Electrodynamics in Riemannian space with Torsion

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Abstract

Based on the fact that electromagnetic radiation has energy and momentum, and it creates curvature in the space time, we have used the covariant derivative of second rank tensor , thus we show that its possible to derive an explicit expression for Maxwell's equations in curved space time with and without torsion as well as—This is a coupling between gravity to electromagnetism. We show that the coupling introduced an extra amount of charge and current density–the electromagnetic and gravitoelectric–which resulting in non vanishing divergence of the magnetic filed tensor, this is equivalent to a magnetic monopole density. This is similar to the result that found by Piplowski, which states that such a coupling breaks the symmetry of $U(1)$ group, and has only significant at early time of the universe or inside black holes where the energy is very high.

1 Introduction

The electromagnetic force is one of the fundamental four forces in nature. It is governed by a set of equations, these are: the Gauss's law for electric and magnetic field, the Faraday's law, and the Ampere's law. In the year 1865 J.C. Maxwell published a revolutionary article on a dynamical theory of the electromagnetic field [1]. In that article he had amended Ampere's law by adding an extra term, which has led him to demonstrated that the electric and magnetic field satisfied the wave equation and propagate with velocity equals to the speed of light in space. Then he concluded that the light is a combination of a perpendicular components of electric and magnetic field that are perpendicular to direction of propagation of the electromagnetic waves. This is why the field equations of electromagnetic theory are called the Maxwell's equations.

Since then, the idea of finding an expression for Maxwell's equations in curved space with and without torsion has extensively been discussed in many articles and literature [2, 3, 4]. For instant, Nikodem and J. Poplawski [5] have considered a Lagrangian density for a free Maxwell field, where the electromagnetic field tensor minimally coupled to the affine connection. Then he derived a formula for the torsion and electromagnetic field tensors in terms of the electromagnetic potential. He has shown that the photon-torsion coupling acted like an effective magnetic monopole density. From his calculations it becomes clear that such a coupling breaks the symmetry of $U(1)$ group,

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and it has only significance inside the black holes or at early stage of our universe where the temperature was very high.

In addition to that, Fresneda [6] has coupled electromagnetism to torsion by proposing a generalized Abelian gauge invariance at the action level. In this approach, he has introduced a free parameter in such way that the new gauge is compatible with minimal coupling procedure, and is constrained by the available experimental data. This is an alternative model to HRRSs [8], where the gauge transformation has been modified by introducing a scalar field to the vector potential whose gradient gives the trace part of the torsion tensor. Further more, Redkov V.M [11] has triad a possibility of writing Maxwell's equation in a curved space-time by considering an effective media of which its properties are determined by metrical structure of the initial curved model. It was found that the metrical structure of the curved space-time generates the material equations for electromagnetic fields in terms of four symmetrical tensors. These tensors are explicitly written for a general case in arbitrary Riemann space-time geometry. Although their matrial equation has been tested for serveral geometries, still the covaraint derivative form of Maxwell's equations is replaced by the usual partial derivative, specially the Bianchi identity.

Based on the fact that the energy and momentum of electromagnetic radiation generates curvature on the space time, thus, the main objective of this paper is to use the covariant derivative of second rank tensor and derive an explicit formula for Maxwell's equations in curved space time with an without torsion, and see how the new form of electric and magnetic field vector.

The structure of this paper is as follows: we begin by introduction, in section two we represent the field equations, in section three we reproduce the Maxwell's equation in curved space time with and without torsion, in section four we consider some spacial cases. And we ends up with our concluding remarks.

2 Introduction

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3 The Field Equations

The action of electromagnetic field equations in curved space-time can be written in terms of the Lagrangian density as follows

$$S = \int \left(\frac{-1}{4\mu_0} F^{\mu\nu}\, F_{\mu\nu} - A_\mu J^\mu \right) \sqrt{-g}\; d^4x \ , \tag{1}$$

where

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \ , \tag{2}$$

and A_μ is the four vector potential. To obtain an expression for Maxwell's equations in curved space, we vary this action with respect to A_ν, to viz

$$\nabla_\nu F^{\mu\nu} = \mu_0 J^\mu \ , \tag{3}$$

The second pair of electromagnetic filed equation can be obtained directly from Bianchi identity

$$\nabla_\sigma F_{\mu\nu} + \nabla_\mu F_{\sigma\nu} + \nabla_\nu F_{\mu\sigma} = 0 \ . \tag{4}$$

Eqs (3) and (4) are the generalized Maxwell's equations. The question arised here is how does **B** and **E** field propagate when the geometry is no longer flat— Minkowski space. Recalling the covariant derivative of antisymmetric second rank tensor $F_{\mu\nu}$, (3) becomes

$$\nabla_\nu F^{\mu\nu} = \partial_\nu F^{\mu\nu} + \Gamma^\mu_{\nu\alpha} F^{\alpha\nu} + \Gamma^\nu_{\nu\alpha} F^{\mu\alpha} = \mu_0 J^\mu \ . \tag{5}$$

By introducing the daul field strength tensor $G^{\alpha\beta}$, which is related to the field tensor $F^{\alpha\beta}$ via

$$G^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \ . \tag{6}$$

Where

$$\varepsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{for even permutations} \\ 0 & \text{when any of the two indices ar equal } \textit{dimetion} \\ -1 & \text{for odd permutations} \end{cases} \tag{7}$$

Hence, an equivalent formula of (4) is obtained by introducing Levi-Civita symbol in four dimension as

$$\nabla_\nu G^{\mu\nu} = \partial_\nu G^{\mu\nu} + \Gamma^\mu{}_{\nu\alpha} G^{\alpha\nu} + \Gamma^\nu{}_{\nu\alpha} G^{\mu\alpha} = 0 \ . \tag{8}$$

4 Reproducing Maxwell's Equations

The first pair of Maxwell's equation can be obtain by setting $\mu = 0$ in (5). Since there are many terms, we extract each term individually, thus

$$\frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} = \frac{\partial F^{0j}}{\partial x^j} = \partial_j F^{0j} \ . \tag{9}$$

Where j runs over $1, 2, 3$. The second term also reads

$$\begin{aligned}
\Gamma^0{}_{\nu\alpha} F^{\alpha\nu} &= \Gamma^0{}_{0\alpha} F^{\alpha 0} + \Gamma^0{}_{j\alpha} F^{\alpha j} \ , \\
&= \Gamma^0{}_{00} F^{00} + \Gamma^0{}_{01} F^{10} + \Gamma^0{}_{02} F^{20} + \Gamma^0{}_{03} F^{30} + \Gamma^0{}_{j0} F^{0j} + \Gamma^0{}_{ji} F^{ij} \ , \\
&= \Gamma^0{}_{01} F^{10} + \Gamma^0{}_{02} F^{20} + \Gamma^0{}_{03} F^{30} + \Gamma^0{}_{10} F^{01} + \Gamma^0{}_{20} F^{02} + \Gamma^0{}_{30} F^{03} \\
&+ \Gamma^0{}_{12} F^{21} + \Gamma^0{}_{13} F^{31} + \Gamma^0{}_{21} F^{12} + \Gamma^0{}_{23} F^{32} + \Gamma^0{}_{31} F^{13} + \Gamma^0{}_{32} F^{23} \ . \\
\Gamma^0{}_{\nu\alpha} F^{\alpha\nu} &= \left(\Gamma^0{}_{01} - \Gamma^0{}_{10}\right) F^{10} + \left(\Gamma^0{}_{02} - \Gamma^0{}_{20}\right) F^{20} + \left(\Gamma^0{}_{03} - \Gamma^0{}_{30}\right) F^{30} \\
&+ \left(\Gamma^0{}_{12} - \Gamma^0{}_{21}\right) F^{12} + \left(\Gamma^0{}_{31} - \Gamma^0{}_{13}\right) F^{13} + \left(\Gamma^0{}_{32} - \Gamma^0{}_{23}\right) F^{23} \ , \\
&= T^0{}_{10} F^{10} + T^0{}_{20} F^{20} + T^0{}_{30} F^{30} + T^0{}_{12} F^{12} + T^0{}_{31} F^{13} + T^0{}_{32} F^{23} \ , \\
&= T^0{}_{j0} F^{j0} + \Gamma^0{}_{ij} F^{ij} \ , \\
&= T^0{}_{j0} F^{j0} + T^0{}_{ij} F^{ij} \ .
\end{aligned} \tag{10}$$

Where T^0_{j0} represents the torsion of the space, and in the the last step, the antisymmetric property of $F^{\mu\nu}$ has been also used. Finally, the third gives

$$\begin{aligned}
\Gamma^\nu{}_{\nu\alpha} F^{0\alpha} &= \Gamma^\nu{}_{\nu 0} F^{00} + \Gamma^\nu{}_{\nu j} F^{0j} = \Gamma^0{}_{0j} F^{0j} + \Gamma^1{}_{1j} F^{0j} + \Gamma^2{}_{2j} F^{0j} + \Gamma^3{}_{3j} F^{0j} \ , \\
&= \frac{1}{2\, g_{\nu\lambda}} \frac{\partial g_{\nu\lambda}}{\partial x^j} F^{0j} \ , \\
&= \frac{\partial \left(\ln \sqrt{|-g|}\right)}{\partial x^j} F^{0j} \ , \\
&= \partial_j \left(\ln \sqrt{|-g|}\right) F^{0j} \ .
\end{aligned} \tag{11}$$

Where $g = \det(g_{\mu\nu})$. Now, adding equations (9), (10), and (11) and collect the zero component of the current density J^μ, we obtain

$$\boxed{\partial_j F^{0j} + \left\{T^0{}_{j0} + \partial_j \left(\ln \sqrt{|-g|}\right)\right\} F^{0j} + T^0{}_{ij} F^{ij} = \frac{1}{\epsilon_0}\rho \ ,} \tag{12}$$

as the generalized Gauss's law. The additional term that appears on the L.H.S of this equation represent the coupling between electromagnetism and curvature plus torsion that is linearly proportional to the electric field vector. In other words, the effect of the curvature and torsion mimics the effect of an external electromagnetic field upon dielectric martial. Also the torsion components on hyper surface of constant time produce kind of a gravito-magnetic filed that is similar to a circulating magnetic field around each point in a long wire.

Similarly, the second Maxwell's equation can be achieved by putting $\mu = k = 1, 2, 3$ in the same equation, thus

$$
\begin{aligned}
\partial_\nu F^{k\nu} &= \frac{\partial F^{k0}}{\partial x^0} + \frac{\partial F^{k1}}{\partial x^1} + \frac{\partial F^{k2}}{\partial x^2} + \frac{\partial F^{k3}}{\partial x^3} = \frac{\partial F^{k0}}{\partial x^0} + \frac{\partial F^{ki}}{\partial x^i} \\
&= \frac{\partial F^{k0}}{\partial t} + \frac{\partial F^{kj}}{\partial x^j} \ , \\
&= \partial_0 F^{k0} + \partial_j F^{kj} \ ,
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
\Gamma^k{}_{\nu\alpha}\, F^{\alpha\nu} &= \Gamma^k{}_{0\alpha}\, F^{\alpha 0} + \Gamma^k{}_{j\alpha}\, F^{\alpha j} \ , \\
&= \Gamma^k{}_{00}\, F^{00} + \Gamma^k{}_{01}\, F^{10} + \Gamma^k{}_{02}\, F^{20} + \Gamma^k{}_{03}\, F^{30} + \Gamma^k{}_{j0}\, F^{0j} + \Gamma^k{}_{ji}\, F^{ij} \ , \\
&= \Gamma^k{}_{01}\, F^{10} + \Gamma^k{}_{02}\, F^{20} + \Gamma^k{}_{03}\, F^{30} + \Gamma^k{}_{10}\, F^{01} + \Gamma^k{}_{20}\, F^{02} + \Gamma^k{}_{30}\, F^{03} \\
&+ \Gamma^k{}_{12}\, F^{21} + \Gamma^k{}_{13}\, F^{31} + \Gamma^k{}_{21}\, F^{12} + \Gamma^k{}_{23}\, F^{32} + \Gamma^k{}_{31}\, F^{13} + \Gamma^k{}_{32}\, F^{23} \ , \\
\Gamma^k{}_{\nu\alpha}\, F^{\alpha\nu} &= \left(\Gamma^k{}_{01} - \Gamma^k{}_{10}\right) F^{10} + \left(\Gamma^k{}_{02} - \Gamma^k{}_{20}\right) F^{20} + \left(\Gamma^k{}_{03} - \Gamma^k{}_{30}\right) F^{30} \\
&+ \left(\Gamma^k{}_{12} - \Gamma^k{}_{21}\right) F^{12} + \left(\Gamma^k{}_{31} - \Gamma^k{}_{13}\right) F^{13} + \left(\Gamma^k{}_{32} - \Gamma^k{}_{23}\right) F^{23} \ , \\
&= T^k{}_{10} F^{10} + T^k{}_{02} F^{20} + T^k{}_{03} F^{30} + T^k{}_{21} F^{12} + T^k{}_{13} F^{13} + T^k{}_{32} F^{23} \ , \\
&= T^k{}_{j0} F^{j0} + \Gamma^k{}_{ij} F^{ij} = T^k{}_{j0} F^{j0} + \frac{1}{2}\left(\Gamma^k{}_{ij} - \Gamma^k{}_{ji}\right) F^{ij} \ , \\
&= T^k{}_{j0} F^{j0} + \frac{1}{2}\, T^k{}_{ij}\, F^{ij} \ .
\end{aligned} \tag{14}
$$

the last term produces

$$
\begin{aligned}
\Gamma^\nu{}_{\nu\alpha}\, F^{1\alpha} &= \Gamma^\nu{}_{\nu 0}\, F^{10} + \Gamma^\nu{}_{\nu 1}\, F^{11} + \Gamma^\nu{}_{\nu 2}\, F^{12} + \Gamma^\nu{}_{\nu 1}\, F^{13} \\
&= \frac{\partial \ln\sqrt{|-g|}}{\partial x^0}\, F^{10} + \frac{\partial \ln\sqrt{|-g|}}{\partial x^1}\, F^{11} + \left(\frac{\partial \ln\sqrt{|-g|}}{\partial x^2}\, F^{12} + \frac{\ln\sqrt{|-g|}}{\partial x^3}\, F^{13}\right) \ , \\
&= \partial_\alpha\left(\ln\sqrt{|-g|}\right)\, F^{1\alpha} \ .
\end{aligned} \tag{15}
$$

Similarly, the corresponding result for $\mu = 2$ and $\mu = 3$ are

$$
\begin{aligned}
\Gamma^\nu{}_{\nu\alpha}\, F^{2\alpha} &= \frac{\partial \ln\sqrt{|-g|}}{\partial x^0}\, F^{20} + \frac{\partial \ln\sqrt{|-g|}}{\partial x^2}\, F^{22} + \left(\frac{\partial \ln\sqrt{|-g|}}{\partial x^1}\, F^{21} + \frac{\ln\sqrt{|-g|}}{\partial x^3}\, F^{23}\right) \ , \\
&= \partial_\alpha\left(\ln\sqrt{|-g|}\right)\, F^{2\alpha} \ .
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\Gamma^\nu{}_{\nu\alpha}\, F^{3\alpha} &= \frac{\partial \ln\sqrt{|-g|}}{\partial x^0}\, F^{30} + \frac{\partial \ln\sqrt{|-g|}}{\partial x^3}\, F^{33} + \left(\frac{\partial \ln\sqrt{|-g|}}{\partial x^1}\, F^{31} + \frac{\ln\sqrt{|-g|}}{\partial x^2}\, F^{32}\right) \ , \\
&= \partial_\alpha\left(\ln\sqrt{|-g|}\right)\, F^{3\alpha} \ .
\end{aligned} \tag{17}
$$

Now, combining all these terms together, we get

$$
\Gamma^\nu{}_{\nu\alpha}\, F^{k\alpha} = \partial_\alpha\left(\ln\sqrt{|-g|}\right)\, F^{k\alpha} \ ,
$$

$$= \partial_0 \left(\ln \sqrt{|-g|} \right) F^{k0} + \partial_j \left(\ln \sqrt{|-g|} \right) F^{kj} \ , \tag{18}$$

Adding equation (13), (14) & (18) with k component of J^μ "the spatial", we obtain the generalized Ampere's Maxwell law, as

$$\boxed{\partial_j F^{kj} + \partial_0 F^{k0} + T^k{}_{j0} F^{j0} + \frac{1}{2}\, T^k{}_{ij}\, F^{ij} + \partial_0 \left(\ln \sqrt{|-g|} \right) F^{k0} + \partial_j \left(\ln \sqrt{|-g|} \right) F^{kj} = \mu_0 \mathbf{J}\ .} \tag{19}$$

From (8) gives rise to the second pair of Maxwell's equations. Following the same approach as before, the generalized Gauss's law for magnetic field can be calculated by setting $\mu = 0$ in (8), as

$$\frac{\partial G^{0\nu}}{\partial x^\nu} = \frac{\partial G^{00}}{\partial t} + \frac{\partial G^{0j}}{\partial x^j} = \partial_j G^{0j} \ , \tag{20}$$

and

$$\begin{aligned}
\Gamma^0{}_{\nu\alpha}\, G^{\alpha\nu} &= \Gamma^0{}_{0\alpha}\, G^{\alpha 0} + \Gamma^0{}_{j\alpha}\, G^{\alpha j} \ , \\
&= \Gamma^0{}_{00}\, G^{00} + \Gamma^0{}_{01}\, G^{10} + \Gamma^0{}_{02}\, G^{20} + \Gamma^0{}_{03}\, G^{30} + \Gamma^0{}_{j0}\, G^{0j} + \Gamma^0{}_{ji}\, G^{ij} \ , \\
&= \Gamma^0{}_{01}\, G^{10} + \Gamma^0{}_{02}\, G^{20} + \Gamma^0{}_{03}\, G^{30} + \Gamma^0{}_{10}\, G^{01} + \Gamma^0{}_{20}\, G^{02} + \Gamma^0{}_{30}\, G^{03} \\
\ln \sqrt{|-g|} &+ \Gamma^0{}_{12}\, G^{21} + \Gamma^0{}_{13}\, G^{31} + \Gamma^0{}_{21}\, G^{12} + \Gamma^0{}_{23}\, G^{32} + \Gamma^0{}_{31}\, G^{13} + \Gamma^0{}_{32}\, G^{23} \ . \\
\Gamma^0{}_{\nu\alpha}\, G^{\alpha\nu} &= \left(\Gamma^0{}_{01} - \Gamma^0{}_{10} \right) G^{10} + \left(\Gamma^0{}_{02} - \Gamma^0{}_{20} \right) G^{20} + \left(\Gamma^0{}_{03} - \Gamma^0{}_{30} \right) G^{30} \\
&+ \left(\Gamma^0{}_{12} - \Gamma^0{}_{21} \right) G^{12} + \left(\Gamma^0{}_{31} - \Gamma^0{}_{13} \right) G^{13} + \left(\Gamma^0{}_{32} - \Gamma^0{}_{23} \right) G^{23} \ , \\
&= T^0{}_{10} G^{10} + T^0{}_{20} G^{20} + T^0{}_{30} G^{30} + T^0{}_{12} G^{12} + T^0{}_{31} G^{13} + T^0{}_{32} G^{23} \ , \\
&= T^0{}_{j0} G^{j0} + \Gamma^0{}_{ij} G^{ij} \\
&= T^0{}_{j0} G^{j0} + T^0{}_{ij} G^{ij} \ ,
\end{aligned} \tag{21}$$

and the third term produces

$$\Gamma^\nu{}_{\nu\alpha}\, G^{0\alpha} = \Gamma^\nu{}_{\nu 0}\, G^{00} + \Gamma^\nu{}_{\nu j}\, G^{0j} = \frac{\partial \left(\ln \sqrt{|-g|} \right)}{\partial x^j}\, G^{0j} = \partial_j \left(\ln \sqrt{|-g|} \right) G^{0j} \ . \tag{22}$$

Thus, from (20), (21) & (22), we get

$$\boxed{\partial_j G^{0j} + \left\{ T^0{}_{j0} + \partial_j \left(\ln \sqrt{|-g|} \right) \right\} G^{0j} + T^0{}_{ij} G^{ij} = 0\ .} \tag{23}$$

We see that the divergence of G^{0j} is not zero. This will be discussed later. The generalized Faraday's law, can be obtain by substitute $\mu = k = 1, 2, 3$ in (8), as

$$\frac{\partial G^{k\nu}}{\partial x^\nu} = \frac{\partial G^{k0}}{\partial x^0} + \frac{\partial G^{kj}}{\partial x^j} = \partial_0 G^{k0} + \partial_j G^{kj} \ . \tag{24}$$

while setting $\mu = k = 1, 2, 3$ in the second term gives

$$\begin{aligned}
\Gamma^k{}_{\nu\alpha}\, G^{\alpha\nu} &= \Gamma^k{}_{0\alpha}\, G^{\alpha 0} + \Gamma^k{}_{j\alpha}\, G^{\alpha j} = \Gamma^k{}_{0j}\, G^{j0} + \Gamma^k{}_{j0}\, G^{0j} + \Gamma^k{}_{ij}\, G^{ji} \ , \\
&= T^k{}_{j0} G^{j0} + \frac{1}{2} T^k{}_{ij} G^{ij} \ ,
\end{aligned} \tag{25}$$

Similary, the last term reads

$$\Gamma^\nu{}_{\nu\alpha}\, G^{1\alpha} = \frac{\partial \ln \sqrt{|-g|}}{\partial x^0}\, G^{10} + \frac{\partial \ln \sqrt{|-g|}}{\partial x^1}\, G^{11} + \left(\frac{\partial \ln \sqrt{|-g|}}{\partial x^2}\, G^{12} + \frac{\ln \sqrt{|-g|}}{\partial x^1}\, G^{13} \right) ,$$

$$= \partial_\alpha \left(\ln\sqrt{|-g|}\right) G^{1\alpha} \ . \tag{26}$$

$$\Gamma^\nu{}_{\nu\alpha} G^{2\alpha} = \frac{\partial \ln\sqrt{|-g|}}{\partial x^0} G^{20} + \frac{\partial \ln\sqrt{|-g|}}{\partial x^2} G^{22} + \left(\frac{\partial \ln\sqrt{|-g|}}{\partial x^1} G^{21} + \frac{\ln\sqrt{|-g|}}{\partial x^3} G^{23}\right) ,$$

$$= \partial_\alpha \left(\ln\sqrt{|-g|}\right) G^{2\alpha} \ . \tag{27}$$

$$\Gamma^\nu{}_{\nu\alpha} G^{3\alpha} = \frac{\partial \ln\sqrt{|-g|}}{\partial x^0} G^{30} + \frac{\partial \ln\sqrt{|-g|}}{\partial x^3} G^{33} + \left(\frac{\partial \ln\sqrt{|-g|}}{\partial x^1} G^{31} + \frac{\ln\sqrt{|-g|}}{\partial x^2} G^{32}\right) ,$$

$$= \partial_\alpha \left(\ln\sqrt{|-g|}\right) G^{3\alpha} \ . \tag{28}$$

Which is equivalent to

$$\Gamma^\nu{}_{\nu\alpha} G^{k\alpha} = \partial_\alpha \left(\ln\sqrt{|-g|}\right) G^{k\alpha} = \partial_0 \left(\ln\sqrt{|-g|}\right) G^{k0} + \partial_j \left(\ln\sqrt{|-g|}\right) G^{kj} \ . \tag{29}$$

Finally, from (24), (25) and (29), we get the Ampere's Maxwell equation as:

$$\boxed{\partial_0 G^{k0} + \partial_j G^{kj} + T^k{}_{j0} G^{j0} + \frac{1}{2} T^k{}_{ij} G^{ij} + \partial_0 \left(\ln\sqrt{|-g|}\right) G^{k0} + \partial_j \left(\ln\sqrt{|-g|}\right) G^{kj} = 0 \ .} \tag{30}$$

Equation (12), (19), (23) & (30) are the generalized set of Maxwell's equations in curved space-time with torsion.

4.1 Torsion Free Space

In the case when the space has no torsion, the above set of Maxwell's equations in §(4) will be reduced to the following:

$$\partial_j F^{0j} + \partial_j \varphi \, F^{0j} = \frac{1}{\epsilon_0} \rho \ , \qquad \partial_j F^{kj} + \partial_j \varphi \ F^{kj} + \left(\partial_0 \varphi \, F^{k0} + \partial_0 F^{k0}\right) = \mu_0 \mathbf{J} \ . \tag{31}$$

$$\partial_j G^{0j} + \partial_j \varphi \, G^{0j} = 0 \ , \qquad \partial_j G^{kj} + \partial_j \varphi G^{kj} = -\left(\partial_0 \varphi \, G^{k0} + \partial_0 G^{k0}\right) \ . \tag{32}$$

where we have used

$$\dot{\varphi} = \partial_0 \left(\ln\sqrt{|-g|}\right) \ , \quad \nabla\varphi = \partial_j \left(\ln\sqrt{|-g|}\right) \ , \quad \text{and} \quad \partial_j(AB) = B\partial_j A + A\partial_j B \ . \tag{33}$$

Notes that ∇ here stand for ∂_j not the covariant derivative, and a letter with dot stand for derivative with respect to time. With these notation in mind, we may also use the following defintions

$$\begin{aligned} D^{k0} &= F^{k0} + \varphi F^{k0} = \chi_\varphi F^{k0} \ , \qquad H^{kj} = F^{kj} + \varphi \, F^{kj} = \chi_\varphi F^{kj} \ , \qquad D^{kj} = G^{kj} + \varphi \, G^{kj} \ , \\ H^{0j} &= G^{0j} + \varphi \, G^{0j} \ , \qquad \mathbf{J}_g = \varphi \left(\partial_j F^{kj} + \partial_0 F^{k0}\right) \ , \qquad \widetilde{\mathbf{J}} = \frac{\varphi}{\mu_0} \mathbf{J}_g + \mathbf{J} \ , \\ \rho_\varphi &= \varphi \, \partial_j F^{0j} \ , \qquad \widetilde{\rho} = \epsilon_0 \rho_\varphi + \rho \ , \qquad \widetilde{\rho}_m = \varphi \, \partial_j G^{0j} \ , \end{aligned} \tag{34}$$

It is clear that the coupling between gravity and electromagnetic generates a kind of gravito-electric susceptibility χ_φ. This is similar to the electric susceptibility of materials, D^{0j} and H^{kj}

are the gravito-electric displacement field and gravito-magnetic field strength respectively. Now, equations (31)-(32) can be simplified to

$$\begin{aligned}
\partial_j D^{0j} &= \frac{\widetilde{\rho}}{\epsilon_0} \ . \quad \text{Gauss's law} && (35)\\
\partial_j H^{0j} &= \widetilde{\rho}_m \ . \quad \text{Gauss's law for magnetism} && (36)\\
\partial_j H^{kj} &= -\partial_0 D^{k0} + \mu_0 \widetilde{\mathbf{J}} \ . \quad \text{Ampere's law} && (37)\\
\partial_j D^{kj} &= -\partial_0 H^{k0} + \mathbf{J}_g \ . \quad \text{Faraday's law} && (38)
\end{aligned}$$

These equations represent the generalized Maxwell's equations in curved space with zero torsion. The coupling between gravity and electromagnetism generates an extra charge and current densities, these are; ρ_φ, ρ_m and $\mathbf{J}_g$. The ρ_m in Gauss's law for magnetism is kind of magnetic monopole density, which leads to a non-vanishing divergence of G^{0j}, in other words, it breaks the symmetry of $U(1)$ group, and it may be exist inside black holes or at early time stage of the universe where the energy is extremly very high. As it was mentioned in [5], but in a diffent approch, such kind of ρ_m may provide a way for studying Dirac's quantization of electric charge [7].

5 Special Cases:

In order to produce some special cases from the above set of equations–The generalized Maxwell's equations, first, we may set $E^j = F^{0j}$, $B_j = \frac{1}{2}\varepsilon_{jkl}F^{kl}$, $G^{0j} = \frac{1}{2}\varepsilon^{0jkl}F_{kl} = B^j$, and $G^{ij} = \frac{1}{2}\left(\varepsilon^{ijk0}F_{k0} + \varepsilon^{ij0k}F_{0k}\right) = \varepsilon^{ijk}F_{k0} = -\varepsilon^{ijk}E_k$, then make a propraite choice of $g_{\nu\nu}$ in each case. These are:

1. If $g_{\mu\nu} = \text{diag}(1,-1,-1,-1) \quad \rightarrow \quad \sqrt{|-g|} = 1 \quad \rightarrow \quad \ln\sqrt{|-g|} = 0$, we obtain the ordinary Maxwell's equations (i.e. the electrodynamics in Minkawiki space).

$$\begin{aligned}
\nabla_j E^j &= \frac{J^0}{\epsilon_0} \ . && (39)\\
\nabla_j \ B^j &= 0 \ . && (40)\\
\varepsilon^{jkl} \ \nabla_k E_l &= -\partial_0 B^j \ . && (41)\\
\varepsilon^{jkl} \ \nabla_k B_l &= \partial_0 E^j + \mu_0 J^j \ . && (42)
\end{aligned}$$

2. Choosing $g_{\mu\nu} = \text{diag}\left(1, -e^{\theta(t,r)}, -e^{\theta(t,r)}, -e^{\theta(t,r)}\right) \quad \rightarrow \quad \sqrt{|-g|} = e^{3\theta(t,r)} \quad \rightarrow \quad \ln\sqrt{|-g|} = \frac{3}{2}\theta(t,r)$, thia will produces

$$\begin{aligned}
\nabla_j E^j &= -\frac{3}{2}\ \nabla_j\theta\ E^j + \frac{J^0}{\epsilon_0} \ . && (43)\\
\nabla_j \ B^j &= -\frac{3}{2}\ \nabla_j\theta\ B^j \ . && (44)\\
\varepsilon^{jkl}\ \nabla_k E_l &= -\partial_0 B^j - \frac{3}{2}\ \partial_0\theta\ B^j - \frac{3}{2}\ \varepsilon^{jkl}\nabla_k\theta\ E_l \ . && (45)\\
\varepsilon^{jkl}\ \nabla_k B_l &= \partial_0 E^j - \frac{3}{2}\ \partial_0\theta\ E^j - \frac{3}{2}\ \varepsilon^{jkl}\nabla_k\theta\ B_l + \mu_0 J^j \ . && (46)
\end{aligned}$$

Equations (43) and (46) are similar to the equations of the Axion electrodynamics which was proposed by Frank Wilczek [9] to solve the dark matter. In this theory, Maxwell's equations are coupled with Axions, and under certain metric can be put in a form similar to Axions electrodynamics [9]. In such way it may indicate that the Axion interact gravitationally as well as electromagnetically.

3. If $g_{\mu\nu} = \text{diag}\left(1, -e^{\theta(t)}, -e^{\theta(t)}, -e^{\theta(t)}\right) \quad \rightarrow \quad \sqrt{|-g|} = e^{3\theta(t)} \quad \rightarrow \quad \ln\sqrt{|-g|} = \frac{3}{2}\theta(t)$, then we get

$$\nabla_j E^j = \frac{J^0}{\epsilon_0} \ . \tag{47}$$

$$\nabla_j \ B^j = 0 \ . \tag{48}$$

$$\varepsilon^{jkl} \ \nabla_k E_l = -\partial_0 B^j - \frac{3}{2} \ \partial_0 \theta \ B^j \ . \tag{49}$$

$$\varepsilon^{jkl} \ \nabla_k B_l = \partial_0 E^j - \frac{3}{2} \ \partial_0 \theta \ E^j + \mu_0 J^j \ . \tag{50}$$

These equations are of the same type as those studied in [10] that describe the generation and evolution of the cosmological magnetic elds in inationary universe.

4. Finally, choosing $g_{\mu\nu} = \text{diag}\left(1, -e^{\theta(r)}, -e^{\theta(r)}, -e^{\theta(r)}\right) \quad \rightarrow \quad \sqrt{|-g|} = e^{3\theta(r)} \quad \rightarrow \quad \ln\sqrt{|-g|} = \frac{3}{2}\theta(r)$, this leads to

$$\nabla_j E^j = -\frac{3}{2} \ \nabla_j \theta \ E^j + \frac{J^0}{\epsilon_0} \ . \tag{51}$$

$$\nabla_j \ B^j = -\frac{3}{2} \ \nabla_j \theta \ B^j \ . \tag{52}$$

$$\varepsilon^{jkl} \ \nabla_k E_l = -\partial_0 B^j - \frac{3}{2} \ \varepsilon^{jkl} \nabla_k \theta \ E_l \ . \tag{53}$$

$$\varepsilon^{jkl} \ \nabla_k B_l = \partial_0 E^j - \frac{3}{2} \ \varepsilon^{jkl} \nabla_k \theta \ B_l + \mu_0 J^j \ . \tag{54}$$

These equations describe the evolution of electromagnetic filed in static space that has a geometry similar to Schwarzschild.

6 Concluding remarks

We have studied in this work the Maxwell's equations in Riemannian space-time with torsion. A particular metric parameterized by the metric determinant g was considered. The ordinary Maxwell's equations are restored when g is constant "flat". Space and time variation of g give rise to magnetic charge and current densities as well as electric charge density. The resulting Maxwell's equations bear some similarities with the axion electrodynamics studied by Frank Wilczek.

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References

[1] J. C. Maxwell, *Phil. Trans. R. Soc. Lond.* **155** (1865) 459-512.

[2] G. F. Rubilar, Y. N. Obukhov and F. W. Hehl, *Class. Quant. Grav.* **20** (2003) L185 .

[3] Y. Itin and F. W. Hehl, *Phys. Rev. D.* **68** (2003) 127701 .

[4] M. Blagojevic and B. Cvetkovic, *Phys. Rev. D.* **78** (2008) 044037 .

[5] N. J. Poplawski, arXiv:1108.6100 [gr-qc].

[6] R. Fresneda, M. C. Baldiotti and T. S. Pereira, *Braz. J. Phys.* **45** (2015) 353 .

[7] P. A. M. Dirac, *Proc. Roy. Soc.* **A 133** (1931) 60

[8] C Mukku and W.A. Sayed, $\widehat{\mathrm{P}}$hys. Let. B. **82(3-4)** (1979) 382386

[9] Frank Wilczek, *Phys. Rev. Lett.* **58** (1987) 1799

[10] Z. x. Dai, gr-qc/0309113.

[11] Redkov V.M. and Tokarevskaya N.G., E.M. Bychkouskaya, *NPCS.* **12** (2009) 232–250