Goldbach and twin prime conjectures implied by strong Goldbach number sequence

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Abstract

In this note, we present a new and direct approach to prove the Goldbach conjecture that if the existence of the limit of $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ being 1/2 can be confirmed by asymptotic result arising from large-scale observation data for status of $\xi(P)$ then the Goldbach conjecture is true, where P is a prime greater than 3 but N_{SGL} and $2P$ are separately the largest strong Goldbach number and the largest Goldbach number generated by P. Further, the existence of the limit also implies the twin prime conjecture by means of the existence of good approximate function form to $\rho_2(A)$ such as our introduced $\rho_2(A) \approx C_2/A^{1/2}$ as an attempt, which tends to lower order infinitesimal as $1/A$ approaches infinitesimal, where $A = N_{SGL}–P$ but $\rho_2(A)$ is the density of strong Goldbach numbers generated by distinct twin prime pairs (p, p+2) among all strong Goldbach numbers from P+1 to P+A and C_2 is the twin prime constant in the first Hardy-Littlewood conjecture.

Keywords: prime, Goldbach number, strong Goldbach number sequence, Goldbach conjecture, twin prime conjecture.

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1. Introduction

Goldbach and twin prime conjectures are two subproblems of Hilbert's 8th problem in mathematics. The Goldbach conjecture states that every even number greater than 2 can be written as the sum of two primes, which is equivalent to the statement that every even number greater than 4 can be written as the sum of two odd primes. The twin prime conjecture states that there are infinitely many primes p such that p+2 is also prime. Chen showed that every sufficiently large even integer is the sum of either two primes, or a prime and the product of two primes in 1973[1] and showed that there are infinitely many primes p with $p+2$ being a prime or the product of two primes but such primes p are called Chen prime[2]. Zhang proved that for some integers N less than 7×10^7, there is an infinite number of pairs of primes that differ by N, and according to the Polymath project, the bound has been reduced to 246 without assuming. There are some researches on the exceptional set of Goldbach numbers to come close to the Goldbach conjecture[3,4,5,6,7,8,9]. By Li[7,8], Goldbach number is defined as a positive number to be a sum of two odd primes and the exceptional set of Goldbach numbers is usually written as $E(x)$ to denote the number of even numbers not exceeding x which cannot be represented as the sum of two odd primes. Thus the Goldbach conjecture is equivalent to proving that $E(x) = 2$ for every $x \geq 4$ and also equivalent to proving that all even numbers greater than 4 are Goldbach numbers. In this note we built a mathematical framework in which every original continuous odd prime number sequence $\{ 3, 5, \ldots, P \}$ will generate a corresponding strong Goldbach number sequence $\{ 6, 8, \ldots, N_{SGL} \}$ in the set of Goldbach numbers $N_G = p_1+p_2$ arising from $\{ 3, 5, \ldots, P \}$. Basing on such a framework, we get a result that if the existence of the limit of $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ being 1/2 is confirmable then the Goldbach conjecture is true, since the existence of the limit implies all even numbers greater than 4 are Goldbach numbers. The existence of the limit also implies the twin prime conjecture if it can be confirmed that there exists a good approximate function form to $\rho_2(A)$ which tends to lower order infinitesimal as $1/A$ approaches infinitesimal.

2. Strong Goldbach number sequence

Definition 2.1 *Let P be prime greater than 3. For a continuous odd prime number sequence, if its first term is 3 then the sequence is called an original continuous odd prime number sequence and written as $\{ 3, 5, \ldots, P \}$, where P is the last term of the sequence.*

Obviously, $\{ 3, 5, \ldots, P \}$ contains all odd primes from 3 to P for any given P.

Definition 2.2 *Let p_1 and p_2 be two same or distinct odd primes not greater than P, then $N_G = p_1+p_2$ is called a Goldbach number generated from $\{ 3, 5, \ldots, P \}$.*

By Definition 2.1 and Definition 2.2, $6 = 3+3$ is the smallest Goldbach number but $2P = P+P$ is the largest Goldbach number for a given $\{ 3, 5, \ldots, P \}$. Obviously, the number of pairs of odd primes not greater than P to generate a Goldbach number N_G should be equal to or greater than 1 for a given $\{ 3, 5, \ldots, P \}$.

Definition 2.3 *Let Q be even number greater than 6. For a continuous even number sequence, if its first term is 6 then the sequence is called an original continuous even number sequence and written as $\{ 6, 8, \ldots, Q \}$, where Q is the last term of the sequence.*

Obviously, $\{ 6, 8, \ldots, Q \}$ contains all even numbers from 6 to Q for any given Q.

Definition 2.4 *In the set of Goldbach numbers N_G arising from a given { 3, 5, …, P }, if all terms of { 6, 8, …, Q } are Goldbach numbers N_G in this set then { 6, 8, …, Q } is called strong Goldbach number sequence generated by { 3, 5, …, P } and written as { 6, 8, …, N_{SGL} }, where N_{SGL} is the last term of the sequence, in which every term is called strong Goldbach number and written as N_{SG}.*

According to Definition 2.4, the last term N_{SGL} is the largest strong Goldbach number generated by { 3, 5, …, P }. By Definition 2.2 and Definition 2.4 every term N_{SG} in { 6, 8, …, N_{SGL} } must be also a Goldbach number N_G but every Goldbach number N_G outside { 6, 8, …, N_{SGL} } is not a strong Goldbach number N_{SG} for a given { 3, 5, …, P }.

Lemma 2.5 *There is a strong Goldbach number sequence { 6, 8, …, N_{SGL} } in the set of Goldbach numbers N_G for any given { 3, 5, …, P }.*

Proof For the first { 3, 5, …, P } i.e. { 3, 5 }, there is a strong Goldbach number sequence { 6, 8, 10 } and N_{SGL} = 10 is the last term of the sequence. Since every { 3, 5, …, P } is contained by next { 3, 5, …, P } so that { 6, 8, …, N_{SGL} } generated by every { 3, 5, …, P } will remain in the set of Goldbach numbers N_G arising from next { 3, 5, …, P } as the first part (including complete sequence) of { 6, 8, …, N_{SGL} } generated by next { 3, 5, …, P }. Hence the lemma holds.

Put simply, the lemma means that any given P will generate a corresponding N_{SGL}.

Observation 2.6 *Status of { 6, 8, …, N_{SGL} } generated by { 3, 5, …, P } for P less than 500.*

By Lemma 2.5 we can give an observation for status of { 6, 8, …, N_{SGL} } generated by { 3, 5, …, P } for P less than 500. In the following observation, $\xi(P) = N_{SGL}/2P$ is the ratio of the largest strong Goldbach number to the largest Goldbach number generated by a given P, and $A = N_{SGL}–P$ but $\delta_2(A)$ is the number of strong Goldbach numbers N_{SG} generated by twin prime pairs ($N_{SG}/2–1$, $N_{SG}/2+1$) for $P+1 \le N_{SG} \le P+A$ and $\rho_2(A) = \delta_2(A)/k$ is the density of strong Goldbach numbers generated by twin prime pairs among the strong Goldbach numbers from P+1 to P+A, whose number is $k = [(P+A)–(P–1)]/2 = (A+1)/2$. In the observation, $C_2/A^{1/2}$ is assumed as an approximate function to $\rho_2(A)$, where C_2 (≈ 0.6601618…) is the twin prime constant in the first Hardy-Littlewood conjecture, and $(C_2/A^{1/2}–\rho_2(A))/\rho_2(A)$ is relative error using $C_2/A^{1/2}$. We give a suitable example for the observation data as follows

Status of { 6, 8, …, N_{SGL} } generated by { 3, 5, …, 251 }:
N_{SG}: 6, 8, 10, …, 470, 472, 474. N_G (Goldbach numbers outside the strong Goldbach number sequence): 478, 480, 482, 490, 492, 502. Strong Goldbach numbers N_{SG} generated by distinct twin prime pairs for (251+1) ≤ N_{SG} ≤ (251+223): 276 = 137+139, 300 = 149+151, 360 = 179+181, 384 = 191+193, 396 = 197+199.
P = 251, N_{SGL} = 474, A = 223, $\xi(251)$ = 0.94422, $\delta_2(A)$ = 5, k = (223+1)/2 = 112, $\rho_2(A)$ = 5/112 = 0.04464, $C_2/A^{1/2}$ = 0.04421, $(C_2/A^{1/2}–\rho_2(A))/\rho_2(A)$ = – 0.00963.

P	$N_{SGL}=P+A$	$\xi(P)$	$\rho_2(A)$	$\delta_2(A)$	$C_2/A^{1/2}$	$(C_2/A^{1/2}-\rho_2(A))/\rho_2(A)$
5	10=5+5	1.00000	0.33333	1	0.29523	– 0.11430
7	14=7+7	1.00000	0.50000	2	0.24952	– 0.50096
11	18=11+7	0.81818	0.25000	1	0.24952	– 0.00192
13	26=13+13	1.00000	0.14286	1	0.18310	0.28167
17	30=17+13	0.88235	0.14286	1	0.18310	0.28167
19	38=19+19	1.00000	0.20000	2	0.15145	– 0.24257
23	42=23+19	0.91304	0.20000	2	0.15145	– 0.24257
29	42=29+13	0.72414	0.14286	1	0.18310	0.28167
31	54=31+23	0.87097	0.08333	1	0.13765	0.65187
37	62=37+25	0.83784	0.07692	1	0.13203	0.71646
41	74=41+33	0.90244	0.05882	1	0.11492	0.95376
43	74=43+31	0.86047	0.06250	1	0.11857	0.89712
47	90=47+43	0.95745	0.09091	2	0.10067	0.10736
53	90=53+37	0.84906	0.10526	2	0.10853	0.03107
59	90=59+31	0.76271	0.12500	2	0.11857	– 0.05144
61	108=61+47	0.88525	0.04167	1	0.09629	1.31078
67	114=67+47	0.85075	0.04167	1	0.09629	1.31078
71	114=71+43	0.80282	0.04545	1	0.10067	1.21496
73	134=73+61	0.91781	0.06451	2	0.08452	0.31018
79	134=79+55	0.84810	0.07143	2	0.08902	0.24626
83	146=83+63	0.87952	0.09375	3	0.08317	– 0.11285
89	162=89+73	0.91011	0.05405	2	0.07727	0.42960
97	172=97+75	0.88660	0.05263	2	0.07623	0.44841
101	180=101+79	0.89109	0.05000	2	0.07427	0.48540
103	186=103+83	0.90291	0.04762	2	0.07246	0.52163
107	186=107+79	0.86916	0.05000	2	0.07427	0.48540
109	218=109+109	1.00000	0.07273	4	0.06323	– 0.13062
113	222=113+109	0.98230	0.07273	4	0.06323	– 0.13062
127	230=127+103	0.90551	0.05769	3	0.06048	0.04836
131	240=131+109	0.91603	0.05454	3	0.06223	0.14100
137	240=137+103	0.87591	0.05769	3	0.06048	0.04836
139	254=139+115	0.91367	0.05172	3	0.06560	0.26837
149	258=149+109	0.86577	0.03636	2	0.06323	0.73900
151	270=151+119	0.89404	0.03333	2	0.06517	0.95530
157	270=157+113	0.85987	0.03509	2	0.06210	0.76973
163	290=163+127	0.88957	0.04688	3	0.05858	0.24957
167	290=167+123	0.86826	0.04839	3	0.05952	0.23001
173	290=173+117	0.83815	0.05085	3	0.06103	0.20020
179	330=179+151	0.92179	0.05263	4	0.05372	0.02071
181	348=181+167	0.96133	0.04762	4	0.05109	0.07287
191	348=191+157	0.91099	0.05633	4	0.05269	– 0.06462
193	366=193+173	0.94819	0.05747	5	0.05019	– 0.12667
197	366=197+169	0.92893	0.05882	5	0.05078	– 0.13669

199	366=199+167	0.91960	0.05952	5	0.05108	– 0.14180
211	398=211+187	0.94313	0.06383	6	0.04828	– 0.24361
223	398=223+175	0.89238	0.05682	5	0.04990	– 0.12179
227	410=227+183	0.90308	0.05435	5	0.04880	– 0.10212
229	410=229+181	0.89520	0.05495	5	0.04907	– 0.10701
233	434=233+201	0.93133	0.04950	5	0.04656	– 0.05939
239	440=239+201	0.92050	0.04950	5	0.04656	– 0.05939
241	440=241+199	0.91286	0.05000	5	0.04680	– 0.06400
251	474=251+223	0.94422	0.04464	5	0.04421	– 0.00963
257	474=257+217	0.92218	0.05505	6	0.04481	– 0.18601
263	474=263+211	0.90114	0.05660	6	0.04545	– 0.19700
269	474=269+205	0.88104	0.05825	6	0.04611	– 0.20841
271	474=271+203	0.87454	0.05882	6	0.04633	– 0.21234
277	522=277+245	0.94224	0.04878	6	0.04218	– 0.13530
281	522=281+241	0.92883	0.04959	6	0.04252	– 0.14257
283	528=283+245	0.93286	0.04878	6	0.04218	– 0.13530
293	528=293+235	0.90102	0.05085	6	0.04306	– 0.15320
307	566=307+259	0.92182	0.05385	7	0.04102	– 0.23825
311	570=311+259	0.91640	0.05385	7	0.04102	– 0.23825
313	570=313+257	0.91054	0.05423	7	0.04118	– 0.24064
317	570=317+253	0.89905	0.05512	7	0.04150	– 0.24710
331	614=331+283	0.92749	0.04930	7	0.03924	– 0.20406
337	614=337+277	0.91098	0.05036	7	0.03967	– 0.21227
347	630=347+283	0.90778	0.05634	8	0.03924	– 0.30351
349	634=349+285	0.90831	0.05594	8	0.03910	– 0.30104
353	650=353+297	0.92068	0.05370	8	0.03831	– 0.28659
359	680=359+321	0.94708	0.04969	8	0.03685	– 0.25840
367	680=367+313	0.92643	0.04459	7	0.03731	– 0.16326
373	680=373+307	0.91153	0.04545	7	0.03768	– 0.17096
379	680=379+301	0.89710	0.04636	7	0.03805	– 0.17925
383	680=383+297	0.88773	0.04698	7	0.03831	– 0.18455
389	686=389+297	0.88175	0.04027	6	0.03831	– 0.04867
397	686=397+289	0.86398	0.04138	6	0.03883	– 0.06162
401	686=401+285	0.85536	0.03497	5	0.03910	0.11810
409	686=409+277	0.83863	0.03597	5	0.03967	0.10286
419	722=419+303	0.86158	0.03947	6	0.03792	– 0.03927
421	722=421+301	0.85748	0.03974	6	0.03805	– 0.04253
431	794=431+363	0.92111	0.03297	6	0.03465	0.05096
433	794=433+361	0.91686	0.03315	6	0.03476	0.04857
439	794=439+355	0.90433	0.03371	6	0.03504	0.03945
443	822=443+379	0.92777	0.03158	6	0.03391	0.07378
449	822=449+373	0.91537	0.03209	6	0.03418	0.06513
457	854=457+397	0.93435	0.02513	5	0.03330	0.32511
461	854=461+393	0.92625	0.02538	5	0.03330	0.31206

463	854=463+391	0.92225	0.02551	5	0.03339	0.30890
467	906=467+439	0.97002	0.02727	6	0.03151	0.15548
479	906=479+427	0.94572	0.02804	6	0.03195	0.13944
487	930=487+443	0.95483	0.02703	6	0.03137	0.16056
491	930=491+439	0.94705	0.02727	6	0.03151	0.15548
499	962=499+463	0.96393	0.02586	6	0.03068	0.18639

In above observation table, we see there is $0.5 < \xi(P) \leq 1$ and N_{SGL} generated by every P can be written as $P+A$ for P less than 500, where A is an odd number greater than 3 and takes on an obvious and relatively stable growth trend with the growth of value of P, such as $A(5) = 5$, $A(97) = 75$, $A(197) = 169$, $A(293) = 235$, $A(397) = 289$ and $A(499) = 463$, though there are some small fluctuations in the growth process. The observation indicates that there exist connections between the exceptional set of Goldbach numbers and the strong Goldbach number sequence generated by P. Considering every A to be an odd number greater than 3, $N_{SGL} = P+A > P$ and $N_{SGL} = P+A > P–1$ for P less than 500. In the observation, the largest P is 499 and $P–1 = 498$ is the largest even number not exceeding $P = 499$. Taking $P = 499$ as x, we see $N_{SGL} = 962$ and $x–1 = P−1 = 498$ so that $x–1 = 498$ is contained by $N_{SGL} = 962$ as a strong Goldbach number, therefore, all even numbers from 6 to 498 are strong Goldbach numbers and also Goldbach numbers, which implies $E(x) = 2$ for $4 \leq x \leq 499$. It is an example which means $E(x) = 2$ is true for a given finite area of x using a known strong Goldbach number sequence. In the observation, we also see $\delta_2(A) \geq 1$ for every A generated by P and value of $\delta_2(A)$ takes on a slow and obvious growth trend but value of $\rho_2(A)$ takes on a slow reduction trend with the growth of P. Trying to introduce the function $C_2/A^{1/2}$ as an approximation to $\rho_2(A)$, we see relative errors are small in general for P less than 500, in which the best data arise from the relative errors calculated as 0.192% for $P = 11$ and – 0.963% for $P = 251$.

3. A basic proposition about the Goldbach conjecture

Proposition 3.1 *N_{SGL} generated by every $P \geq 5$ can be written as $N_{SGL} = P+A$, where A is an odd number greater than 3 and an undetermined function $A(P)$ which approaches lower order infinity as $P \to \infty$.*

Remark 3.2 Taking $P+1$ as x, from $E(x) = 2$ for every $x \geq 4$ we get $E(P+1) = 2$, which means all even numbers from 6 to $P+1$ are Goldbach numbers and also strong Goldbach numbers generated by P. If A is taken as a negative odd number $A = −a$ then there is $N_{SGL} = P+A = P−a$, which means $N_{SGL} = P−a$ is the largest strong Goldbach number for a given P so that $N_{SGL}+2 = P−a+2$ is not a Goldbach number and also not a strong Goldbach number generated by P. Since $P+1 \geq P−a+2 = (P+1)−(a−1)$, $P+1$ is not a strong Goldbach number generated by P. It contradicts to above result that $P+1$ is a strong Goldbach number generated by P. Thus A cannot be taken as negative odd number. In addition, if $A = 0$ then $N_{SGL} = P+A = P$, which is not true since N_{SGL} is an even number but P is an odd number. Hence A should remain positive in Proposition 3.1.

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Remark 3.3 By Remark 3.2 A can be taken as 1, which makes P+1 become a Goldbach number and also a strong Goldbach number generated by P. However, P+1 cannot become the largest strong Goldbach number for a given P, since if P+1 is a strong Goldbach number then P+3, P+5, P+7 are all strong Goldbach numbers so that N_{SGL} cannot be written as $N_{SGL} = P+1$ for any given P. Considering P to be defined as a prime greater than 3, even if $N_{SGL} = 2P = P+P$ is the largest strong Goldbach number for a given P, there is no a value of P which makes N_{SGL} written as $N_{SGL} = P+3$. But for $A < P$, if P+3 is a strong Goldbach number then P+5, P+7 are all strong Goldbach numbers for a given P. Hence A should be an odd number greater than 3 in Proposition 3.1.

Remark 3.4 In Observation 2.6 we have seen value of A takes on a growth trend with the growth of value of P, therefore, we assume there is a growth trend of A in general case. Thus A can be assumed as an undetermined function $A = A(P)$ whose value will take on a growth trend with the growth of P up to $P \to \infty$. By the prime number theorem we see the density of prime numbers $\rho(x) \approx 1/(\text{In } x)$ will be lower and lower with the growth of x. It means the density of prime numbers will be lower and lower with the growth of P up to $P \to \infty$. Thus the growth rate of $A(P)$ will be much lower than the growth rate of P so that $A(P)$ will approach lower order infinity as $P \to \infty$. However, we should see the observation scale for status of $A(P)$ generated by P in Observation 2.6 is too small to imply existence of an approximate function form of $A(P)$. Hence $A = A(P)$ should be assumed as an undetermined function which will approach lower order infinity as $P \to \infty$ in Proposition 3.1.

Corollary 3.5 (Proposition 3.1) *$N_{SGL} \to \infty$ as $P \to \infty$.*

Proof By Proposition 3.1 $N_{SGL} = P+A$ and undetermined function $A(P)$ approaches lower order infinity as $P \to \infty$. Since the limit of the ratio A/P as $P \to \infty$ is 0, the limit of the ratio $(P+A)/P$ as $P \to \infty$ is 1 so that $N_{SGL} = P+A \to \infty$ as $P \to \infty$. Thus the corollary holds.

Corollary 3.6 (Proposition 3.1) *The limit of the ratio $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ is 1/2.*

Proof By Proposition 3.1 $N_{SGL} = P+A$ and undetermined function $A(P)$ approaches lower order infinity as $P \to \infty$. Since the limit of the ratio $A/2P$ as $P \to \infty$ is 0, the limit of $\xi(P) = N_{SGL}/2P = P/2P+A/2P$ as $P \to \infty$ is 1/2. Thus the corollary holds.

Conjecture 3.7 (Goldbach conjecture） *Every even number greater than 4 can be written as the sum of two odd primes.*

Proposition 3.8 *Corollary 3.5 is equivalent to Conjecture 3.7.*

Proof Let $P \to \infty$, by Corollary 3.5 we obtain $N_{SGL} \to \infty$ as $P \to \infty$. Since N_{SGL} is the largest strong Goldbach number generated by P according to Definition 2.4, the strong Goldbach number sequence { 6, 8, 10, …, $N_{SGL}-4$, $N_{SGL}-2$, N_{SGL} } generated by $P \to \infty$ will become an infinite sequence by $N_{SGL} \to \infty$ as $P \to \infty$. Thus all even numbers greater than 4 will become strong Goldbach numbers N_{SG} and also Goldbach numbers N_G as $P \to \infty$ by Definition 2.2 and Definition 2.4, in which every even number is the sum of two odd primes not greater than P such as 6 = 3+3. It implies every even number greater than 4 is the sum of two odd primes. Hence the Goldbach conjecture is true and the proposition holds.

Proposition 3.9 *Corollary 3.6 is equivalent to Conjecture 3.7.*

Proof By Corollary 3.6 the limit of the ratio $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ is 1/2. The result means that the limit of the ratio N_{SGL}/P as $P \to \infty$ is 1, which clearly implies $N_{SGL} \to \infty$ as $P \to \infty$. By Proposition 3.8 every even number greater than 4 is the sum of two odd primes. Hence the Goldbach conjecture is true and the proposition holds.

Lemma 3.10 *In a given strong Goldbach number sequence { 6, 8, …, N_{SGL} }, if N_{SG} is greater than 6 but contained by N_{SGL} then all even numbers from 6 to N_{SG} are strong Goldbach numbers and also Goldbach numbers.*

Proof Since strong Goldbach number N_{SG} is greater than 6 but contained by N_{SGL} in { 6, 8, …, N_{SGL} }. Hence { 6, 8, …, N_{SG} } is a subsequence of { 6, 8, …, N_{SGL} } so that all even numbers from 6 to N_{SG} are strong Goldbach numbers and also Goldbach numbers by Definition 2.2 and Definition 2.4. Thus the lemma holds.

Corollary 3.11 (Proposition 3.1) *$E(x) = 2$ for every $x \geq 4$.*

Proof By Proposition 3.1 A remains positive for any given P including $P \to \infty$, therefore, $N_{SGL} = P+A > P$ and $N_{SGL} = P+A > P-1$ for any given P including $P \to \infty$. Taking P as x, $x-1 = P-1$ is the largest even number not exceeding $x = P$. By Corollary 3.5 $N_{SGL} \to \infty$ as $x = P \to \infty$. Thus $x-1 = P-1$ is contained by N_{SGL} as a strong Goldbach number as $x = P \to \infty$. By Lemma 3.10 all even numbers from 6 to $x-1 = P-1$ are strong Goldbach numbers and also Goldbach numbers as $x = P \to \infty$. The result implies $E(x) = 2$ for every $x \geq 4$, which is equivalent to the Goldbach conjecture. Thus the corollary holds.

Remark 3.12 Although Proposition 3.1 remains unproved, it has become a basic proposition, whose hardcore is the existence of undetermined function $A = A(P)$ approaching lower order infinity as $P \to \infty$ to be a product of the prime number theorem, and brought us many results of concern to support the Goldbach conjecture. However, Proposition 3.1 seems to be not the only proposition to imply the Goldbach conjecture, and there is another one as the following proposition.

Proposition 3.13 *Let P_n be the nth prime greater than 3, then $P_{n+1}+1$ is a strong Goldbach number G_{SG} generated by P_n.*

Corollary 3.14 (Proposition 3.13) *The Goldbach conjecture is true.*

Proof By Proposition 3.13 $P_{n+1}+1$ is a strong Goldbach number G_{SG} generated by P_n as $n \to \infty$. Hence all even numbers from 6 to $P_{n+1}+1$ are strong Goldbach numbers and also Goldbach numbers generated by P_n as $n \to \infty$. It implies every even number greater than 4 is the sum of two odd primes, that is, the Goldbach conjecture is true.

Remark 3.15 We have seen all data arising from Observation 2.6 support Proposition 3.13. However, it needs a proof that Proposition 3.13 holds in general case specially as $n \to \infty$. The problem is relative to the gap between P_n and P_{n+1} as $n \to \infty$. Zhang proved the limit of the lower bound of gap between P_n and P_{n+1} as $n \to \infty$ is 7×10^7 and the limit of the lower bound has been reduced to 246 without assuming. Relying on the existence of so small limit of the lower bound of gap between P_n and P_{n+1} as $n \to \infty$, it is certain that $P_{n+1}+1$ is a strong Goldbach number G_{SG} generated by P_n as $n \to \infty$ so that all even numbers from 6 to $P_{n+1}+1$ are strong Goldbach numbers and also Goldbach numbers as $n \to \infty$, therefore, the Goldbach conjecture is true.

Proposition 3.16 *If (P_n, P_{n+1}) is a pair of twin primes, then $P_{n+1}+1$ is a strong Goldbach number generated by P_n.*

Proof Since (P_n, P_{n+1}) is a pair of twin primes. Hence $P_{n+1}+1 = P_n+3$ is a Goldbach number generated by P_n. If P_n+1 is a strong Goldbach number generated by P_n then P_n+3 must be a strong Goldbach number generated by P_n, therefore, $P_{n+1}+1$ is a strong Goldbach number generated by P_n.

4. The Goldbach conjecture attributed to the limit of $\xi(P)$

In this note we proposed a basic proposition (Proposition 3.1), whose basis is the prime number theorem, to make the three results, including $N_{SGL} \to \infty$ as $P \to \infty$, the limit of the ratio $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ being 1/2 and $E(x) = 2$ for every $x \geq 4$, become corollaries of the proposition. These corollaries are clearly equivalent to the Goldbach conjecture. We also gave an argument for the rationality of proposing the proposition by Remark 3.2, Remark 3.3 and Remark 3.4 but Remark 3.2 relies on the Goldbach conjecture itself. Hence there is no an independent argument for the rationality of the proposition. However, the observation scale for status of $A(P)$ generated by P in Observation 2.6 is too small to imply existence of a function form which is good approximation to $A(P)$ and will approach lower order infinity as $P \to \infty$. Therefore, a large-scale observation for status of $A(P)$ generated by P is necessary, for example, all primes P less than 10^6 will be considered by computer programs to find the function form. If there is really a function form being a good approximation to

$A(P)$ and approaching lower order infinity as $P \to \infty$ then the asymptotic result arising from it will independently give an explanation for the rationality of Proposition 3.1 so that the basic proposition will be proven and the Goldbach conjecture will be proven by Corollary 3.5, Corollary 3.6 or Corollary 3.11. From the result we see there are very close connections between the Goldbach conjecture and the prime number theorem. We should also see there is a similar result which will appear in a large-scale observation for status of $\xi(P)$ generated by P, that is, if there is really a function form being a good approximation to $\xi(P)$ whose limit as $P \to \infty$ is 1/2 then the Goldbach conjecture will be proven by Corollary 3.5, Corollary 3.6 or Corollary 3.11 since Proposition 3.1 will be proven by the following theorem.

Theorem 4.1 *The limit of the ratio ($P+A$)/$2P$ as $P \to \infty$ is 1/2 if and only if $A = A(P)$ approaches lower order infinity as $P \to \infty$.*

Proof If $A = A(P)$ approaches lower order infinity as $P \to \infty$, then the limit of the ratio ($P+A$)/$2P = P/2P+A/2P$ as $P \to \infty$ is 1/2 since the limit of the ratio $A/2P$ as $P \to \infty$ is 0. If the limit of the ratio ($P+A$)/$2P = P/2P+A/2P$ as $P \to \infty$ is 1/2, then the limit of the ratio $A/2P$ as $P \to \infty$ is 0 since the limit of the ratio $P/2P$ as $P \to \infty$ is 1/2, which implies $A = A(P)$ approaches lower order infinity as $P \to \infty$. Hence the theorem holds.

Remark 4.2 Theorem 4.1 satisfies necessary and sufficient conditions, therefore, finding a good approximate function form to $A(P)$ which approaches lower order infinity as $P \to \infty$ is equivalent to finding a good approximate function form to $\xi(P)$ whose limit as $P \to \infty$ is 1/2. It implies there is a direct approach to prove the Goldbach conjecture without assuming as the following proposition shows.

Proposition 4.3 *The correctness of the Goldbach conjecture is directly attributed to the existence of the limit of $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ being 1/2.*

Proof Since the limit of the ratio $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ being 1/2 implies the limit of ratio N_{SGL}/P as $P \to \infty$ being 1, and the limit of ratio N_{SGL}/P as $P \to \infty$ being 1 implies $N_{SGL} \to \infty$ as $P \to \infty$ but $N_{SGL} \to \infty$ as $P \to \infty$ is equivalent to the Goldbach conjecture. Hence the proposition holds.

Remark 4.4 Proposition 4.3 means we may directly get $N_{SGL} \to \infty$ as $P \to \infty$ without assuming, since the result can be directly obtained from the existence of the limit of the ratio $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ being 1/2 arising from large-scale observation data for status of $\xi(P)$, even if we know nothing about Proposition 3.1 whose basis is the existence of $A = A(P)$ approaching lower order infinity as $P \to \infty$ to be only an assumption before making a large-scale observation for status of $A(P)$ as Remark 3.4 states. Although Proposition 4.3 seems to be similar to Proposition 3.9, the latter contains above assumption but the former is presented without the assumption. However, by Theorem 4.1 the existence of the limit of the ratio $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ being 1/2 implies the existence of $A = A(P)$ approaching lower order infinity as $P \to \infty$, therefore, the existence of the limit of the ratio $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ being 1/2 must remain essentially a positive result arising from the prime number theorem, whether or not $A = A(P)$ approaching lower order infinity as $P \to \infty$ is mentioned in a specific computation model.

5. A basic proposition about the twin prime conjecture

Proposition 5.1 *Let $\rho_2(A)$ be an undetermined function which tends to infinitesimal as $A = A(P)$ approaches infinity, then $\rho_2(A)$ tends to lower order infinitesimal as $1/A$ approaches infinitesimal.*

Remark 5.2 In Observation 2.6 we have seen $\delta_2(A) \geq 1$ for every $P \geq 5$ and there are a slow but obvious growth trend of $\delta_2(A)$ and a slow reduction trend of $\rho_2(A)$ for P less than 500, therefore, we can assume $\rho_2(A)$ as an undetermined function which tends to infinitesimal as A approaches infinity. However, the first Hardy-Littlewood conjecture[10], a stronger form of the twin prime conjecture, postulates there is a distribution law for twin primes akin to the prime number theorem, and the conjecture can be expressed as $\pi_2(x) \approx 2C_2x/(\text{In } x)^2$, where C_2 is the twin prime constant (60 digits listed) but $\pi_2(x)$ denotes the number of twin primes p and p+2 such that $p \leq x$. From the conjecture we see the density of twin primes $\rho_2(x) \approx 2C_2/(\text{In } x)^2$ will be lower and lower with the growth of x so that the density of twin primes will be lower and lower with the growth of P up to $P \to \infty$. Comparing it with $\rho(x) \approx 1/(\text{In } x)$, it seems certain that $\rho_2(A)$ should tend to lower order infinitesimal as $1/A$ approaches infinitesimal. In fact, we can give a few examples arising from the data in Observation 2.6 to support the basic proposition, such as $\rho_2(A)$ = 0.33333 and $1/A$ = 0.20000 for P = 5, $\rho_2(A)$ = 0.12500 and $1/A$ = 0.03226 for P = 59, $\rho_2(A)$ = 0.07273 and $1/A$ = 0.00917 for P = 113, $\rho_2(A)$ = 0.05495 and $1/A$ = 0.00552 for P = 229, $\rho_2(A)$ = 0.04930 and $1/A$ = 0.00353 for P = 331, $\rho_2(A)$ = 0.03315 and $1/A$ = 0.00277 for P = 433, $\rho_2(A)$ = 0.02586 and $1/A$ = 0.00216 for P = 499.

Corollary 5.3 (Proposition 5.1) *$\delta_2(A)$ tends to lower order infinity as A approaches infinity.*

Proof By Proposition 5.1 $\rho_2(A)$ tends to lower order infinitesimal as $1/A$ approaches infinitesimal. Considering $\delta_2(A) = k\rho_2(A) = (A+1)\rho_2(A)/2 = \rho_2(A)/(2/A)+\rho_2(A)/2$, we see $\rho_2(A)/2$ tends to infinitesimal as A approaches infinity but $\rho_2(A)$ tends to lower order infinitesimal as $2/A$ approaches infinitesimal. Hence $\delta_2(A) = \rho_2(A)/(2/A)$ tends to lower order infinity as A approaches infinity. Thus the corollary holds.

Conjecture 5.4 (twin prime conjecture) *There are infinitely many twin primes.*

Proposition 5.5 *Corollary 5.3 is equivalent to Conjecture 5.4.*

Proof By Corollary 5.3 $\delta_2(A)$ tends to lower order infinity as A approaches infinity. It implies there are infinitely many strong Goldbach numbers N_{SG} for $P+1 \le N_{SG} \le P+A$ as $P \to \infty$ which can be represented as sums of distinct twin prime pairs ($N_{SG}/2-1$, $N_{SG}/2+1$), therefore, there are infinitely many twin primes. Thus Conjecture 5.4 is true and the proposition holds.

Proposition 5.6 *The existence of the limit of the ratio $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ being 1/2 implies Conjecture 5.4 if it is confirmed that there is a good approximate function form to $\rho_2(A)$ which tends to lower order infinitesimal as $1/A$ approaches infinitesimal.*

Proof If the existence of the limit of the ratio $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ being 1/2 is confirmed by an asymptotic result arising from the large-scale observation data for status of $\xi(P)$ then by Theorem 4.1 we get $A = A(P)$ approaching lower order infinity as $P \to \infty$. Since the result is the precondition of proposing Proposition 5.1. Hence if there is a function form being good approximation to $\rho_2(A)$ which tends to lower order infinitesimal as $1/A$ approaches infinitesimal then Proposition 5.1 holds. Therefore, by Corollary 5.3 and Proposition 5.5 there are infinitely many twin primes. Thus the proposition holds.

Remark 5.7 Proposition 5.6 means that it is necessary for proving the twin prime conjecture to find a good approximate function form to $\rho_2(A)$ which tends to lower order infinitesimal as $1/A$ approaches infinitesimal. We have seen the observation scale of Observation 2.6 is too small to find good approximate functions to $A(P)$ and $\xi(P)$, however, we discovered really there is a suitable function $C_2/A^{1/2}$ which can be thought as an acceptable approximation to $\rho_2(A)$ at so small observation scale of Observation 2.6, in which we got the data with small relative errors using $C_2/A^{1/2}$. If the data calculated by $C_2/A^{1/2}$ are more close to $\rho_2(A)$, which means relative errors are smaller, in a large-scale observation for status of $\rho_2(A)$ then the function $C_2/A^{1/2}$ can be confirmed as a good approximation to $\rho_2(A)$, therefore, the twin prime conjecture will be implied by both the existence of the limit of the ratio $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ being 1/2 and the existence of the approximate function $\rho_2(A) \approx C_2/A^{1/2}$, since $\delta_2(A) \approx kC_2/A^{1/2} \approx (C_2/2)A^{1/2}+(C_2/2)/A^{1/2}$ tends to lower order infinity as A approaches infinity. Considering $(C_2/2)/A^{1/2}$ tending to infinitesimal as A approaching infinity, we obtain $\delta_2(A) \approx (C_2/2)A^{1/2}$ for large P up to $P \to \infty$. The twin prime constant C_2 in the first Hardy-Littlewood conjecture can be expressed as $C_2 = \exp\{(2-2^n)[P(n) - 2^{-n}]/n\}$, where $P(n)$ is the prime zeta function to be a generalization of the Riemann zeta function. Therefore, it seems not accidental that there are small relative errors in general using $C_2/A^{1/2}$ in Observation 2.6.

6. Conclusion

By Proposition 4.3 and Remark 4.4 we have found a simple, independent and direct approach to prove the Goldbach conjecture, that is, as long as the existence of the limit of the ratio $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ being 1/2 has been confirmed by an asymptotic result arising from the large-scale observation data for status of $\xi(P)$ through a specific computation model, we are sure that the Goldbach conjecture has been proven. Further, by Proposition 5.6 and Remark 5.7, If not only the existence of the limit of the ratio $\xi(P) = N_{SGL}/2P$ as $P \to \infty$ being 1/2 is confirmable but also the function $C_2/A^{1/2}$ can be confirmed as a good approximation to $\rho_2(A)$ by a large-scale observation for status of $\rho_2(A)$, which implies $\delta_2(A) \approx (C_2/2)A^{1/2}$ tends to lower order infinity as A approaches infinity, then the twin prime conjecture is true.

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