A Very Brief Introduction to Reflections in 2D Geometric Algebra, and their Use in Solving “Construction” Problems

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Abstract

This document is intended to be a convenient collection of explanations and techniques given elsewhere ([1]-[3]) in the course of solving tangency problems via Geometric Algebra.
Geometric-Algebra Formulas
for Plane (2D) Geometry

The Geometric Product, and Relations Derived from It
For any two vectors \( a \) and \( b \),
\[
\begin{align*}
\mathbf{a} \cdot \mathbf{b} &= \mathbf{b} \cdot \mathbf{a} \\
\mathbf{b} \wedge \mathbf{a} &= -\mathbf{a} \wedge \mathbf{b} \\
\mathbf{a} \mathbf{b} &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \\
\mathbf{b} \mathbf{a} &= \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \wedge \mathbf{a} = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \wedge \mathbf{b} \\
\mathbf{a} \mathbf{b} + \mathbf{b} \mathbf{a} &= 2\mathbf{a} \cdot \mathbf{b} \\
\mathbf{a} \mathbf{b} - \mathbf{b} \mathbf{a} &= 2\mathbf{a} \wedge \mathbf{b} \\
\mathbf{a} \mathbf{b} = 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \wedge \mathbf{a} \\
\mathbf{b} \mathbf{a} = 2\mathbf{a} \wedge \mathbf{b} - \mathbf{a} \cdot \mathbf{b}
\end{align*}
\]

Definitions of Inner and Outer Products (Macdonald A. 2010 p. 101.)
The inner product
The inner product of a \( j \)-vector \( A \) and a \( k \)-vector \( B \) is
\[
\mathbf{A} \cdot \mathbf{B} = \langle \mathbf{AB} \rangle_{k-j}.
\]
Note that if \( j > k \), then the inner product doesn’t exist.
However, in such a case \( \mathbf{B} \cdot \mathbf{A} = \langle \mathbf{BA} \rangle_{j-k} \) does exist.

The outer product
The outer product of a \( j \)-vector \( \mathbf{A} \) and a \( k \)-vector \( \mathbf{B} \) is
\[
\mathbf{A} \wedge \mathbf{B} = \langle \mathbf{AB} \rangle_{k+j}.
\]

Relations Involving the Outer Product and the Unit Bivector, \( \mathbf{i} \)
For any two vectors \( \mathbf{a} \) and \( \mathbf{b} \),
\[
\begin{align*}
\mathbf{i} \mathbf{a} &= -\mathbf{a} \mathbf{i} \\
\mathbf{a} \wedge \mathbf{b} &= [(\mathbf{a} \cdot \mathbf{i}) \cdot \mathbf{b}] \mathbf{i} = -[\mathbf{a} \cdot (\mathbf{b} \mathbf{i})] \mathbf{i} = -\mathbf{b} \wedge \mathbf{a}
\end{align*}
\]

Equality of Multivectors
For any two multivectors \( \mathcal{M} \) and \( \mathcal{N} \),
\( \mathcal{M} = \mathcal{N} \) if and only if for all \( k, \langle \mathcal{M} \rangle_k = \langle \mathcal{N} \rangle_k \).

Formulas Derived from Projections of Vectors
and Equality of Multivectors
Any two vectors \( \mathbf{a} \) and \( \mathbf{b} \) can be written in the form of “Fourier expansions”
with respect to a third vector, \( \mathbf{v} \):
\[
\begin{align*}
\mathbf{a} &= (\mathbf{a} \cdot \mathbf{v}) \mathbf{v} + [\mathbf{a} \cdot (\mathbf{v} \mathbf{i})] \mathbf{i} \mathbf{v} \\
\mathbf{b} &= (\mathbf{b} \cdot \mathbf{v}) \mathbf{v} + [\mathbf{b} \cdot (\mathbf{v} \mathbf{i})] \mathbf{i} \mathbf{v}
\end{align*}
\]
Using these expansions,
\[
\begin{align*}
\mathbf{a} \mathbf{b} &= \{[(\mathbf{a} \cdot \mathbf{v}) \mathbf{v} + [\mathbf{a} \cdot (\mathbf{v} \mathbf{i})] \mathbf{i} \mathbf{v}] \mathbf{v} + [\mathbf{b} \cdot (\mathbf{v} \mathbf{i})] \mathbf{i} \mathbf{v}] \}
\end{align*}
\]
Equating the scalar parts of both sides of that equation,
\[ a \cdot b = [a \cdot \hat{v}] [b \cdot \hat{v}] + [a \cdot (\hat{v}i)] [b \cdot (\hat{v}i)], \text{ and} \\
\quad a \land b = \{(a \cdot \hat{v}) [b \cdot (\hat{v}i)] - [a \cdot (\hat{v}i)] [b \cdot (\hat{v}i)]\} i. \]

Also, \(a^2 = [a \cdot \hat{v}]^2 + [a \cdot (\hat{v}i)]^2\), and \(b^2 = [b \cdot \hat{v}]^2 + [b \cdot (\hat{v}i)]^2\).

**Reflections of Vectors, Geometric Products, and Rotation operators**

For any vector \(a\), the product \(\hat{v}a\hat{v}\) is the reflection of \(a\) with respect to the direction \(\hat{v}\).

For any two vectors \(a\) and \(b\), \(\hat{v}ab\hat{v} = ba\), and \(vabv = v^2 ba\). Therefore, \(\hat{v}e^{\theta i} \hat{v} = e^{-\theta i}\), and \(ve^{\theta i}v = v^2 e^{-\theta i}\).

**A useful relationship that is valid only in plane geometry:** \(abc = cba\).

Here is a brief proof:

\[
abc = \{a \cdot b + a \land b\} c \\
= \{a \cdot b + [(ai) \cdot b] i\} c \\
= (a \cdot b) c + [(ai) \cdot b] ic \\
= c (a \cdot b) - c [(ai) \cdot b] i \\
= c (a \cdot b) + c [a \cdot (bi)] i \\
= c (b \cdot a) + c [(bi) \cdot a] i \\
= c \{b \cdot a + [(bi) \cdot a] i\} \\
= c \{b \cdot a + b \land a\} \\
= cba.
\]
1 Introduction

This document discusses reflections of vectors and of geometrical products of two vectors, in two-dimensional Geometric Algebra (GA). It then uses reflections to solve a simple tangency problem.

2 Reflections in 2D GA

2.1 Reflections of a single vector

For any two vectors \( \hat{u} \) and \( v \), the product \( \hat{u}v\hat{u} \) is

\[
\hat{u}v\hat{u} = \{2\hat{u} \wedge v + v\hat{u} \} \hat{u} \tag{2.1}
\]

\[
= v + 2 [(\hat{u}\cdot v) \hat{u}] \hat{u} \tag{2.2}
\]

\[
= v - 2 [v \cdot (\hat{u}\hat{u})] \hat{u}, \tag{2.3}
\]

which evaluates to the reflection of the reflection of \( v \) with respect to \( \hat{u} \) (Fig. 2.1).

![Figure 2.1: Geometric interpretation of \( \hat{u}v\hat{u} \), showing why it evaluates to the reflection of \( v \) with respect to \( \hat{u} \).](image)

We also note that because \( u = |u| \hat{u} \),

\[
uvu = u^2 (\hat{u}v\hat{u}) = u^2 v - 2 [v \cdot (u\hat{u})] u\hat{u}. \tag{2.4}\]
2.2 Reflections of a bivector, and of a geometric product of two vectors

The product $\hat{u}vw\hat{u}$ is

$$\hat{u}vw\hat{u} = \hat{u}(v \cdot w + v \wedge w)\hat{u}$$

$$= \hat{u}(v \cdot w)\hat{u} + \hat{u}(v \wedge w)\hat{u}$$

$$= \hat{u}^2(v \cdot w) + \hat{u}([-v \cdot (\hat{u}i)](-\hat{u}i))$$

$$= v \cdot w + \hat{u}^2[(\hat{u}i) \cdot v]i$$

$$= w \cdot v + w \wedge v$$

$$= vw.$$

In other words, the reflection of the geometric product $vw$ is $wv$, and does not depend on the direction of the vector with respect to which it is reflected. We saw that the scalar part of $vw$ was unaffected by the reflection, but the bivector part was reversed.

Further to that point, the reflection of geometric product of $v$ and $w$ is equal to the geometric product of the two vectors’ reflections:

$$\hat{u}vw\hat{u} = \hat{u}v(\hat{u}\hat{u})w\hat{u}$$

$$= (\hat{u}v\hat{u})(\hat{u}w\hat{u}).$$

That observation provides a geometric interpretation (Fig. 2.2) of why reflecting a bivector changes its sign: the direction of the turn from $v$ to $w$ reverses.

![Figure 2.2: Geometric interpretation of $\hat{u}vw\hat{u}$, showing why it evaluates to the reflection of $v$ with respect to $\hat{u}$. Note that $\hat{u}vw\hat{u} = \hat{u}v(\hat{u}\hat{u})w\hat{u} = (\hat{u}v\hat{u})(\hat{u}w\hat{u})$.](image)
3 Use of reflections to solve a simple tangency problem

The problem that we will solve is

“Given two coplanar circles, with a point $Q$ on one of them, construct the circles that are tangent to both of the given circles, with point $Q$ as one of the points of tangency” (Fig. 3.1).

Several solutions that use rotations are given by [1], but here we will use reflections. The triangle $TQC_3$ is isosceles, so $\hat{t}$ is the reflection of $\hat{w}$ with respect to the mediatrix of segment $QT$. In order to make use of that fact, we need to express the direction of that mediatrix as a vector written in terms of known quantities. We can do so by constructing another isosceles triangle $(C_1SC_3)$ that has the same mediatrix (Fig. 3.2).

The vector from $C_1$ to $S$ is $c_2 + (r_2 - r_1) \hat{w}$, so the direction of the mediatrix of $QT$ is the vector $[c_2 + (r_2 - r_1) \hat{w}] i$. The unit vector with that direction is $\frac{[c_2 + (r_2 - r_1) \hat{w}] i}{\|c_2 + (r_2 - r_1) \hat{w}\|}$. Therefore, to express $\hat{t}$ as the reflection of $\hat{w}$ with respect
We’ll use point $C_1$ as the origin.

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**Figure 3.2:** Adding segment $C_1S$ to Fig. 3.1 to produce a new isosceles triangle with the same mediatrix as $QT$.

to the mediatrix, we write

$$\hat{t} = \left[ \frac{c_2 + (r_2 - r_1) \hat{w}}{\|c_2 + (r_2 - r_1) \hat{w}\|} \right] \hat{w} \left[ \frac{c_2 + (r_2 - r_1) \hat{w}}{\|c_2 + (r_2 - r_1) \hat{w}\|} \right]$$

$$= \frac{[c_2 + (r_2 - r_1) \hat{w}] \hat{w}}{[c_2 + (r_2 - r_1) \hat{w}]^2} \left[ c_2 + (r_2 - r_1) \hat{w} \right] \hat{w} \left[ c_2 + (r_2 - r_1) \hat{w} \right] i i$$

from which

$$t \ (= r_1 \hat{t}) = -r_1 \left\{ \frac{[c_2 + (r_2 - r_1) \hat{w}] \hat{w}}{[c_2 + (r_2 - r_1) \hat{w}]^2} \right\} . \quad (3.1)$$

Interestingly, the geometric interpretation of that result is that $\hat{t}$ and $-\hat{w}$ are reflections of each other with respect to the vector $c_2 + (r_2 - r_1) \hat{w}$. After expanding and rearranging the numerator and denominator of (3.1), then using $w = r_2 \hat{w}$, we obtain

$$t = r_1 \left\{ \frac{c_2^2 - (r_2 - r_1)^2}{r_2c_2^2 + 2 (r_2 - r_1) c_2 \cdot w + r_2 (r_2 - r_1)^2} \right\} w - 2 \left[ c_2 \cdot w + r_2 (r_2 - r_1) \right] c_2 . \quad (3.2)$$

**References**
