A POLYNOMIAL RECURSION FOR PRIME CONSTELLATIONS

SCOTT B. GUTHERY

Abstract. An algorithm for recursively generating the sequence of solutions of a prime constellation is described. The algorithm is based on a polynomial equation formed from the first \( n \) elements of the constellation. A root of this equation is the next element of the sequence.

1. Introduction

Hypothesis \( H \) is one of the few mathematics conjectures that is distinguished by having its own Wikipedia page. The hypothesis, proposed independently by Schinzel-Sierpinski [1] and Bateman-Horn [2], describes a pattern of integers and then hypothesizes that there is an instance of the pattern such that all the integers in the pattern are prime numbers. It is a small step to conjecture that there are an infinite number of such occurrences.

The twin prime pattern, \( n, n + 2 \), is one of the forms characterized Hypothesis \( H \) but the hypothesis also subsumes the conjectures of de Polignac [3], Bunyakovskii [4], Hardy-Littlewood [5], Dickson [6], Shanks [7], and many others regarding the infinitude and density of patterns of primes.

Hypothesis \( H \). Let \( m \) be a positive integer and let \( F = \{ f_1(x), f_2(x), \ldots, f_m(x) \} \) be a set of irreducible polynomials with integral coefficients and positive leading coefficients such that there is not a prime \( p \) which divides the product

\[
 f_1(n) \cdot f_2(n) \cdot \ldots \cdot f_i(n) = \prod_{i=1}^{m} f_i(n) \tag{1}
\]

for every integer \( n \). Then there exists an integer \( q \) such that \( f_1(q), f_2(q), \ldots, f_m(q) \) are all prime numbers.

A sequence of functions \( F \) which satisfies Hypothesis \( H \) is traditionally called a prime constellation. A value \( q \) such that \( f_1(q), f_2(q), \ldots, f_m(q) \) are all prime numbers is called a solution of \( F \) while \( F \) is said to be solved by \( q \). Table 1 lists some familiar examples of prime constellations.

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<table>
<thead>
<tr>
<th>Familiar Name</th>
<th>Pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td>Twin Primes</td>
<td>{x, x + 2}</td>
</tr>
<tr>
<td>Sophie Germain Primes</td>
<td>{x, 2x + 1}</td>
</tr>
<tr>
<td>Shanks Primes</td>
<td>{x^4 + a}</td>
</tr>
<tr>
<td>Hardy-Littlewood Primes</td>
<td>{ax^2 + bx + c}</td>
</tr>
<tr>
<td>Dickson Chains</td>
<td>{a_i x + b_i}</td>
</tr>
<tr>
<td>Cunningham Chains</td>
<td>{2^{i-1}x + (2^{i-1} - 1)}</td>
</tr>
</tbody>
</table>

Table 1. Examples of Prime Constellations

Given the first $n$ solutions of a prime constellation we describe a polynomial one of whose roots is the next solution in this sequence. The polynomial can be regarded as a generalization of Rowland [8] which is, in turn, based on the formula for generating the next prime of Gandhi [9]. See also Golomb [10] and [11], Vanden Eynden [12], and Ellis [13]. An interpretation of the recursion is that the first $n$ solutions of an instance of Hypothesis H algebraically encode the $(n+1)^{st}$ solution.

2. Generation of Prime Constellations

The recursion for prime constellation generation is based on the following primality test:

**Lemma 1.** Let

$$Q_d(x) = \sum_{k=1}^{d-1} \gcd(x, x-k) - 1 = \sum_{i=1}^{d-1} \gcd(i, x-i) - 1.$$  (2)

$p$ is prime if and only if $Q_p(p) = 0$.

Let $F = \{f_1(x), f_2(x), \ldots, f_m(x)\}$ be a prime constellation and let $p$ be a solution of $F$. Set

$$Q_{F,p}(x) = \sum_{i=1}^{m} Q_{f_i,p}(f_i(x)).$$  (3)

As an example of a $Q_{F,p}(x)$, take $F = \{x, x + 2, x + 6\}$. This prime constellation is solved by $n = 5$, viz., (5, 7, 11). In this case,

$$Q_{F,5}(x) = Q_5(x) + Q_7(x + 2) + Q_{11}(x + 6).$$
Recursion. Let \( p \) be solution of the prime constellation \( F \) so that
\[
Q_{F,p}(p) = 0.
\]
If \( q \) is the next integer greater than \( p \) such that \( Q_{F,p}(q) = 0 \), then \( q \) is a solution of the prime constellation \( F \).

Example. The sequence of prime numbers
If \( F = \{ x \} \), then
\[
Q_{F,p}(x) = Q_p(x).
\]
According to the above recursion, if \( p \) is the \( i \)th prime and \( q \) is the next larger root of \( Q_p(x) \) beyond \( p \), then \( q \) is the \( i + 1 \)st prime.

It is straightforward to show that this recursion yields the sequence of primes using Bertrand’s Postulate ([14], [15], [16], [17]) that guarantees there is always a prime between \( n \) and \( 2n \).

Example. The sequence of twin primes
If \( F = \{ x, x + 2 \} \), then
\[
Q_{F,p}(x) = Q_p(x) + Q_{p+2}(x + 2).
\]
According to the above conjecture, if \( (p, p + 2) \) is a twin prime and \( q \) is the next larger root of \( Q_{F,p}(x) \) beyond \( p \), then \( (q, q + 2) \) is a twin prime.

3. Continuations

A continuous rendering of \( Q_{F,p}(x) \) permits existing equation-solving methods to be used in finding its roots.

As one possibility, take
\[
P_d(x) = \prod_{\substack{n \leq d \leq x \atop n \mid d}} \sin^2 \left( \frac{\pi(x - n)}{d} \right).
\]
Then, \( P_d(x) \) is zero if and only if \( \gcd(x, d) = 1 \). If we set
\[
\tilde{Q}_d(x) = \sum_{k=1}^{d-1} P_k(x),
\]
then \( \tilde{Q}_d(x) \) is zero if and only if \( Q_d(x) \) is zero so \( \tilde{Q}_d(x) \) can be used in Equation 3 as well as \( Q_d(x) \). Since \( \tilde{Q}_d(x) \) is continuous and periodic a next larger is guaranteed to exist.

As a second possibility, Slavin [18] has shown that for odd \( n \)
\[
\gcd(n, m) = \log_2 \prod_{k=0}^{n-1} \left( 1 + e^{-2\pi i km/n} \right) = n + \log_2 \left( \prod_{k=1}^{(n-1)/2} \cos \frac{km\pi}{n} \right)^2.
\]
When both arguments of \( \gcd \) in Equation 3 are even, Slavin’s formula produces a negative infinity so it can also be used to find roots of \( Q_{F,p}(x) \).
4. The Dual

The recursion states that given solution $p$ for a prime constellation $F$, the next element in the sequence of solutions is obtained by finding the next larger root of $Q_{F,p}$. One can also formulate this recursion using the divisors of the integers between 1 and $p$ rather than the non-divisors. Since the number of divisors grows slightly more quickly than the number of non-divisors, this may yield computational efficiency by reducing the complexity of $Q_{F,p}$.

To take this dual approach, we set

$$P_d(x) = \prod_{1 \leq n < d \atop n \mid d} (x \mod d - n)^2$$

and

$$Q_d(x) = \prod_{k=1}^{d-1} P_k(x).$$

To generate a sequence of prime constellation solutions using this formulation, we seek non-zero values of a product over the constellation functions rather than a zero value over a sum. The difficulty of seeking a non-zero value as compared to seeking a root may, of course, offset the reduction in complexity of the function being analyzed.
5. The Computation

The next larger root of $Q_{F,p}(x)$ is readily computed and easily checked as the next solution $F$ after $p$. The following Mathematica routine computes the next $n$ sequence elements satisfying constellation after the solution start:

```mathematica
Sieve[constellation_, start_, n_] := Module[{f, i, j, q = start, l},
    For[i = 1, i <= n, i++,
        f[x_] :=
            Sum[Q[constellation[[i]]][x], constellation[[i]][q]],
                {i, 1, Length[constellation]}];
        q = NextZero[f, q];
        l = {};
        For[j = 1, j <= Length[constellation], j++,
            p = pattern[[j]][q];
            AppendTo[l, {p, PrimeQ[p]}];
            ];
        Print[{q, l}];
    ];
]
```

Table 2 below lists some prime constellations for which sequences of solutions have been generated using this routine. The starting value Table 2 is a value which when substituted into the pattern yields a prime sequence satisfying the pattern. Thus, for example, when looking for Shank’s primes of the form $n^2 + 1$ a starting value could be 4. Tables 3 and 5 lists some other types of prime sequences to which the routine has been applied.
<table>
<thead>
<tr>
<th><strong>Familiar Name</strong></th>
<th><strong>Pattern</strong></th>
<th><strong>Start</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Primes</td>
<td>( { # } )</td>
<td>5</td>
</tr>
<tr>
<td>Twin Primes</td>
<td>( { # , # + 2 } )</td>
<td>3</td>
</tr>
<tr>
<td>Cousin Primes</td>
<td>( { # , # + 4 } )</td>
<td>3</td>
</tr>
<tr>
<td>Prime Constellation</td>
<td>( { # , # + 2 , # + 6 } )</td>
<td>5</td>
</tr>
<tr>
<td>Sophie Germain Primes</td>
<td>( { # , 2 # + 1 } )</td>
<td>5</td>
</tr>
<tr>
<td>Gaussian Primes</td>
<td>( { # , 4 # + 3 } )</td>
<td>5</td>
</tr>
<tr>
<td>Cunningham Chain</td>
<td>( { # , 2 # + 1 , 4 # + 3 } )</td>
<td>5</td>
</tr>
<tr>
<td>Dickson Chain</td>
<td>( { # , 2 # + 1 , 3 # + 4 } )</td>
<td>5</td>
</tr>
<tr>
<td>Star Primes</td>
<td>( { 6 # (# - 1) + 1 } )</td>
<td>2</td>
</tr>
<tr>
<td>Shanks Primes</td>
<td>( { # ^ 2 + 1 } )</td>
<td>4</td>
</tr>
<tr>
<td>Shanks Twins</td>
<td>( { (# - 1) ^ 2 + 1 \ &amp; , (# + 1) ^ 2 + 1 \ &amp; } )</td>
<td>3</td>
</tr>
<tr>
<td>Shanks Quads</td>
<td>( { (# - 1) ^ 2 + 1 \ &amp; , (# + 1) ^ 2 + 1 \ &amp; } )</td>
<td>4</td>
</tr>
<tr>
<td>Hardy-Littlewood Primes</td>
<td>( { # ^ 2 + # + 1 } )</td>
<td>3</td>
</tr>
<tr>
<td>Safe Primes</td>
<td>( { # , (# - 1) / 2 } )</td>
<td>11</td>
</tr>
<tr>
<td>Centered Heptagonal Primes</td>
<td>( { (7 # ^ 2 - 7 # + 2) / 2 } )</td>
<td>4</td>
</tr>
<tr>
<td>Centered Square Primes</td>
<td>( { # ^ 2 + (# + 1) ^ 2 } )</td>
<td>3</td>
</tr>
<tr>
<td>Centered Triangular Primes</td>
<td>( { 3 # ^ 2 + 3 # + 2 } / 2 )</td>
<td>3</td>
</tr>
<tr>
<td>Centered Decagonal Primes</td>
<td>( { 5 (# ^ 2 - #) + 1 } )</td>
<td>2</td>
</tr>
<tr>
<td>Pythagorean Primes</td>
<td>( { 4 # + 1 } )</td>
<td>0</td>
</tr>
<tr>
<td>Prime Quadruplets</td>
<td>( { # , # + 2 , # + 6 , # + 8 } )</td>
<td>3</td>
</tr>
<tr>
<td>Sexy Primes</td>
<td>( { # , # + 6 } )</td>
<td>5</td>
</tr>
</tbody>
</table>

**Table 2.** Hypothesis H Constellations
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Familiar Name Pattern Start
Thabit Primes \(3 \times 2^{\#} - 1\) 3
Wagstaff Primes \((2^\# + 1)/3\) 5
Proth Primes \(2^\# + 1, 3\) 4
Kynea Primes \((2^\# + 1)^2 - 2\) 2
Mersenne Primes \(2^\# - 1\) 1
Double Mersenne Primes \(2(2^\# - 1) - 1\) 2
Mersenne Prime Exponents \(\#^\#, 2^\# - 1\) 2
Carol Primes \((2^\# - 1)^2 - 2\) 2
Cullen Primes \(\#(2^\#) + 1\) 1
Fermat Primes \(2(2^\#) + 1\) 0
Generalized Fermat Primes Base 10 \(10^{\#+1}\) 0
Factorial Primes \(# + 1\) or \(# - 1\) 0

Table 3. Other Single-Variable Prime Sequences

Familiar Name Pattern Start
Leyland Primes \(#_1^{#_2} + #_2^{#_1}\) 0
Pierpont Primes \(2^{#_1} 3^{#_2}\) 0
Solinas Primes \(2^{#_1} \pm 2^{#_2} \pm 1\) 0
Primes of Binary Quadratic Form \(#_1^2 + #_1 \#_2 + 2 \#_2^2\) 0
Quartan Primes \(#_1^4 + #_2^4\) 0

Table 4. Two-Variable Prime Sequences

The following Mathematica routine implements the dual.

```mathematica
Sieve[constellation_, start_, n_] := Module[{f, i, j, q = start, l},
  For[i = 1, i <= n, i++,
    f[x_] :=
      Sum[Q[constellation[[i]][x], constellation[[i]][q]],
        {i, 1, Length[constellation]}];
    q = NextNonZero[f, q];
    l = {};
    For[j = 1, j <= Length[constellation], j++,
      p = pattern[[j]][q];
      AppendTo[l, {p, PrimeQ[p]}];
    ];
    Print[{q, l}];
  ];
]```

The following Mathematica routine implements the dual.
References


E-mail address: sbg@acw.com