

and the system of equations

$$\begin{gathered}-b_2 + o_9 \cdot \tfrac{32}{10} = 0\\ b_3 - o_9 \cdot \tfrac{40}{10} + o_7 \cdot \tfrac{6}{3} = 0\\ -b_4 + o_9 \cdot \tfrac{18}{10} - o_7 \cdot \tfrac{3}{3} + o_5 = 0.\end{gathered}$$

We chose $b_1 = 32$ and $o_{11} = 6$, which allowed us to write

$$o_5 + o_7 + o_9 = 26.$$

In the system of equations, if we add the expressions together then we get

$$-b_2 + b_3 - b_4 + \left(\frac{32}{10} - \frac{40}{10} + \frac{18}{10}\right) \cdot o_9 + \left(\frac{6}{3} - \frac{3}{3}\right) \cdot o_7 + \frac{3}{3} \cdot o_5 = 0,$$

which is

$$-b_2 + b_3 - b_4 = -\left(o_9 + o_7 + o_5\right).$$

From expression 83 it follows that

$$\begin{aligned}6 \quad = \quad & o_{11} = 32 - (o_5 + o_7 + o_9)\\ & = b_1 - b_2 + b_3 - b_4.\end{aligned}$$

What this means is, if we start with the choices of $b_1 = 32$ and $o_{11} = 6$, for *any* o_5, o_7, o_9 which satisfy $o_5 + o_7 + o_9 = 26$ it will follow that $b_1 - b_2 + b_3 - b_4 = 6$. For example, if we choose

$$o_5 = 24,\ o_7 = 1,\ o_9 = 1,$$

which are the *wrong* values for o_5, o_7, o_9, the system of equations tells us

$$\begin{aligned}
b_2 &= 1 \cdot \frac{32}{10} = \frac{16}{5} \\
b_3 &= 1 \cdot \frac{40}{10} - 1 \cdot 2 = 2 \\
b_4 &= 1 \cdot \frac{18}{10} - 1 \cdot \frac{3}{3} + 24 = \frac{124}{5},
\end{aligned}$$

which yields

$$O_{11} = \frac{1}{6} \cdot \left(32 \cdot \left(\frac{n(n+1)}{2} \right)^4 - \frac{16}{5} \cdot \left(\frac{n(n+1)}{2} \right)^3 - 2 \cdot \left(\frac{n(n+1)}{2} \right)^2 - \frac{124}{5} \cdot O_5 \right). \tag{84}$$

We still have

$$6 = 32 - \frac{16}{5} + 2 - \frac{124}{5},$$

but of course expressions 82 and 84 produce different values. Therefore such an observation would have been insufficient to suggest we had found the correct expression.

Exercise prove the result for the general case: if $b_1, b_2, b_3, \ldots, b_{m-1}$ are the coefficients for the terms in O_{2m+1} and the conjectures of (83) are true,

$$b_1 - b_2 + b_3 \mp \cdots \mp b_{m-1} = m + 1.$$

For E_{2m} and its coefficients, state and prove the analogous result.

24.4 Shaping the Conjectures

Our previous work suggests that if we want to establish the algorithm then we need to prove the following conjectures:

Conjecture 1

$$\sum_{k=1}^{n} k^{2m} = E_{2m}(n) \cdot \sum_{k=1}^{n} k^2$$

$$\sum_{k=1}^{n} k^{2m+1} = O_{2m+1}(n) \cdot \sum_{k=1}^{n} k^3,$$

where m is a positive integer and $E_{2m}(n)$ and $O_{2m+1}(n)$ are rational expressions involving n.

Conjecture 2.0 $E_{2m}(n)$ satisfies

$$E_2(n) = 1$$

$$E_2(n) + 5 \cdot E_4(n) = 2 \cdot 3 \cdot \frac{n(n+1)}{2}$$

$$5 \cdot E_4(n) + 7 \cdot E_6(n) = 2^2 \cdot 3 \cdot \left(\frac{n\,(n+1)}{2}\right)^2, \ldots,$$

$$e_2 E_2(n) + e_4 E_4(n) + e_6 E_6(n) + \cdots + e_{2m} E_{2m}(n) = 2^{m-1} \cdot 3 \cdot \left(\frac{n(n+1)}{2}\right)^{m-1},$$

where $e_2, e_4, e_6, \ldots, e_{2m}$ are integers such that

$$e_2 + e_4 + e_6 + \cdots + e_{2m} = 2^{m-1} \cdot 3.$$

$O_{2m+1}(n)$ satisfies

$$O_3(n) = 1$$

$$O_3(n) + 3 \cdot O_5(n) = 2^2 \cdot \frac{n(n+1)}{2}$$

$$4 \cdot O_5(n) + 4 \cdot O_7(n) = 2^3 \cdot \left(\frac{n(n+1)}{2}\right)^2, \ldots,$$

$$o_3 O_3(n) + o_5 O_5(n) + o_7 O_7(n) + \cdots + o_{2m+1} O_{2m+1}(n) = 2^m \cdot \left(\frac{n(n+1)}{2}\right)^{m-1},$$

where $o_3, o_5, o_7, \ldots, o_{2m+1}$ are integers such that

$$o_3 + o_5 + o_7 + \cdots + o_{2m+1} = 2^m.$$

Conjecture 3.0 the sums for $e_2, e_4, e_6, \ldots, e_{2m}$ and $o_3, o_5, o_7, \ldots, o_{2m+1}$ can be expressed as

$$\begin{aligned}
1 &= 1\\
2\cdot 3 &= e_2 + e_4\\
&= \left[\binom{2}{-1} + \binom{3}{0}\right] + \left[\binom{2}{1} + \binom{3}{2}\right]\\
2^2\cdot 3 &= e_4 + e_6\\
&= \left[\binom{3}{0} + \binom{4}{1}\right] + \left[\binom{3}{2} + \binom{4}{3}\right]\\
&\vdots\\
2^{m-1}\cdot 3 &= e_2 + e_4 + e_6 + \cdots + e_{2m}\\
&= \left[\binom{m}{-1} + \binom{m+1}{0}\right] + \left[\binom{m}{1} + \binom{m+1}{2}\right] + \left[\binom{m}{3} + \binom{m+1}{4}\right]\\
&\quad + \cdots + \left[\binom{m}{m-1} + \binom{m+1}{m}\right]\\
2^{m}\cdot 3 &= e_2 + e_4 + e_6 + \cdots + e_{2m} + e_{2(m+1)}\\
&= \left[\binom{m+1}{0} + \binom{m+2}{1}\right] + \left[\binom{m+1}{2} + \binom{m+2}{3}\right] + \left[\binom{m+1}{4} + \binom{m+2}{5}\right]\\
&\quad + \cdots + \left[\binom{m+1}{m} + \binom{m+2}{m+1}\right]
\end{aligned}$$

and

$$\begin{aligned}
1 &= 1\\
2^2 &= o_3 + o_5\\
&= \binom{3}{0} + \binom{3}{2}\\
2^3 &= o_5 + o_7\\
&= \binom{4}{1} + \binom{4}{3}\\
&\vdots\\
2^m &= o_3 + o_5 + o_7 + \cdots + o_{2m+1}\\
&= \binom{m+1}{0} + \binom{m+1}{2} + \binom{m+1}{4} + \cdots + \binom{m+1}{m}\\
2^{m+1} &= o_3 + o_5 + o_7 + \cdots + o_{2m+1} + o_{2(m+1)+1}\\
&= \binom{m+2}{1} + \binom{m+2}{3} + \binom{m+2}{5} + \cdots + \binom{m+2}{m+1},
\end{aligned}$$

where some e_i and/or o_j are equal to zero.

We are going to keep Conjecture 1 but refine Conjectures 2.0 and 3.0. The reason? The term $2^{m-1} \cdot 3$ sticks out. For the previous calculations that was irrelevant, but now we will address it.

At the beginning of Part 4 we made a breakthrough by noticing we could rewrite the coefficients for O_{2m+1} and E_{2m},

$$\begin{aligned} 1 &= 1 \\ 4 &= 1+3 \\ 8 &= 0+4+4 \\ 16 &= 0+1+10+5 \\ 32 &= 0+0+6+20+6 \\ 64 &= 0+0+1+21+35+7 \\ 128 &= 0+0+0+8+56+56+8 \end{aligned} \tag{85}$$

and

$$\begin{aligned} 1 &= 1 \\ 6 &= 1+5 \\ 12 &= 0+5+7 \\ 24 &= 0+1+14+9 \\ 48 &= 0+0+7+30+11 \\ 96 &= 0+0+1+27+55+13 \\ 192 &= 0+0+0+9+77+91+15, \end{aligned} \tag{86}$$

in terms of entries from Pascal's Triangle,

$$\begin{aligned} 1 &= 1 \\ 2^2 &= \binom{3}{0} + \binom{3}{2} \\ 2^3 &= \binom{4}{1} + \binom{4}{3} \\ 2^4 &= \binom{5}{0} + \binom{5}{2} + \binom{5}{4} \\ 2^5 &= \binom{6}{1} + \binom{6}{3} + \binom{6}{5} \\ 2^6 &= \binom{7}{0} + \binom{7}{2} + \binom{7}{4} + \binom{7}{6} \\ 2^7 &= \binom{8}{1} + \binom{8}{3} + \binom{8}{5} + \binom{8}{7} \end{aligned}$$

and

$$
\begin{aligned}
1 &= 1\\
2\cdot 3 &= \left[\binom{2}{-1}+\binom{3}{0}\right]+\left[\binom{2}{1}+\binom{3}{2}\right]\\
2^2\cdot 3 &= \left[\binom{3}{0}+\binom{4}{1}\right]+\left[\binom{3}{2}+\binom{4}{3}\right]\\
2^3\cdot 3 &= \left[\binom{4}{-1}+\binom{5}{0}\right]+\left[\binom{4}{1}+\binom{5}{2}\right]+\left[\binom{4}{3}+\binom{5}{4}\right]\\
2^4\cdot 3 &= \left[\binom{5}{0}+\binom{6}{1}\right]+\left[\binom{5}{2}+\binom{6}{3}\right]+\left[\binom{5}{4}+\binom{6}{5}\right]\\
2^5\cdot 3 &= \left[\binom{6}{-1}+\binom{7}{0}\right]+\left[\binom{6}{1}+\binom{7}{2}\right]+\left[\binom{6}{3}+\binom{7}{4}\right]+\left[\binom{6}{5}+\binom{7}{6}\right]\\
2^6\cdot 3 &= \left[\binom{7}{0}+\binom{8}{1}\right]+\left[\binom{7}{2}+\binom{8}{3}\right]+\left[\binom{7}{4}+\binom{8}{5}\right]+\left[\binom{7}{6}+\binom{8}{7}\right].
\end{aligned}
$$

If we look again at these expressions then we notice, for example,

$$
\begin{aligned}
2^3\cdot 3 &= \left[\binom{4}{-1}+\binom{5}{0}\right]+\left[\binom{4}{1}+\binom{5}{2}\right]+\left[\binom{4}{3}+\binom{5}{4}\right]\\
&= \binom{4}{1}+\binom{4}{3}+\binom{5}{0}+\binom{5}{2}+\binom{5}{4}
\end{aligned}
$$

and

$$
\begin{aligned}
2^4\cdot 3 &= \left[\binom{5}{0}+\binom{6}{1}\right]+\left[\binom{5}{2}+\binom{6}{3}\right]+\left[\binom{5}{4}+\binom{6}{5}\right]\\
&= \binom{5}{0}+\binom{5}{2}+\binom{5}{4}+\binom{6}{1}+\binom{6}{3}+\binom{6}{5}.
\end{aligned}
$$

The coefficients for O_{2m+1} appear in those for E_{2m}. We can write

$$
\begin{aligned}
2^3\cdot 3 &= \binom{4}{1}+\binom{4}{3}+\binom{5}{0}+\binom{5}{2}+\binom{5}{4}\\
&= 2^3+2^4
\end{aligned}
$$

and

$$
\begin{aligned}
2^4\cdot 3 &= \binom{5}{0}+\binom{5}{2}+\binom{5}{4}+\binom{6}{1}+\binom{6}{3}+\binom{6}{5}\\
&= 2^4+2^5.
\end{aligned}
$$

In other words,

$$3 = 1 + 2. \tag{87}$$

Therefore we will write the previous expressions

$$\begin{gathered}
E_2 = 1 \\
E_2 + 5E_4 = 2 \cdot 3 \cdot \tfrac{n(n+1)}{2} \\
5E_4 + 7E_6 = 2^2 \cdot 3 \cdot \left(\tfrac{n(n+1)}{2}\right)^2 \\
E_4 + 14E_6 + 9E_8 = 2^3 \cdot 3 \cdot \left(\tfrac{n(n+1)}{2}\right)^3 \\
7E_6 + 30E_8 + 11E_{10} = 2^4 \cdot 3 \cdot \left(\tfrac{n(n+1)}{2}\right)^4 \\
E_6 + 27E_8 + 55E_{10} + 13E_{12} = 2^5 \cdot 3 \cdot \left(\tfrac{n(n+1)}{2}\right)^5
\end{gathered}$$

instead as

$$\begin{gathered}
E_2 = 1 \\
E_2 + 5E_4 = \left(2 + 2^2\right) \cdot \tfrac{n(n+1)}{2} \\
5E_4 + 7E_6 = \left(2^2 + 2^3\right) \cdot \left(\tfrac{n(n+1)}{2}\right)^2 \\
E_4 + 14E_6 + 9E_8 = \left(2^3 + 2^4\right) \cdot \left(\tfrac{n(n+1)}{2}\right)^3 \\
7E_6 + 30E_8 + 11E_{10} = \left(2^4 + 2^5\right) \cdot \left(\tfrac{n(n+1)}{2}\right)^4 \\
E_6 + 27E_8 + 55E_{10} + 13E_{12} = \left(2^5 + 2^6\right) \cdot \left(\tfrac{n(n+1)}{2}\right)^5
\end{gathered} \tag{88}$$

and restate Conjecture 2.0 as

Conjecture 2.1 $O_{2m+1}(n)$ satisfies

$$O_3(n) = 1$$

$$O_3(n) + 3 \cdot O_5(n) = 2^2 \cdot \frac{n(n+1)}{2}$$

$$4 \cdot O_5(n) + 4 \cdot O_7(n) = 2^3 \cdot \left(\frac{n(n+1)}{2}\right)^2, \ldots,$$

$$o_3 O_3(n) + o_5 O_5(n) + o_7 O_7(n) + \cdots + o_{2m+1} O_{2m+1}(n) = 2^m \cdot \left(\frac{n(n+1)}{2}\right)^{m-1},$$

where $o_3, o_5, o_7, \ldots, o_{2m+1}$ are integers such that

$$o_3 + o_5 + o_7 + \cdots + o_{2m+1} = 2^m.$$

$E_{2m}(n)$ satisfies

$$E_2(n) = 1$$

$$E_2(n) + 5 \cdot E_4(n) = \left(2 + 2^2\right) \cdot \frac{n(n+1)}{2}$$

$$5 \cdot E_4(n) + 7 \cdot E_6(n) = \left(2^2 + 2^3\right) \cdot \left(\frac{n\,(n+1)}{2}\right)^2, \ldots,$$

$$e_2E_2(n)+e_4E_4(n)+e_6E_6(n)+\cdots+e_{2m}E_{2m}(n) = \left(2^{m-1}+2^m\right)\cdot\left(\frac{n(n+1)}{2}\right)^{m-1},$$

where $e_2, e_4, e_6, \ldots, e_{2m}$ are integers such that

$$e_2 + e_4 + e_6 + \cdots + e_{2m} = \left(2^{m-1} + 2^m\right).$$

We will restate Conjecture 3.0 as

Conjecture 3.1 the sums for $o_3, o_5, o_7, \ldots, o_{2m+1}$ and $e_2, e_4, e_6, \ldots, e_{2m}$ can be expressed as

$$\begin{aligned}
1 &= 1 \\
2^2 &= o_3 + o_5 \\
&= \binom{3}{0} + \binom{3}{2} \\
2^3 &= o_5 + o_7 \\
&= \binom{4}{1} + \binom{4}{3} \\
&\vdots \\
2^m &= o_3 + o_5 + o_7 + \cdots + o_{2m+1} \\
&= \binom{m+1}{0} + \binom{m+1}{2} + \binom{m+1}{4} + \cdots + \binom{m+1}{m} \\
2^{m+1} &= o_3 + o_5 + o_7 + \cdots + o_{2m+1} + o_{2(m+1)+1} \\
&= \binom{m+2}{1} + \binom{m+2}{3} + \binom{m+2}{5} + \cdots + \binom{m+2}{m+1}
\end{aligned}$$

and

$$\begin{aligned}
1 &= 1 \\
2 + 2^2 &= e_2 + e_4 \\
&= \binom{2}{1} + \binom{3}{0} + \binom{3}{2} \\
2^2 + 2^3 &= e_4 + e_6 \\
&= \binom{3}{0} + \binom{3}{2} + \binom{4}{1} + \binom{4}{3} \\
&\vdots \\
2^{m-1} + 2^m &= e_2 + e_4 + e_6 + \cdots + e_{2m} \\
&= \binom{m}{-1} + \binom{m}{1} + \binom{m}{3} + \cdots + \binom{m}{m-1} \\
&+ \binom{m+1}{0} + \binom{m+1}{2} + \binom{m+1}{4} + \cdots + \binom{m+1}{m} \\
2^m + 2^{m+1} &= e_2 + e_4 + e_6 + \cdots + e_{2m} + e_{2(m+1)} \\
&= \binom{m+1}{0} + \binom{m+1}{2} + \binom{m+1}{4} + \cdots + \binom{m+1}{m} \\
&+ \binom{m+2}{1} + \binom{m+2}{3} + \binom{m+2}{5} + \cdots + \binom{m+2}{m+1},
\end{aligned}$$

where some e_i and/or o_j are equal to zero.

24.4.1 Looking Back on Our Work

How did we miss this? In order to notice we could write

$$2^3 \cdot 3 = 2^3 \cdot (1+2) = 2^3 + 2^4,$$

why did we need to wait until we introduced Pascal's Triangle? Well, the aim of the paper was to find a general expression for $\sum k^m$. Focused on that singular purpose, in the calculations a $2^3 \cdot 3$ was as good as a $2^3 + 2^4$, right? It is more nuanced than that.

In the previous calculations we needed to know the exact values for e_i and o_j. For example, in the calculation for E_{12} in Section 20.1 we used the relationship

$$\begin{aligned}
96 \quad = \quad & e_6 + e_8 + e_{10} + e_{12} \\
& = 1 + 27 + 55 + 13.
\end{aligned}$$

We needed to know $e_6 = 1, e_8 = 27, e_{10} = 55$, and $e_{12} = 13$. The new observation does something different. Look at Conjecture 3.1 in terms of the sums in (85) and (86):

$$
\begin{aligned}
1 &= 1 \\
\\
2 + 2^2 &= 1 + 5 \\
&= 2 + (1 + 3) \\
\\
2^2 + 2^3 &= 0 + 5 + 7 \\
&= (1 + 3) + (4 + 4) \\
\\
2^3 + 2^4 &= 0 + 1 + 14 + 9 \\
&= (4 + 4) + (1 + 10 + 5) \\
\\
2^4 + 2^5 &= 0 + 0 + 7 + 30 + 11 \\
&= (1 + 10 + 5) + (6 + 20 + 6) \\
\\
2^5 + 2^6 &= 0 + 0 + 1 + 27 + 55 + 13 \\
&= (6 + 20 + 6) + (1 + 21 + 35 + 7) \\
\\
2^6 + 2^7 &= 0 + 0 + 0 + 9 + 77 + 91 + 15 \\
&= (1 + 21 + 35 + 7) + (8 + 56 + 56 + 8). \qquad (89)
\end{aligned}
$$

Shocking, right? Unfortunately, it tells us

$$
\begin{aligned}
2^5 + 2^6 \quad = \quad & 1 + 27 + 55 + 13 \\
&= (6 + 20 + 6) + (1 + 21 + 35 + 7),
\end{aligned}
$$

which is a general sum concerning $e_2 + e_4 + e_6 + e_8 + e_{10} + e_{12}$. It does not tell us any information about each e_i. We might have been content to overlook a relationship which, when expressed in the entries in Pascal's Triangle, becomes painfully obvious because for calculating expressions for E_{2m} and O_{2m+1} it would have been of little use. Nevertheless, now that we are aware of it, can we make use of it?

Looking back on our entire solution, if there is one regret, it is we failed to bring together the separate patterns for E_{2m} and O_{2m+1}. The final answer to the initial question of the paper, which we gave in Section 23, is just as correct with two cases as with one, but in an intellectual or aesthetic sense we know something is missing. If we persevere one last time and look closely at the new conjectures then we will be able to bridge the gap.

In (88) we have

$$
\begin{aligned}
E_2 &= 1\\
E_2 + 5E_4 &= \left(2 + 2^2\right) \cdot \tfrac{n(n+1)}{2}\\
5E_4 + 7E_6 &= \left(2^2 + 2^3\right) \cdot \left(\tfrac{n(n+1)}{2}\right)^2\\
E_4 + 14E_6 + 9E_8 &= \left(2^3 + 2^4\right) \cdot \left(\tfrac{n(n+1)}{2}\right)^3\\
7E_6 + 30E_8 + 11E_{10} &= \left(2^4 + 2^5\right) \cdot \left(\tfrac{n(n+1)}{2}\right)^4\\
E_6 + 27E_8 + 55E_{10} + 13E_{12} &= \left(2^5 + 2^6\right) \cdot \left(\tfrac{n(n+1)}{2}\right)^5 .
\end{aligned}
$$

The corresponding relationships for O_{2m+1} are

$$
\begin{aligned}
O_3 &= 1\\
O_3 + 3O_5 &= 2^2 \cdot \tfrac{n(n+1)}{2}\\
4O_5 + 4O_7 &= 2^3 \cdot \left(\tfrac{n(n+1)}{2}\right)^2\\
O_5 + 10 \cdot O_7 + 5O_9 &= 2^4 \cdot \left(\tfrac{n(n+1)}{2}\right)^3\\
6O_7 + 20 \cdot O_9 + 6O_{11} &= 2^5 \cdot \left(\tfrac{n(n+1)}{2}\right)^4\\
O_7 + 21 \cdot O_9 + 35 \cdot O_{11} + 7 \cdot O_{13} &= 2^6 \cdot \left(\tfrac{n(n+1)}{2}\right)^5 .
\end{aligned}
$$

If we simply subtract one from the other then we get

$$
\begin{aligned}
\left(2 + 2^2\right) \cdot \frac{n\,(n+1)}{2} - 2^2 \cdot \frac{n\,(n+1)}{2} &= E_2 + 5E_4\\
&\quad - (O_3 + 3O_5)\\
\left(2^2 + 2^3\right) \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - 2^3 \cdot \left(\frac{n(n+1)}{2}\right)^2 &= 5E_4 + 7E_6\\
&\quad - (4O_5 + 4O_7)\\
\left(2^3 + 2^4\right) \cdot \left(\frac{n(n+1)}{2}\right)^3 - 2^4 \cdot \left(\frac{n(n+1)}{2}\right)^3 &= E_4 + 14E_6 + 9E_8\\
&\quad - (O_5 + 10 \cdot O_7 + 5O_9)\\
\left(2^4 + 2^5\right) \cdot \left(\frac{n(n+1)}{2}\right)^4 - 2^5 \cdot \left(\frac{n(n+1)}{2}\right)^4 &= 7E_6 + 30E_8 + 11E_{10}\\
&\quad - (6O_7 + 20 \cdot O_9 + 6O_{11})\\
\left(2^5 + 2^6\right) \cdot \left(\frac{n(n+1)}{2}\right)^5 - 2^6 \cdot \left(\frac{n(n+1)}{2}\right)^5 &= E_6 + 27E_8 + 55E_{10} + 13E_{12}\\
&\quad - (O_7 + 21 \cdot O_9 + 35 \cdot O_{11} + 7 \cdot O_{13})\,,
\end{aligned}
$$

which we rewrite as[10]

$$
\begin{aligned}
3O_5 + 2 \cdot \frac{n\,(n+1)}{2} &= 5E_4 \\
4O_5 + 4O_7 + 2^2 \cdot \left(\frac{n(n+1)}{2}\right)^2 &= 5E_4 + 7E_6 \\
O_5 + 10 \cdot O_7 + 5O_9 + 2^3 \cdot \left(\frac{n(n+1)}{2}\right)^3 &= E_4 + 14E_6 + 9E_8 \\
6O_7 + 20 \cdot O_9 + 6O_{11} + 2^4 \cdot \left(\frac{n(n+1)}{2}\right)^4 &= 7E_6 + 30E_8 + 11E_{10} \\
O_7 + 21 \cdot O_9 + 35 \cdot O_{11} + 7 \cdot O_{13} + 2^5 \cdot \left(\frac{n(n+1)}{2}\right)^5 &= E_6 + 27E_8 + 55E_{10} + 13E_{12}.
\end{aligned}
$$

Let us pause for a moment to look at this.

With regard to the previous calculations, now we can write the expressions for E_{2m} in terms of those for O_{2m+1}:

$$
\begin{aligned}
5E_4 &= 3O_5 + 2 \cdot \frac{n\,(n+1)}{2} \\
7E_6 &= 4O_5 + 4O_7 + 2^2 \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - 5E_4 \\
9E_8 &= O_5 + 10 \cdot O_7 + 5O_9 + 2^3 \cdot \left(\frac{n\,(n+1)}{2}\right)^3 - (E_4 + 14E_6) \\
11E_{10} &= 6O_7 + 20 \cdot O_9 + 6O_{11} + 2^4 \cdot \left(\frac{n\,(n+1)}{2}\right)^4 - (7E_6 + 30E_8) \\
13E_{12} &= O_7 + 21 \cdot O_9 + 35 \cdot O_{11} + 7 \cdot O_{13} + 2^5 \cdot \left(\frac{n\,(n+1)}{2}\right)^5 - (E_6 + 27E_8 + 55E_{10})\,.
\end{aligned}
\tag{90}
$$

Therefore, if we calculate the expressions for *only* O_{2m+1} then we get the ones for E_{2m} as a bonus. For example, if we calculate O_5 then we get E_4; if we calculate O_7 then we get E_6; if we calculate O_9 then we get E_8; and so on an so forth. Also, as the previous discussions have pointed out, it is easier to find the coefficients for O_{2m+1} than for E_{2m}: each o_j corresponds to a single entry in Pascal's Triangle. Unfortunately, on the right side of the equations in (90) we still have o_j and e_i. Can we get rid of the e_i?

Suppose we take a cross between Conjectures 3.0 and 3.1 and write the

[10]Since $O_3 = E_2 = 1$, $O_3 + 3O_5 + 2 \cdot \frac{n(n+1)}{2} = E_2 + 5E_4$ is equivalent to $3O_5 + 2 \cdot \frac{n(n+1)}{2} = 5E_4$.

expressions for e_i as

$$
\begin{aligned}
1 &= 1 \\
2+2^2 &= e_2+e_4 \\
&= \binom{2}{1}+\binom{3}{0}+\binom{3}{2} \\
2^2+2^3 &= e_4+e_6 \\
&= \binom{3}{0}+\binom{3}{2}+\binom{4}{1}+\binom{4}{3} \\
&= o_3+o_5+\left(o_5^{'}+o_7^{'}\right) \\
&= \left(o_3+o_5^{'}\right)+\left(o_5+o_7^{'}\right) \\
&\vdots \\
2^{m-1}+2^m &= e_2+e_4+e_6+\cdots+e_{2m} \\
&= \binom{m}{-1}+\binom{m}{1}+\binom{m}{3}+\cdots+\binom{m}{m-1} \\
&+\binom{m+1}{0}+\binom{m+1}{2}+\binom{m+1}{4}+\cdots+\binom{m+1}{m} \\
&= o_3+o_5+o_7+\cdots+o_{2(m-1)+1}+\left(o_5^{'}+o_7^{'}+o_9^{'}+\cdots+o_{2(m-1)+1}^{'}+o_{2m+1}^{'}\right) \\
&= \left(o_3+o_5^{'}\right)+\left(o_5+o_7^{'}\right)+\left(o_7+o_9^{'}\right)+\cdots+\left(o_{2(m-1)+1}+o_{2m+1}^{'}\right) \\
2^m+2^{m+1} &= e_2+e_4+e_6+\cdots+e_{2m}+e_{2(m+1)} \\
&= \binom{m+1}{0}+\binom{m+1}{2}+\binom{m+1}{4}+\cdots+\binom{m+1}{m} \\
&+\binom{m+2}{1}+\binom{m+2}{3}+\binom{m+2}{5}+\cdots+\binom{m+2}{m+1} \\
&= o_3+o_5+o_7+\cdots+o_{2m+1}+\left(o_5^{'}+o_7^{'}+o_9^{'}+\cdots+o_{2m+1}^{'}+o_{2(m+1)+1}^{'}\right) \\
&= \left(o_3+o_5^{'}\right)+\left(o_5+o_7^{'}\right)+\left(o_7+o_9^{'}\right)+\cdots+\left(o_{2m+1}+o_{2(m+1)+1}^{'}\right).
\end{aligned}
$$

Then we can rewrite the expressions in (90) as

$$
\begin{aligned}
5E_4 &= 3O_5 + 2 \cdot \frac{n\,(n+1)}{2} \\
(3+4) \cdot E_6 &= 4O_5 + 4O_7 + 2^2 \cdot \left(\frac{n\,(n+1)}{2}\right)^2 - (1+4) \cdot E_4 \\
(4+5) \cdot E_8 &= O_5 + 10 \cdot O_7 + 5O_9 + 2^3 \cdot \left(\frac{n\,(n+1)}{2}\right)^3 - ((0+1) \cdot E_4 + (4+10) \cdot E_6) \\
(5+6) \cdot E_{10} &= 6O_7 + 20 \cdot O_9 + 6O_{11} + 2^4 \cdot \left(\frac{n\,(n+1)}{2}\right)^4 - ((1+6) \cdot E_6 + (10+20) \cdot E_8) \\
(6+7) \cdot E_{12} &= O_7 + 21 \cdot O_9 + 35 \cdot O_{11} + 7 \cdot O_{13} + 2^5 \cdot \left(\frac{n\,(n+1)}{2}\right)^5 \\
&\quad - ((0+1) \cdot E_6 + (6+21) \cdot E_8 + (20+35) \cdot E_{10}) \,.
\end{aligned}
$$

We have eliminated the e_i. At long last we have brought together the two patterns. We will restate Conjecture 3.1 as

Conjecture 3.2 the sums for $o_3, o_5, o_7, \ldots, o_{2m+1}$ can be expressed as

$$
\begin{aligned}
1 &= 1 \\
2^2 &= o_3 + o_5 \\
&= \binom{3}{0} + \binom{3}{2} \\
2^3 &= o_5 + o_7 \\
&= \binom{4}{1} + \binom{4}{3} \\
&\vdots \\
2^m &= o_3 + o_5 + o_7 + \cdots + o_{2m+1} \\
&= \binom{m+1}{0} + \binom{m+1}{2} + \binom{m+1}{4} + \cdots + \binom{m+1}{m} \\
2^{m+1} &= o_3 + o_5 + o_7 + \cdots + o_{2m+1} + o_{2(m+1)+1} \\
&= \binom{m+2}{1} + \binom{m+2}{3} + \binom{m+2}{5} + \cdots + \binom{m+2}{m+1},
\end{aligned}
$$

where some o_j are equal to zero.

We will restate Conjecture 2.1 as

Conjecture 2.2 $O_{2m+1}(n)$ satisfies

$$O_3(n) = 1$$

$$O_3(n) + 3 \cdot O_5(n) = 2^2 \cdot \frac{n(n+1)}{2}$$

$$4 \cdot O_5(n) + 4 \cdot O_7(n) = 2^3 \cdot \left(\frac{n(n+1)}{2}\right)^2, \dots,$$

$$o_3 O_3(n) + o_5 O_5(n) + o_7 O_7(n) + \cdots + o_{2(m-1)+1} O_{2(m-1)+1}(n) = 2^{m-1} \cdot \left(\frac{n(n+1)}{2}\right)^{m-2},$$

$$o_3^{'} O_3(n) + o_5^{'} O_5(n) + o_7^{'} O_7(n) + \cdots + o_{2(m-1)+1}^{'} O_{2(m-1)+1}(n)$$
$$+ o_{2m+1}^{'} O_{2m+1}(n) = 2^m \cdot \left(\frac{n(n+1)}{2}\right)^{m-1},$$

where

$$o_3, o_5, o_7, \dots, o_{2(m-1)+1},$$
$$o_3^{'}, o_5^{'}, o_7^{'}, \dots, o_{2(m-1)+1}^{'}, o_{2m+1}^{'}$$

are integers such that

$$o_3 + o_5 + o_7 + \cdots + o_{2(m-1)+1} = 2^{m-1},$$

$$o_3^{'} + o_5^{'} + o_7^{'} + \cdots + o_{2(m-1)+1}^{'} + o_{2m+1}^{'} = 2^m.$$

For the same o_j and $o_k^{'}$, $O_{2m+1}(n)$ and $E_{2m}(n)$ satisfy

$$E_2(n) = 1$$

$$3 \cdot O_5(n) + 2 \cdot \frac{n\,(n+1)}{2} = 5 \cdot E_4(n)$$

$$4 \cdot O_5(n) + 4 \cdot O_7(n) + 2^2 \cdot \left(\frac{n(n+1)}{2}\right)^2 = \left(o_3 + o_5^{'}\right) \cdot E_4(n) + \left(o_5 + o_7^{'}\right) \cdot E_6(n), \dots,$$

$$o_3 O_3(n) + o_5 O_5(n) + o_7 O_7(n) + \cdots + o_{2m+1} O_{2m+1}(n) + 2^{m-1} \cdot \left(\frac{n(n+1)}{2}\right)^{m-1}$$

$$= \left(o_3 + o_5^{'}\right) \cdot E_2(n) + \left(o_5 + o_7^{'}\right) \cdot E_4(n) + \left(o_7 + o_9^{'}\right) \cdot E_6(n)$$
$$+ \cdots + \left(o_{2(m-1)+1} + o_{2m+1}^{'}\right) \cdot E_{2m}(n).$$

Exercise examine the strengths and weaknesses of having the conjectures in two cases versus having them in one.