A Note About The Resolution Of Navier-Stokes Equations

Par

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Abstract

This note represents an attempt of solving the Navier-Stokes equations under the assumptions (A) of the problem as described by the Clay Institute (C.L. Fefferman, 2006).

1 Introduction

As it was described in the paper cited above, the Euler and Navier-Stokes equations describe the motion of a fluid in $\mathbb{R}^n$ ($n = 2$ or 3). These equations are to be solved for an unknown velocity vector $u(x,t) = (u_i(x,t))_{i=1,n} \in \mathbb{R}^n$ and pressure $p(x,t) \in \mathbb{R}$ defined for position $x \in \mathbb{R}^n$ and time $t \geq 0$.

Here we are concerned with incompressible fluids filling all of $\mathbb{R}^n$. The Navier-Stokes equations are given by:

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \quad i \in \{1,.,n\} \ (x \in \mathbb{R}^n,\ t \geq 0) \tag{1}$$

$$divu = \sum_{i=1}^{i=n} \frac{\partial u_i}{\partial x_i} = 0 \ \ (x \in \mathbb{R}^n,\ t \geq 0) \tag{2}$$

with the initial conditions:

$$u(x,0) = u^o(x) \ \ (x \in \mathbb{R}^n) \tag{3}$$

where $u^o(x)$ a given vector function of class C^∞, $f_i(x,t)$ are the components of a given external force (e.g gravity), ν is a positive coefficient (viscosity), and Δ is the Laplacian in the space variables. Euler equations are equations (1) (2) (3) with $\nu = 0$.

2 The Navier-Stokes Equations

We try to present a solution to the Navier-Stokes equations following assumptions (A) as described in (C.L. Fefferman, 2006) that summarized here:

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* (A) **Existence and smooth solutions $\in \mathbb{R}^3$ the Navier-Stokes equations:**

- Take $\nu > 0$. Let $u^0(x)$ a smooth function such that $div(u^0(x)) = 0$ and satisfying:

$$||\partial_{x_j}^{\delta} u^0(x)|| \leq C_{\delta K}(1 + ||x||)^{-K} \ sur \ \mathbb{R}^3 \quad \forall \delta,\ K \tag{4}$$

- Take $f \equiv 0$. Then show that there are functions $p(x,t), u(x,t)$ of class C^∞ on $\mathbb{R}^3 \times [0, +\infty)$ satisfying (1),(2),(3),(4) and:

$$\int_{\mathbb{R}^3} ||u(x,t)||^2 dx < C \ \forall t \geq 0, \ \text{(bounded energy)} \tag{5}$$

We consider the Navier-Stokes equations. It takes $\nu > 0$ and $f_i \equiv 0$, then equations (1) are written:

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} - \nu \Delta u_i = -\frac{\partial p}{\partial x_i} \tag{6}$$

Considering the case n = 3, we write:

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} - \nu \Delta u_1 = -\frac{\partial p}{\partial x} \tag{7}$$

$$\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} - \nu \Delta u_2 = -\frac{\partial p}{\partial y} \tag{8}$$

$$\frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} - \nu \Delta u_3 = -\frac{\partial p}{\partial z} \tag{9}$$

As:

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz + \frac{\partial p}{\partial t} dt \tag{10}$$

Using equations (7 - 8 - 9), we get:

$$\begin{aligned} dp = &- \left(\frac{\partial u_1}{\partial t} - \nu \Delta u_1 + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} \right) dx \\ &- \left(\frac{\partial u_2}{\partial t} - \nu \Delta u_2 + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} \right) dy \\ &- \left(\frac{\partial u_3}{\partial t} - \nu \Delta u_3 + u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} \right) dz + \frac{\partial p}{\partial t} dt \end{aligned} \tag{11}$$

But:

$$\frac{du^2}{2} = \frac{d(u_1^2 + u_2^2 + u_3^3)}{2} = \sum_i u_i du_i = \sum_i u_i (\partial_x u_i dx + \partial_y u_i dy + \partial_z u_i dz + \partial_t u_i dt) \tag{12}$$

noting $\partial_x = \dfrac{\partial}{\partial x}$. Then equation (11) becomes:

$$\begin{aligned}
-dp + \partial_t p.dt = &\left(\frac{\partial u_1}{\partial t} - \nu\Delta u_1 + u_2\frac{\partial u_1}{\partial y} + u_3\frac{\partial u_1}{\partial z} - u_2\frac{\partial u_2}{\partial x} - u_3\frac{\partial u_3}{\partial x}\right) dx \\
&+ \left(\frac{\partial u_2}{\partial t} - \nu\Delta u_2 + u_1\frac{\partial u_2}{\partial x} + u_3\frac{\partial u_2}{\partial z} - u_1\frac{\partial u_1}{\partial y} - u_3\frac{\partial u_3}{\partial y}\right) dy \\
&+ \left(\frac{\partial u_3}{\partial t} - \nu\Delta u_3 + u_1\frac{\partial u_3}{\partial x} + u_2\frac{\partial u_3}{\partial y} - u_1\frac{\partial u_1}{\partial z} - u_2\frac{\partial u_2}{\partial z}\right) dz \\
&- \left(u_1\frac{\partial u_1}{\partial t} + u_2\frac{\partial u_2}{\partial t} + u_3\frac{\partial u_3}{\partial t}\right) dt + d\left(\frac{u^2}{2}\right)
\end{aligned} \tag{13}$$

Let Ω the vector $curl(u)$, then:

$$\Omega = \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = \begin{vmatrix}\partial_x\\ \partial_y\\ \partial_z\end{vmatrix} \wedge \begin{vmatrix}u_1\\ u_2\\ u_3\end{vmatrix} = \begin{pmatrix}\partial_y u_3 - \partial_z u_2\\ \partial_z u_1 - \partial_x u_3\\ \partial_x u_2 - \partial_y u_1\end{pmatrix} \tag{14}$$

Then, equation (13) is written as follows:

$$\begin{aligned}
-d\left(p + \frac{u^2}{2}\right) = -\partial_t(p + \frac{1}{2}u^2)dt + \left(\frac{\partial u_1}{\partial t} - \nu\Delta u_1 - u_2\omega_3 + u_3\omega_2\right) dx+ \\
\left(\frac{\partial u_2}{\partial t} - \nu\Delta u_2 + u_1\omega_3 - u_3\omega_1\right) dy + \left(\frac{\partial u_3}{\partial t} - \nu\Delta u_3 - u_1\omega_2 + u_2\omega_1\right) dz
\end{aligned} \tag{15}$$

We write the above equation in the form:

$$\begin{aligned}
d\left(p + \frac{u^2}{2}\right) = \partial_t(p + \frac{1}{2}u^2)dt + &\left(-\frac{\partial u_1}{\partial t} + \nu\Delta u_1 + u_2\omega_3 - u_3\omega_2\right) dx+ \\
&\left(-\frac{\partial u_2}{\partial t} + \nu\Delta u_2 - u_1\omega_3 + u_3\omega_1\right) dy \\
&+ \left(-\frac{\partial u_3}{\partial t} + \nu\Delta u_3 + u_1\omega_2 - u_2\omega_1\right) dz
\end{aligned} \tag{16}$$

or as:

$$d\left(p + \frac{u^2}{2}\right) = \partial_t(p + \frac{1}{2}u^2)dt + A.dx + B.dy + C.dz \tag{17}$$

with:

$$A = u_2\omega_3 - u_3\omega_2 - \frac{\partial u_1}{\partial t} + \nu\Delta u_1 \tag{18}$$

$$B = u_3\omega_1 - u_1\omega_3 - \frac{\partial u_2}{\partial t} + \nu\Delta u_2 \tag{19}$$

$$C = u_1\omega_2 - u_2\omega_1 - \frac{\partial u_3}{\partial t} + \nu\Delta u_3 \tag{20}$$

Let h the vector:

$$h = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \tag{21}$$

The left member of equation (17) is a total differential, we can write the conditions:

$$\partial_y A = \partial_x B \tag{22}$$
$$\partial_z A = \partial_x C \tag{23}$$
$$\partial_z B = \partial_y C \tag{24}$$

Which give:

$$curl(h) = \begin{pmatrix} \partial_y C - \partial_z B \\ \partial_z A - \partial_x C \\ \partial_x B - \partial_y A \end{pmatrix} = 0 \tag{25}$$

But h is written as:

$$h = \begin{pmatrix} A \\ B \\ C \end{pmatrix} = u \wedge \Omega - \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \\ u_2 \end{pmatrix} + \nu \Delta \begin{pmatrix} u_1 \\ u_2 \\ u_2 \end{pmatrix} = u \wedge \Omega - \frac{\partial u}{\partial t} + \nu \Delta u \tag{26}$$

The conditions (22 - 23 - 24) are summarized by $curl(h) = 0$:

$$\boxed{curl(u \wedge \Omega) = \frac{\partial \Omega}{\partial t} - \nu \Delta \Omega} \tag{27}$$

because $\Omega = curl(u)$. Recall now the formula (Landau and Lifshitz, 1970):

$$curl(a \wedge b) = (b.\nabla).a - (a.\nabla).b + a.divb - b.diva \tag{28}$$

In our study, we have $a = u \Longrightarrow diva = divu = \partial_x u_1 + \partial_y u_2 + \partial_z u_3 = 0$ and $b = \Omega = curl(u)$ then $divb = div\Omega = div(curl(u)) = 0$. As a result:

$$(\Omega.\nabla).u - (u.\nabla).\Omega = \frac{\partial \Omega}{\partial t} - \nu \Delta \Omega \tag{29}$$

Or in matrix form:

$$\begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{pmatrix} . \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} - \begin{pmatrix} \frac{\partial \omega_1}{\partial x} & \frac{\partial \omega_1}{\partial y} & \frac{\partial \omega_1}{\partial z} \\ \frac{\partial \omega_2}{\partial x} & \frac{\partial \omega_2}{\partial y} & \frac{\partial \omega_2}{\partial z} \\ \frac{\partial \omega_3}{\partial x} & \frac{\partial \omega_3}{\partial y} & \frac{\partial \omega_3}{\partial z} \end{pmatrix} . \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \nu \begin{pmatrix} \Delta \omega_1 \\ \Delta \omega_2 \\ \Delta \omega_3 \end{pmatrix} - \begin{pmatrix} \frac{\partial \omega_1}{\partial t} \\ \frac{\partial \omega_2}{\partial t} \\ \frac{\partial \omega_3}{\partial t} \end{pmatrix} \tag{30}$$

Let:

$$A(u) = \begin{pmatrix} \dfrac{\partial u_1}{\partial x} & \dfrac{\partial u_1}{\partial y} & \dfrac{\partial u_1}{\partial z} \\ \\ \dfrac{\partial u_2}{\partial x} & \dfrac{\partial u_2}{\partial y} & \dfrac{\partial u_2}{\partial z} \\ \\ \dfrac{\partial u_3}{\partial x} & \dfrac{\partial u_3}{\partial y} & \dfrac{\partial u_3}{\partial z} \end{pmatrix} \tag{31}$$

$$A(\Omega) = \begin{pmatrix} \dfrac{\partial \omega_1}{\partial x} & \dfrac{\partial \omega_1}{\partial y} & \dfrac{\partial \omega_1}{\partial z} \\ \\ \dfrac{\partial \omega_2}{\partial x} & \dfrac{\partial \omega_2}{\partial y} & \dfrac{\partial \omega_2}{\partial z} \\ \\ \dfrac{\partial \omega_3}{\partial x} & \dfrac{\partial \omega_3}{\partial y} & \dfrac{\partial \omega_3}{\partial z} \end{pmatrix} \tag{32}$$

In this case, equation (30) becomes:

$$\boxed{A(u).\Omega - A(\Omega).u = \nu\Delta\Omega - \frac{\partial \Omega}{\partial t}} \tag{33}$$

The equations (33) are the fundamental equations of this study. These are non-linear partial differential equations of the third order. Their resolutions are the solutions of the Navier-Stokes equations.

3 The Study of The Fundamental Equations (33)

3.1 Preliminaries

Call respectively:

$$F(u,\Omega) = A(u).\Omega - A(\Omega).u \tag{34}$$

$$G(\Omega) = \nu\Delta\Omega - \frac{\partial \Omega}{\partial t} \tag{35}$$

If you exchange u, Ω in $-u, -\Omega$, we get:

$$F(-u,-\Omega) = F(u,\Omega) \tag{36}$$

$$G(-\Omega) = -G(\Omega) \tag{37}$$

According to equation (33), we get:

$$\begin{cases} F(u,\Omega) = G(\Omega) \\ \\ F(-u,-\Omega) = G(-\Omega) = -G(\Omega) = F(u,\Omega) \end{cases} \Longrightarrow G(\Omega) = 0 \Longrightarrow F(u,\Omega) = 0 \tag{38}$$

It was therefore the differential system:

$$\boxed{\begin{cases} \nu\Delta\Omega - \dfrac{\partial\Omega}{\partial t} = 0 \\ \\ A(u).\Omega - A(\Omega).u = 0 \\ with\ \Omega = curl(u) \\ and\ curl(u \wedge \Omega) = \nu\Delta\Omega - \dfrac{\partial\Omega}{\partial t} \Longrightarrow curl(u \wedge \Omega) = 0 \end{cases}} \tag{39}$$

under equation (27).

3.2 Case $\Omega \equiv 0$

In this case, obviously:

$$\nu\Delta\Omega - \frac{\partial\Omega}{\partial t} = 0$$
$$A(u).\Omega - A(\Omega).u = 0$$

So:

$$\Omega = curl(u) = 0 \Longrightarrow \begin{cases} u \equiv 0 \quad which\ is\ a\ contradiction, \\ u = a\ constant\ vector\ which\ is\ a\ contradiction, \\ \exists\ a\ scalaire\ function\ \Phi\ /\ u = grad\Phi \end{cases} \tag{40}$$

In the latter case, as $\Omega = curl(u) \Longrightarrow curl(u) = 0$ then:

$$\begin{cases} \dfrac{\partial u_1}{\partial y} = \dfrac{\partial u_2}{\partial x} \\ \\ \dfrac{\partial u_2}{\partial z} = \dfrac{\partial u_3}{\partial y} \\ \\ \dfrac{\partial u_3}{\partial x} = \dfrac{\partial u_1}{\partial z} \end{cases} \tag{41}$$

and as $div(u) = 0$, it is easily obtained:

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} = \Delta u_1 = 0 \tag{42}$$

Similarly, we have also:

$$\begin{cases} \Delta u_2 = 0 \\ \Delta u_3 = 0 \end{cases} \tag{43}$$

Using $div(u) = 0$, we have also:

$$\Delta\Phi = 0 \tag{44}$$

Thus $\Phi = \Phi(x, y, z, t)$ is a harmonic function of (x, y, z).

Equation (7) becomes:

$$\frac{\partial^2\Phi}{\partial x\partial t}+\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y}+\frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial x\partial z}=$$
$$\nu\frac{\partial}{\partial x}\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\right]-\frac{\partial p}{\partial x} \tag{45}$$

But $\Delta\Phi = 0$ then:

$$\frac{\partial}{2\partial x}\left[\frac{2\partial\Phi}{\partial t}+\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right]=-\frac{\partial p}{\partial x} \tag{46}$$

And integrating with respect to x, we obtain:

$$p=\frac{\partial\Phi}{\partial t}-\frac{1}{2}u^2+\psi_1(y,z,t) \tag{47}$$

Similarly, we also obtain:

$$p=-\frac{\partial\Phi}{\partial t}-\frac{1}{2}u^2+\psi_2(x,z,t) \tag{48}$$
$$p=-\frac{\partial\Phi}{\partial t}-\frac{1}{2}u^2+\psi_3(x,y,t) \tag{49}$$

As a result:

$$p+\frac{1}{2}u^2-\psi_1(y,z,t)=p+\frac{1}{2}u^2-\psi_2(x,z,t)=p+\frac{1}{2}u^2-\psi_3(x,y,t) \tag{50}$$

Which gives:

$$\psi_1(t)=\psi_2(t)=\psi_3(t)=\psi(t) \tag{51}$$

a function that is added to the function Φ, and the result:

$$\Delta\Phi = 0 \tag{52}$$

$$\left.\frac{\partial\Phi(x,y,z,t)}{\partial x}\right|_{t=0}=u_1^0(x,y,z);\quad \left.\frac{\partial\Phi(x,y,z,t)}{\partial y}\right|_{t=0}=u_2^0(x,y,z) \tag{53}$$
$$\left.\frac{\partial\Phi(x,y,z,t)}{\partial z}\right|_{t=0}=u_3^0(x,y,z) \tag{54}$$

and:

$$\Delta u_i(x,y,z,t)|_{t=0}=\Delta u_i^0(x,y,z)=0,\quad i=1,n \tag{55}$$

$$p(x,y,z,t)=-\frac{\partial\Phi}{\partial t}-\frac{1}{2}u^2=-\frac{\partial\Phi}{\partial t}-\frac{1}{2}||grad\Phi||^2 \tag{56}$$

3.3 Case Ω is not the zero function

We rewrite the differential system (39)

$$\begin{cases} \dfrac{\partial \Omega}{\partial t} - \nu \Delta \Omega = 0 \\ \\ A(u).\Omega - A(\Omega).u = 0 \\ with\ \Omega = curl(u) \\ and\ curl(u \wedge \Omega) = \dfrac{\partial \Omega}{\partial t} - \nu \Delta \Omega \Longrightarrow curl(u \wedge \Omega) = 0 \end{cases}$$

From $curl(u \wedge \Omega) = 0$, we deduce that:

1. There is a scalar function $\varphi(x, y, z)$ as $u \wedge \Omega = grad\varphi$.
2. $u \wedge \Omega = C$ where $C = (c_1, c_2, c_3)^T$ is a nonzero constant vector or vector function of t of $\mathbb{R} \longrightarrow \mathbb{R}^3$.
3. $u \wedge \Omega = 0 \Longrightarrow$ as u and Ω are not nuls, it is that u and Ω collinear.

3.3.1 Case 2

As $C = u \wedge \Omega$, one can write:

$$c_1.u_1 + c_2.u_2 + c_3.u_3 = 0 \tag{57}$$

because C is orthogonal to u. let us differentiate the previous equation, respectively, to x, y and z, we get:

$$\begin{cases} c_1.\dfrac{\partial u_1}{\partial x} + c_2.\dfrac{\partial u_2}{\partial x} + c_3.\dfrac{\partial u_3}{\partial x} = 0 \\ \\ c_1.\dfrac{\partial u_1}{\partial y} + c_2.\dfrac{\partial u_2}{\partial y} + c_3.\dfrac{\partial u_3}{\partial y} = 0 \\ \\ c_1.\dfrac{\partial u_1}{\partial z} + c_2.\dfrac{\partial u_2}{\partial z} + c_3.\dfrac{\partial u_3}{\partial z} = 0 \end{cases} \tag{58}$$

that in matrix form:

$$A^T(u).C = 0 \tag{59}$$

where $A(u)$ is the matrix given by (31). However, the matrix $A(u)$ is the Jacobian matrix of $(x, y, z) \longrightarrow u(x, y, z, t)$ therefore its determinant is nonzero. As a result, we deduce from (59) that the vector C is necessarily zero. It is the case 3.

3.3.2 Case where $u//\Omega$

Assume now that u and Ω are collinear. Let $u//\Omega$.

Case $u = \lambda\Omega$ with $\lambda \in \mathbb{R}^*$ Then there is a coefficient $\lambda \neq 0$ such that:

$$u = \lambda \Omega \tag{60}$$

Using the equation:

$$A(u).\Omega - A(\Omega).u = 0$$

it is verified. Then we have the system:

$$\frac{\partial \Omega}{\partial t} - \nu \Delta \Omega = 0$$

But the above equation is the heat equation. Let the change of variables:

$$x = \nu X \tag{61}$$
$$y = \nu Y \tag{62}$$
$$z = \nu Z \tag{63}$$
$$t = \nu T \tag{64}$$
$$u(x, y, z, t) = U(X, Y, Z, T) \tag{65}$$
$$p(x, y, z, t) = P(X, Y, Z, T) \tag{66}$$
$$\Omega(x, y, z, t) = \overline{\Omega}(X, Y, Z, T) \tag{67}$$

Then:

$$\partial_x u dx + \partial_y u dy + \partial_z u dz + \partial_t u dt = \partial_X U dX + \partial_X U dX + \partial_X U dX + \partial_X U dX$$
$$\nu(\partial_x u dX + \partial_y u dY + \partial_z u dZ + \partial_t u dT) = \partial_X U dX + \partial_X U dX + \partial_X U dX + \partial_X U dX$$
$$\partial_x u = \frac{1}{\nu}\partial_X U,\ \partial_y u = \frac{1}{\nu}\partial_Y U,\ \partial_z u = \frac{1}{\nu}\partial_Z U,\ \partial_t u = \frac{1}{\nu}\partial_T U \tag{68}$$

Then the equation

$$\frac{\partial \Omega}{\partial t} - \nu \Delta \Omega = 0$$

becomes:

$$\boxed{\frac{\partial \overline{\Omega}}{\partial T} - \Delta \overline{\Omega} = 0} \tag{69}$$

This is the heat equation!

4 Resolution of the Equation (69)

Noting that $U^0(X, Y, Z) = U^0(\boldsymbol{X}) = U(X, Y, Z, 0) = u(x, y, z, 0) = u^0(x, y, z)$ and $\overline{\Omega}^0 = rotU^0(\boldsymbol{X})$. Then the solution of (69) with $T \geq 0$ satisfying:

$$\overline{\Omega} \in \mathbb{R}^3 \text{ and of class } C^\infty(\mathbb{R}^3 \times [0, +\infty)) \tag{70}$$
$$\overline{\Omega}(\boldsymbol{X}, 0) = \overline{\Omega}^0(\boldsymbol{X}) \tag{71}$$

is given by (S. Godunov, 1973):

$$\overline{\Omega}(\boldsymbol{X}, T) = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}^3} \frac{\overline{\Omega}^0(\alpha, \beta, \gamma)}{\sqrt{T}} e^{-\frac{(X-\alpha)^2 + (Y-\beta)^2 + (Z-\gamma)^2}{4T}} dV \tag{72}$$

where $dV = d\alpha d\beta\gamma$.

4.1 Expression of U

We have:

$$U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \lambda.\overline{\Omega} = \lambda. \begin{pmatrix} \overline{\Omega}_1 \\ \overline{\Omega}_2 \\ \overline{\Omega}_3 \end{pmatrix} \tag{73}$$

Let :

$$U_1 = \lambda.\overline{\Omega}_1 = \frac{\lambda}{2\sqrt{\pi}} \int_{\mathbb{R}^3} \frac{\overline{\Omega}_1^0(\alpha,\beta,\gamma)}{\sqrt{T}} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} dV \tag{74}$$

$$U_2 = \lambda.\overline{\Omega}_2 = \frac{\lambda}{2\sqrt{\pi}} \int_{\mathbb{R}^3} \frac{\overline{\Omega}_2^0(\alpha,\beta,\gamma)}{\sqrt{T}} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} dV \tag{75}$$

$$U_3 = \lambda.\overline{\Omega}_3 = \frac{\lambda}{2\sqrt{\pi}} \int_{\mathbb{R}^3} \frac{\overline{\Omega}_3^0(\alpha,\beta,\gamma)}{\sqrt{T}} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} dV \tag{76}$$

4.2 Checking $div(U) = 0$

Let us calculate $\partial_X U_1$, we get:

$$\frac{\partial U_1}{\partial X} = \frac{-\lambda}{4\sqrt{\pi}} \int_{\mathbb{R}^3} \frac{(X-\alpha)\overline{\Omega}_1^0(\alpha,\beta,\gamma)}{T\sqrt{T}} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} dV \tag{77}$$

We can write the above expression as follows:

$$\frac{\partial U_1}{\partial X} = \frac{-\lambda}{2\sqrt{\pi T}} \int_{\mathbb{R}^2} d\beta d\gamma \int_{\alpha=-\infty}^{\alpha=+\infty} \overline{\Omega}_1^0(\alpha,\beta,\gamma) \frac{\partial}{\partial \alpha} \left(e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} \right) d\alpha \tag{78}$$

Now we do an integration by parts, we get:

$$\begin{aligned} \frac{\partial U_1}{\partial X} &= \frac{-\lambda}{2\sqrt{\pi T}} \int_{\mathbb{R}^2} d\beta d\gamma \left[\overline{\Omega}_1^0(\alpha,\beta,\gamma).e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} \right]_{\alpha=-\infty}^{\alpha=+\infty} + \\ &\frac{\lambda}{2\sqrt{\pi T}} \int_{\mathbb{R}^2} d\beta d\gamma \int_{\alpha=-\infty}^{\alpha=+\infty} e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} \frac{\partial \overline{\Omega}_1^0(\alpha,\beta,\gamma)}{\partial \alpha}.d\alpha \end{aligned} \tag{79}$$

Taking into account the assumption that:

$$||\partial_{X_j}^{\delta} U^0(\boldsymbol{X})|| \le \nu C_{\delta K}(1+\nu||\boldsymbol{X}||)^{-K} \ in \ \mathbb{R}^3 \quad \forall \delta, \, K \tag{80}$$

where X_j denotes one of the coordinates X, Y, Z, and choosing $K > 1$, the first term of the right member is zero. Then:

$$\frac{\partial U_1}{\partial X} = \frac{\lambda}{2\sqrt{\pi T}} \int_{\mathbb{R}^2} d\beta d\gamma \int_{\alpha=-\infty}^{\alpha=+\infty} e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} \frac{\partial \overline{\Omega}_1^0(\alpha,\beta,\gamma)}{\partial \alpha} .d\alpha \tag{81}$$

or:

$$\frac{\partial U_1}{\partial X} = \frac{\lambda}{2\sqrt{\pi T}} \int_{\mathbb{R}^3} e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} \frac{\partial \overline{\Omega}_1^0(\alpha,\beta,\gamma)}{\partial \alpha} .dV \tag{82}$$

As a result:

$$div(U) = \sum_{X_j} \frac{\partial U_j}{\partial X_j} = \frac{\lambda}{2\sqrt{\pi T}} \int_{\mathbb{R}^3} e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} \sum_{\alpha_j} \frac{\partial \overline{\Omega}_j^0(\alpha,\beta,\gamma)}{\partial \alpha} .dV = 0 \tag{83}$$

because $\overline{\Omega}^0(\alpha,\beta,\gamma)$ satisfies $div(\overline{\Omega}^0) = \sum_{\alpha_j} \frac{\partial \overline{\Omega}_j^0(\alpha,\beta,\gamma)}{\partial \alpha_j} = 0$.

4.3 Estimation of $\int_{\mathbb{R}^3} ||U(\boldsymbol{X},T)||^2 dV$

We have:

$$||U(\boldsymbol{X},T)||^2 = \sum_i U_i^2 = \lambda^2 ||\overline{\Omega}(\boldsymbol{X},T)||^2 = \frac{\lambda^2}{4\pi T} \left\| \int_{\mathbb{R}^3} \overline{\Omega}^0(\alpha,\beta,\gamma).e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{4T}} dV \right\|^2$$

$$\leq \frac{\lambda^2}{4\pi T} \int_{\mathbb{R}^3} \left\| \overline{\Omega}^0(\alpha,\beta,\gamma) \right\|^2 .e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{2T}} dV \tag{84}$$

As :

$$||\overline{\Omega}^0(\alpha,\beta,\gamma)||^2 = (\omega_1^{(0)})^2 + (\omega_2^{(0)})^2 + (\omega_3^{(0)})^2$$

and taking into account the assumption that:

$$|\partial_{x_j}^{\delta} u_i^0(\boldsymbol{x})| \leq C_{\delta K}(1 + ||\boldsymbol{x}||)^{-K} \ in \ \mathbb{R}^3 \quad \forall \delta,\ K \ with \ ||\boldsymbol{x}|| = \sqrt{x^2+y^2+z^2}$$

and passing to the coordinates (X, Y, Z), we have the inequalities:

$$\left| \frac{\partial^{\delta} U_i^0(\boldsymbol{X})}{\partial X_j} \right| \leq \nu C_{\delta K}(1 + \nu||\boldsymbol{X}||)^{-K} \ in \ \mathbb{R}^3 \quad \forall \delta,\ K \in \mathbb{R}$$
$$with \ ||\boldsymbol{X}|| = \sqrt{X^2+Y^2+Z^2} \tag{85}$$

But:

$$(\omega_i^{(0)})^2 = \left(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} \right)^2 \leq \left(\left| \frac{\partial u_k}{\partial x_j} \right| + \left| \frac{\partial u_j}{\partial x_k} \right| \right)^2 \leq 4\nu^2 C_K^2 (1 + \nu||X||)^{-2K} \tag{86}$$

then :

$$||\overline{\Omega}^0(\alpha,\beta,\gamma)||^2 \leq 12\nu^2 C_K^2 (1+\nu||X||)^{-2K} = 12\nu^2 C_K^2 (1+\nu||\sqrt{\alpha^2+\beta^2+\gamma^2}||)^{-2K} \tag{87}$$

As a result:

$$||U(\boldsymbol{X},T)||^2 \leq \frac{3\nu^2\lambda^2 C_K^2}{\pi T} \int_{\mathbb{R}^3} \frac{e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{2T}}}{(1+\nu||\sqrt{\alpha^2+\beta^2+\gamma^2}||)^{2K}} d\alpha d\beta d\gamma \tag{88}$$

Let us now majorize $\int_{\mathbb{R}^3} ||u(\boldsymbol{x},t)||^2 dxdydz$:

$$\int_{\mathbb{R}3} ||u(\boldsymbol{x},t)||^2 dxdydz = \int_{\mathbb{R}3} ||U(\boldsymbol{X},T)||^2 dxdydz = \nu^3 \int_{\mathbb{R}3} ||U(\boldsymbol{X},T)||^2 dXdYdZ$$

$$\leq \frac{3\nu^5\lambda^2 C_K^2}{\pi T} \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}^3} \frac{e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{2T}}}{(1+\nu||\sqrt{\alpha^2+\beta^2+\gamma^2}||)^{2K}} d\alpha d\beta d\gamma \right] dXdYdZ \tag{89}$$

As the integral $\int_{\mathbb{R}^3} e^{-X^2-Y^2-Z^2} dXdYdZ < +\infty$, we can permutate the two triple integrals of the above equation. Let:

$$\tau_0 = \frac{3\nu^5\lambda^2 C_K^2}{\pi} \tag{90}$$

we obtain:

$$\int_{\mathbb{R}3} ||u(\boldsymbol{x},t)||^2 dxdydz \leq \frac{\tau_0}{T} \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}^3} e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{2T}} dXdYdZ \right] . \frac{d\alpha d\beta d\gamma}{(1+\nu||\sqrt{\alpha^2+\beta^2+\gamma^2}||)^{2K}} \tag{91}$$

Let:

$$I = \int_{\mathbb{R}^3} e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-\gamma)^2}{2T}} dXdYdZ \tag{92}$$

and let the following change of variables:

$$\begin{cases} \overline{X} = \frac{X-\alpha}{\sqrt{2T}} \Longrightarrow dX = \sqrt{2T} d\overline{X} \quad et\ \overline{X}^2 = \frac{(X-\alpha)^2}{2T} \\ \overline{Y} = \frac{Y-\beta}{\sqrt{2T}} \Longrightarrow dY = \sqrt{2T} d\overline{Y} \quad et\ \overline{Y}^2 = \frac{(Y-\beta)^2}{2T} \\ \overline{Z} = \frac{Z-\gamma}{\sqrt{2T}} \Longrightarrow dZ = \sqrt{2T} d\overline{Z} \quad et\ \overline{Z}^2 = \frac{(Z-\gamma)^2}{2T} \end{cases} \tag{93}$$

I is written as:

$$I = (\sqrt{2T})^3 \left[\int_{-\infty}^{+\infty} e^{-\overline{X}^2} d\overline{X} \right]^3 = 2T\sqrt{2T} \left[2\int_0^{+\infty} e^{-\xi^2} d\xi \right]^3 = 2T\sqrt{T}.\pi\sqrt{\pi} = 2\pi T\sqrt{\pi T} \tag{94}$$

using the formula $2\int_0^{+\infty} e^{-\xi^2} d\xi = \sqrt{\pi}$. then the equation (91) becomes:

$$\int_{\mathbb{R}3} ||u(\boldsymbol{x},t)||^2 dxdydz \leq 2\tau_0\pi\sqrt{\pi T}\int_{\mathbb{R}^3} \frac{d\alpha d\beta d\gamma}{(1+\nu||\sqrt{\alpha^2+\beta^2+\gamma^2}||)^{2K}} \tag{95}$$

Let us now:

$$B = \int_{\mathbb{R}^3} \frac{d\alpha d\beta d\gamma}{(1+\nu||\sqrt{\alpha^2+\beta^2+\gamma^2}||)^{2K}} \tag{96}$$

and we use the spherical coordinates:

$$\begin{cases} \alpha = rsin\theta cos\varphi \\ \beta = rsin\theta sin\varphi \\ \gamma = rcos\theta \end{cases} \tag{97}$$

the form of the volume $d\alpha d\beta d\gamma = r^2 sin\theta dr d\theta d\varphi$ and B becomes:

$$B = \int_{\theta=0}^{\theta=\pi} sin\theta d\theta \int_{\varphi=0}^{\varphi=2\pi} d\varphi \int_0^r \frac{r^2 dr}{(1+\nu r)^{2K}} = 4\pi \int_0^r \frac{r^2 dr}{(1+\nu r)^{2K}} \tag{98}$$

We take $K = 2$, the integral B is convergent when $r \to +\infty$. Let:

$$F = lim_{r\to+\infty} \int_0^r \frac{r^2 dr}{(1+\nu r)^4} = \int_0^{+\infty} \frac{r^2 dr}{(1+\nu r)^4} = \int_0^1 \frac{r^2 dr}{(1+\nu r)^4} + \int_1^{+\infty} \frac{r^2 dr}{(1+\nu r)^4} \tag{99}$$

But :

$$\int_0^1 \frac{r^2 dr}{(1+\nu r)^4} < \int_0^1 r^2 dr = \left[\frac{r^3}{3}\right]_0^1 = \frac{1}{3} \tag{100}$$

We calculate now $\int_1^{+\infty} \frac{r^2 dr}{(1+\nu r)^4}$. Let the change of variables:

$$\xi = 1 + \nu r \Rightarrow r = \frac{\xi - 1}{\nu} \Rightarrow dr = \frac{d\xi}{\nu} \tag{101}$$

then:

$$\int_1^{+\infty} \frac{r^2 dr}{(1+\nu r)^4} = \frac{1}{\nu^3} \int_{1+\nu}^{+\infty} \frac{\xi^2 - 2\xi + 1}{\xi^4} d\xi = l(\nu) \ avec \quad l(\nu) = \frac{3\nu 2 + 9\nu + 5}{\nu^3(1+\nu)^3} \tag{102}$$

As a result:

$$B < 4\pi(\frac{1}{3} + l(\nu)) \tag{103}$$

Hence the important result:

$$\int_{\mathbb{R}3} ||u(\boldsymbol{x},t)||^2 dxdydz < 8\tau_0\pi^2\sqrt{\pi T}\left(\frac{1}{3} + l(\nu)\right) \tag{104}$$

or:

$$\boxed{\int_{\mathbb{R}3} ||u(\boldsymbol{x},t)||^2 dxdydz < +\infty \quad \forall t} \tag{105}$$

let:

$$\boxed{\int_{\mathbb{R}3} ||U(\boldsymbol{X},T)||^2 dXdYdZ < +\infty \quad \forall T} \tag{106}$$

because:

$$\int_{\mathbb{R}3} ||U(\boldsymbol{X},T)||^2 dXdYdZ = \frac{1}{\nu^3} \int_{\mathbb{R}3} ||u(\boldsymbol{x},t)||^2 dxdydz$$

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