Considerations on accelerated systems, with particular application to weak gravitational fields

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Starting with acceleration, the Equivalence Principle is used to argue that the known law of decreasing acceleration for high speed motion in the domain of low acceleration, produces the same result in a weak gravitational field with subsequent implications for stronger fields.

I. INTRODUCTION

The acceleration four-vector defined in special relativity is a compact and elegant way to describe non-inertial motion and is inclusive of the fact that objects cannot reach the light speed. While it is accepted that general relativity provides the correct generalization of special relativity for accelerating frames, nevertheless acceleration in the context of special relativity alone can still provide a very useful form of analysis.

We now briefly introduce the notation we will be using in this paper in order to describe accelerations within the framework of special relativity. We define a spacetime coordinate differential with a four-vector $dX = [cdt, dx]$ where, in general, x is a vector with contribution from three spatial dimensions and t is the time in a particular reference frame and c is the invariant speed of light[1]. In this paper we will be dealing exclusively with one-dimensional motion and so we can suppress two of the space dimensions. In the co-moving frame we have $dx = 0$ that defines τ the proper time. We also define, in general, the four-velocity $V = \frac{dX}{d\tau} = [\gamma c, \gamma v]$, where $v = dx/dt$ and $\gamma = t/\tau = 1/\sqrt{1 - v^2/c^2}$. We then have the proper velocity $\sqrt{V \cdot V} = \sqrt{\gamma^2 c^2 - \gamma^2 v^2} = c$ that is a Lorentz invariant. We also have the acceleration $A = \frac{dV}{d\tau} = [\gamma^4 va/c, \gamma^4 a]$, where we have shown the special case for one-dimensional motion in which v is parallel to a. We then find the Lorentz invariant proper acceleration

$$\sqrt{A \cdot A} = \sqrt{\gamma^8 v^2 a^2/c^2 - \gamma^8 a^2} = \gamma^3 a. \qquad (1)$$

In the momentarily co-moving frame (MCF) we have $v = 0$ and so we have the acceleration vector $A = [0, \alpha]$ and the velocity $V = [c, 0]$. This then gives the expected orthogonality condition $V \cdot A = 0$. Hence, the MCF defines the invariant proper acceleration α so that in different frames we have the acceleration $a = \alpha/\gamma^3$.

We now consider how acceleration appears inside an accelerating rocket from different inertial reference frames that each view the acceleration of the rocket with different initial velocities. Using the principle of equivalence we then transfer our results to a gravity setting.

A. A thought experiment

Consider a rocket deep in outer space far from the effects of gravity. In this effectively flat region of space we place small frames of reference that individually can measure the acceleration of passing objects. We will call these types of frames PG1 for particle group 1. The PG1 frames are currently at rest relative to the rocket and also with respect to each other and they span the space surrounding the rocket. The rocket also has a hole at the top and bottom so that the PG1 can pass straight through and measure the acceleration of the rocket. The rocket also has an inbuilt mechanism so that, when the rocket is accelerating, it will start dropping a second group of particles, labeled PG2, from the top of the rocket, at predetermined fixed time intervals. PG2 can also measure the rockets acceleration.

Now, for the sake of argument, let the rocket be accelerated at 9.8 m s^{-2} and as specified, PG2 will start to drop from the top of the rocket. The rocket now accelerates away from the PG2 frames with acceleration $\alpha = F/m = 9.8\text{ms}^{-2}$, where m is the mass of the rocket and F is the applied thrust. The PG2, once released, are clearly inertial objects not partaking in the rockets acceleration. Additionally, as the rocket continues its acceleration it will encounter PG1 lying in its path that will enter the hole at the top of the rocket and while passing through measure the acceleration of the rocket. Now, as the rocket is maintaining a steady acceleration, clearly the velocity of the rocket will be steadily increasing. Hence the rocket will be encountering the PG1 at higher and higher relative velocities.

The question to be considered is: *Will PG1 and PG2 measure the same acceleration for the rocket?*

Based on known physics, we expect the answer to be negative. This is because special relativity asserts that as the rocket's velocity converges to the light speed upper bound, the acceleration will appear to decrease, as viewed by the surrounding inertial frames PG1. This current example assumes a constant rocket thrust but from the PG1 frames will fail to increase velocities in accordance with $\alpha = F/m$. Indeed, this is a well verified phenomena commonly encountered in particle accelerators where under a constant applied force the particles slowly converge to the light speed bound.

Hence, the one-dimensional relativistic equation for acceleration a measured in the PG1 frames, is

$$a = \frac{\alpha}{\gamma^3} = \frac{F}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2}, \quad (2)$$

where α is the acceleration measured in the co-moving frames PG2, v is the velocity of the rocket relative to PG1.

The results so far are well established result from special relativity when accelerations are considered. However, following on from the above we now ask a more important question: *Given the principle of equivalence will the result in the rocket be the same as under gravity?*

We, presume that for appropriately small regions of the field, that the answer must be yes.

B. Gravity fields

The central role played by the the equivalence principle in the general theory was stated by Einstein in 1907:

we [...] assume the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system.

Einstein's equivalence principle is based firstly on the well established equivalence of gravitational and inertial mass, also called the weak equivalence principle, which has been confirmed by experiment[2] to an accuracy better than 1×10^{-15}.

The full Einstein equivalence principle also incorporates the relativity principles such as Lorentz invariance that is stated as:

The outcome of any local non-gravitational experiment in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime.

The Einstein equivalence principle essentially requires a curved spacetime metric theory of gravity in which particles follow geodesics within this space as described by Einstein in his general theory[3].

Hence, incorporating the equivalence principle, our current proposition is that since Eq. (2) pertains to a reference frame described above with an accelerating rocket then we also must have in a gravitational field

$$a = \frac{GM}{r^2}\left(1 - \frac{v^2}{c^2}\right)^{3/2}. \quad (3)$$

The purest case holds for the proper frame of the inertial falling object. This proposal shows that for gravity the rate of acceleration is a function of initial velocity, the gravitational mass and the position in the field.

II. SCHWARZSCHILD SOLUTION

For a static, non-rotating, spherical mass the field equations of general relativity give the Schwarzschild solution[3] with the metric

$$c^2 d\tau^2 = \left(1 - \frac{2\mu}{r}\right) c^2 dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \cos^2\theta d\phi^2, \quad (4)$$

where $\mu = GM/c^2$ and r measured from the center and outside the mass[3].

Now viewing the Schwarzschild solution as the metric distance we can find a Lagrangian,

$$\mathcal{L} = \left(1 - \frac{2\mu}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \dot{r}^2 = c^2, \quad (5)$$

where $\dot{t} = \frac{dt}{d\tau}$ and $\dot{r} = \frac{dr}{d\tau}$ and for purely radial motion the angular terms are zero. Hence we now extremize the action $S = \int \mathcal{L} d\tau$ and using Lagrange's equations for t we find

$$\frac{d}{d\tau}\left(\left(1 - \frac{2\mu}{r}\right) c^2 \dot{t}\right) = \frac{d\mathcal{L}}{dt} = 0. \quad (6)$$

Hence we have a constant of the motion

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = \frac{E}{mc^2}. \quad (7)$$

Now if a particle at rest slowly enters the field then the particles' energy E is approximately its rest energy mc^2, however if we wish to inject the particle into the field with velocity V then $E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - V^2/c^2}}$. Now, substituting Eq. (7) back into the metric we find

$$\frac{dr}{d\tau} = c\sqrt{\frac{1}{1 - V^2/c^2} - \left(1 - \frac{2\mu}{r}\right)}. \quad (8)$$

We can see that as $r \to \infty$ then $\frac{dr}{d\tau} \to V$ as required.

The second derivative with respect to proper time is

$$\frac{d^2 r}{d\tau^2} = -\frac{Mc^2}{r^2} = -\frac{GM}{r^2}, \quad (9)$$

which shows a constant acceleration as assumed for the rocket frame as measured by PG2, referred to earlier as proper acceleration. This thus corresponds with Eq. (2) when $v = 0$.

Now, for an observer at infinity viewing the falling object we find, substituting back into Eq. (4) that

$$\frac{dr}{dt} = \left(1 - \frac{2\mu}{r}\right)\left(1 - \left(1 - \frac{2\mu}{r}\right)\left(1 - \frac{V^2}{c^2}\right)\right)^{1/2}, \quad (10)$$

where $\frac{dr}{dt} \to V$ as $r \to \infty$, as required. The second derivative gives

$$\frac{d^2 r}{dt^2} = -\frac{m}{r^2}\left(1 - \frac{2\mu}{r}\right)\left(3\left(1 - \frac{2\mu}{r}\right)\left(1 - \frac{V^2}{c^2}\right) - 2\right). \quad (11)$$

Therefore we can see that the Schwarzschild solution also gives a velocity dependent acceleration for observers at rest with respect to the gravitational field coordinates. This implies an apparent weakening of the field strength in gravity, for moving objects relative to stationary observers in weak gravitational fields. Indeed, to a first approximation, we have a velocity dependence $1-\frac{3v^2}{2c^2}\ldots$ compared with a Schwarzschild dependence of $1-\frac{3v^2}{c^2}$. Hence we can see that, in fact, Schwarzschild and general relativity predicts approximately twice the effect to Eq. (2). This implies a possible agreement between the approximate solution and the stronger Schwarzchild solution, and suggests the basic principle to be sound enough to warrant experimental testing. This might be achieved in an earth bound frame, if there are accurate enough clocks to measure such deviations from current expected accelerations.

III. EXPERIMENTAL TESTS

Integrating the expression in Eq. (8), we can find the proper time taken between two heights as

$$\tau=\int_{r_0}^{r}\frac{dr}{\sqrt{\frac{V^2}{1-V^2/c^2}+\frac{2GM}{r}}}. \tag{12}$$

This allows us to calculate the expected time difference for a falling particle based on velocity dependence V, and so allowing an experimental test of this principle.

Also, due to the rockets mild acceleration rate, then inside the rocket frame itself, there will only very small time dilation effects. This allows the stationary frame in gravity, to be the frame of reference to to measure fairly accurately the rates of acceleration of PG1 and PG2. It is therefore proposed that this should be is the reference frame for an experimental test of the principle.

In order to maximize the effect predicted in Eq. (2) we envisage a test on particles falling in the earths gravitational field at velocities approaching the speed of light. For example, an apparatus involving an electron gun oriented vertically with a sensitive measurement of velocity at the top and bottom of the apparatus should be able to detect a variation in the gravitational acceleration as a function of velocity.

IV. DISCUSSION

We show in this paper that by considering accelerating objects within the context of special relativity and using the equivalence principle, the behavior of weak uniform gravitational fields are predicted. Specifically, we have shown that acceleration due to gravity, is a function of particle velocity as shown in Eq. (2). This can also be interpreted as a weakening of the field. As noted, our result based on accelerating frames, leads to an expected effect about half that predicted by general relativity. Hence it would make an interesting experiment to precisely measure this effect, and to account for the discrepancy between the two types of analysis. This test would also thus allow a further verification of the Einstein principle of equivalence.

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[2] S. Schlamminger, K.-Y. Choi, T. A. Wagner, J. H. Gundlach, and E. G. Adelberger, Phys. Rev. Lett. **100**, 041101 (2008).
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