Tidal Asymmetry

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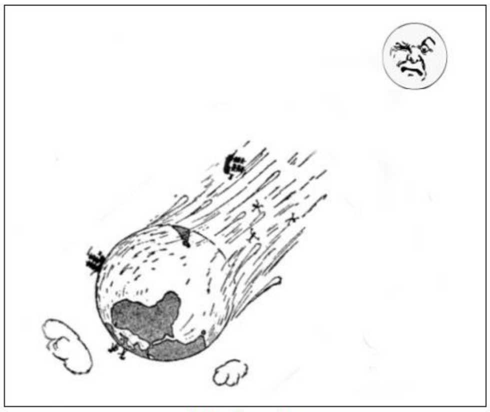
The Earth's diametrically opposed, presumably symmetric, tides are due to the Moon's differential gravitational force varying across the Earth. This is not intuitively obvious, but becomes clear when the physics is examined mathematically. The presumed symmetry is due to an approximation that holds when the radius of the affected body (e.g., The Earth) is much less than its center-to-center distance from the affecting body (e.g., the Moon). The exact solution indicates an asymmetry, which becomes more pronounced as the assumption loses its applicability.

Keywords: Earth-Moon, tides, differential gravity, barycentric rotation, ring-spring

1. Introduction

Explaining why the Earth experiences height tides (or low tides) simultaneously on opposite hemispheres is not intuitively obvious. If due to the gravitational force of the Moon (and, to a lesser extent, that of the Sun),[1] one might expect there to be a tidal bulge solely on the 'near' hemisphere (i.e., the one closer to the Moon), as illustrated in Figure 1 [2].



Tidal Catastrophe.

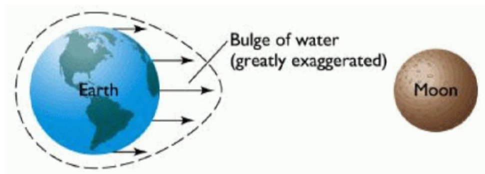


Figure 1. Tidal Misconceptions [2]

This is clearly not observed. Most websites that explain the tides follow the following logic or something similar [1].

"The tidal force is a secondary effect of the force of gravity and is responsible for the tides. It arises because the gravitational force exerted by one body on another is not constant across it; the nearest side is attracted more strongly than the farther side. Thus, the tidal force is differential ... For a given (externally generated) gravitational field, the tidal acceleration at a point with respect to a body is obtained by vectorially subtracting the gravitational acceleration at the center of the body (due to the given externally generated field) from the gravitational acceleration (due to the same field) at the given point. Correspondingly, the term tidal force is used to describe the forces due to tidal acceleration. Note that for these purposes the only gravitational field considered is the external one; the gravitational field of the body is not relevant ...

"By Newton's law of universal gravitation and laws of motion, a body of mass M [i.e., the Earth] at a distance D from the center of a sphere of mass m [i.e., the Moon] feels a force $F = -GMm/D^2$ equivalent to an acceleration $A = -Gm/D^2$ [along] a unit vector pointing from the body m to the body M ... Consider now the acceleration due to the sphere of mass m experienced by a particle in the vicinity of the body of mass M. With D as the distance from the center of m to the center of M, let R be the (relatively small) distance of the particle from the center of the body of mass M. For simplicity, distance are ... considered only in the direction pointing towards or away from the sphere of mass m.

"If the body of mass M is itself a sphere of radius R, then the new particle considered may be located on its surface, at a distance $D \pm R$ from the center of the sphere of mass m, and R may be taken as positive where the par-

[1] Only the Moon's effect is examined in this paper. It has been estimated to be approximately twice that of the Sun [1].

ticle's distance from m is greater than R. Leaving aside whatever gravitational acceleration may be experienced by the particle towards M on account of M's own mass, we have the acceleration on the particle due to gravitational force towards m as $A = -\frac{Gm}{(D \pm R)^2}$. Pulling out the D^2 term from the denominator gives $A = -\frac{GmD^2}{(1 \pm R/D)^2}$, ... [which expands, via the Maclaurin series, into] ... $A = GmD^2 + (2GM/D^2)(R/D) + ...$

"The first term is the gravitational acceleration due to m at the center of the reference body M, i.e., at the point where R is zero [i.e., Earth's center]. This term does not affect the observed acceleration of particles on the surface of M because with respect to m, M (and everything on its surface) is in free fall. When the force on the far particle is subtracted from the force on the near particle, this first term cancels, as do all other even-order terms. The remaining (residual) terms represent the difference mentioned above and are tidal force (acceleration) terms. When R is small compared to D, the terms after the first residual term are very small and can be neglected, giving the approximate tidal acceleration (axial) for the distances R considered, along the axis joining the centers of M and m [as] $A \approx \pm 2GMR/D^3$."

We see equal magnitude accelerations for the maximum tides, implying symmetry. Additional websites that explain the ocean tides often cite the hemispherical opposites as symmetric based on polynomial expansions and neglecting higher-order terms beyond the second power, e.g., "The tide generating force can be decomposed into components perpendicular and parallel to the sea surface. The tides are produced by the horizontal components ... The tidal potential is symmetric about the Earth-moon line, and it produces symmetric bulges [3]." This conclusion implicitly assumes that the ratio between the radius of the affect<u>ed</u> body and its center-to-center distance from the affect<u>ing</u> body is « 1. A common illustration is shown in Figure 2.

2. Tidal Asymmetry?

The goal here is to show that, using the exact, vs. the asymptotic, solution to the differential force between the Moon's gravitational pull at the Earth's surface vs. at its center, an asymmetry between the tides will result for equal angles θ on the Earth's far and near hemispheres. This asymmetry will exist for both the magnitude of the differential force (Δg) and the angle (β). Figure 3 provides the geometry for the comparison. Note that, for the near hemisphere, the Moon's gravitational force at the surface is almost always greater than that at the Earth's center,[2] as indicated by the first forces triangle

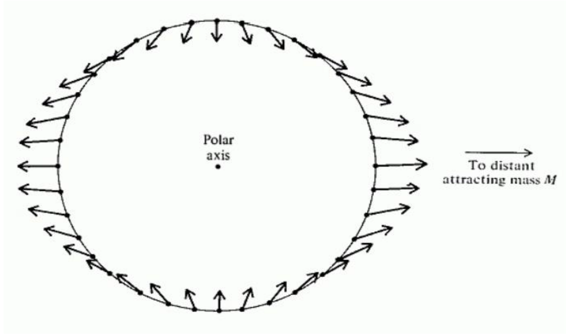


Figure 2. Effect of Differential (Tidal) Forces [4]

for the near hemisphere. The opposite holds exclusively for the far hemisphere, where the Moon's gravitational force at the Earth's center is always greater than at the surface, as indicated by the second forces triangle for the far hemisphere. Calculations for the various parameters are as follows:

$$d_n = \sqrt{(D - Rcos\theta_n)^2 + (Rsin\theta_n)^2} = \sqrt{D^2 - 2DRcos\theta_n + R^2}$$

$$d_f = \sqrt{(D + Rcos\theta_f)^2 + (Rsin\theta_f)^2} = \sqrt{D^2 + 2DRcos\theta_f + R^2}$$

Assuming, for convenience, that G (gravitational constant) and m (Moon's mass) are both unity, $g_{mn} = 1/d_n^2$, $g_{mf} = 1/d_f^2$, $g_c = 1/D^2$. In addition, the Moon's gravitational force is assumed to act on a unit mass of $1kg$ of ocean water on the Earth's surface, so that the force equations developed below can be viewed as characterizing the force per unit of affected mass, essentially an acceleration. therefore, the differential forces between the Moon's gravitational pull at the Earth's surface and at the Earth's center are as follows:

$$\Delta g_n = \sqrt{g_{m,n}^2 + g_c^2 - 2g_{m,n}g_c cos\phi_n}$$

$$\Delta g_f = \sqrt{g_{m,f}^2 + g_c^2 - 2g_{m,f}g_c cos\phi_f}$$

$$cos\phi_n = (D - Rcos\theta_n)/d_n$$

$$cos\phi_f = (D + Rcos\theta_f)/d_f$$

[2] As θ_n approaches 90^o, β_n reaches a maximum then starts to decrease, with the angle at which the maximum occurs being closer to 90^o as R/D decreases. This will be shown later via plots of the differences between the Δg forces for corresponding angles θ on the near and far hemispheres.

Moon is distance D from Earth (center-to-center). Earth radius is R. Define θ_f and θ_n as equal angles on f(ar) and n(ear) hemispheres when looking down from the North Pole (or up from the South), forming distances d_f and d_n between Moon's center and points on Earth's surface at which Moon's gravitational effect on the tides is calculated. These create two triangles with corresponding angles ϕ_f and ϕ_n between the Moon's gravitational force at the surface points and the Earth's center.

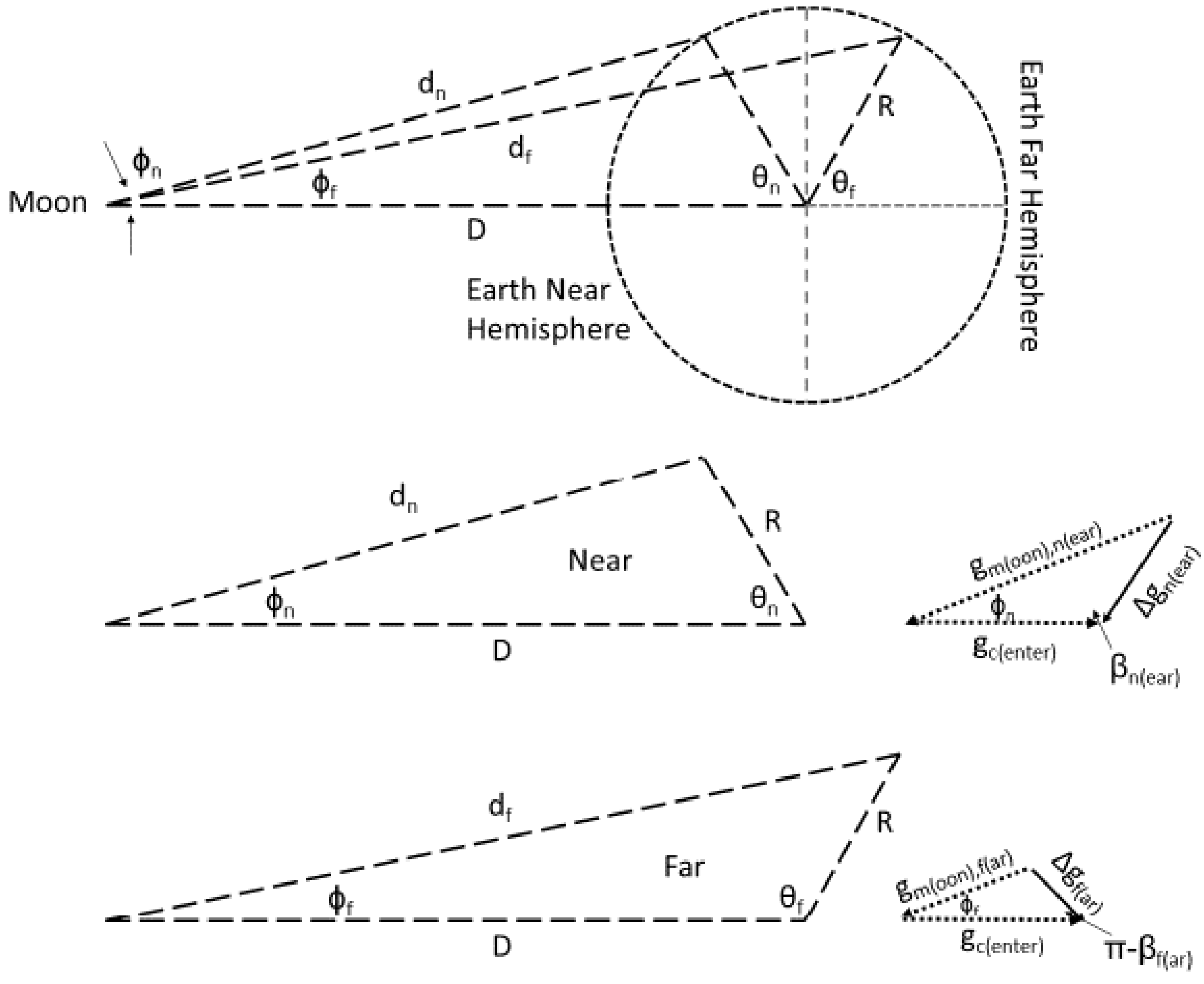


Figure 3. Geometry for Analysis

$g_{m,n}/sin\beta_n = \Delta g_n/sin\phi_n$, yielding
$\beta_n = arcsin(g_{m,n}sin\phi_n/\Delta g_n)$

$g_{m,f}/sin(\pi - \beta_f) = \Delta g_f/sin\phi_f$, yielding
$g_{m,f}/sin\beta_f = \Delta g_f/sin\phi_f$, yielding
$\beta_f = arcsin(g_{m,f}sin\phi_f/\Delta g_f)$.

To compare corresponding angles θ on the near and far hemispheres in terms of the differences between the differential forces in terms of magnitude (Δg) and direction (β), we calculate the following pair of differences for $0 \leq \theta \leq \pi/2$: (1) $\Delta g_n - \Delta g_f$ and (2) $\Delta\beta_n - \Delta\beta_f$. For convenience, we assume D = 1 and express R as a fraction of D ranging from 0.001 to 0.5 and including the ratio for the Earth-Moon system, i.e., $R/D = (6,371km)/(384,400km) = 0.0166$. Figures 4 and 5 plot both pairs of differences over the complete range. Figures 6 and 7 are analogous plots in terms of the percent differences (relative to the average of the corresponding values for the far and near hemispheres). All four include the results for the Earth-Moon system, with g and m set to unity.

The expected trend is that the Moon's gravitational force on the near hemisphere, albeit decreasing from $\theta = 0$ *to* 90^o, vs. the Moon's gravitational force on the far hemisphere, always increasing, is always greater for corresponding values of θ, with equality achieved only when $\theta = 90^o$. As a result, the differential force on the near hemisphere, albeit decreasing from $\theta = 0$ *to* 90^o, always exceeds the differential force on the far hemisphere, also always decreasing, as evidenced by the values remaining positive, albeit decreasing, from $\theta = 0$ *to* 90^o. This trend is evident in Figures 4 and 6, increasing as R/D increases.[3]

The trend for the direction (angle β) of the differential force on the near hemisphere vs. far hemisphere is also evident from Figures 5 and 7. On the far hemisphere, this angle always increases from $\theta = 0$ *to* 90^o. On the near hemisphere, it also increases over nearly the entire range, only showing a slight decrease from $\theta = 89^o$ *to* 90^o. The result is that the difference between the angles of the differential forces is always positive (i.e., $\beta_n > \beta_f$). However, as shown in Figures 5 and 7, this difference reaches a maximum as β approaches 90^o, with the maximum occurring at a lesser angle with increasing R/D.[4] This maximum value occurs where $d_n = D$ (d_f is always > D), i.e.,

$d_n = D$, yielding $\sqrt{D^2 - 2DRcos\theta_n + R^2} = D$

$\theta_n = arccos(R/2D)$

Table 1 lists where these maxima occur.

3. Ring-Spring Analogy

Figure 8 illustrates the assumed tidal effect for the asymptotic case where R/D « 1, such that the tides are symmetric on both hemispheres. A force pulling at one end of the ring-spring (with the other end fixed), such as the Moon, translates into a differential force as if pulling at both ends (neither end fixed). An observer in the middle of the ring-spring before any force is applied sees both ends of the spring as equidistant, and the ring as circular. After the force is applied, the observer still sees both ends equidistant, albeit now equally farther away, and the ring stretched to form a symmetrical ellipse. This is the assumed behavior of the tides when R « D.

Figure 9 assumes the ring-spring starts in 'deep space' where there is no gravity. There, no deformation will occur. If the bottom is pulled, uniform deformation will occur, analogous to the deformation in Figure 8 since there is still no gravity. However, as the ring-spring enters a gravitational field, it acquires weight, with the weight being proportional to the length of the spring such that, towards the top, the coils feel a greater pull (more coils) than near the bottom (less coils). Now the deformation is not uniform and an observer originally at the middle of the spring when the ends were equidistant now will see the upper end farther away than the lower. The ring also deforms into more of an egg-shape than a symmetric ellipse. This is the analogy for the case where R is not « D. This parallels the results from the analysis as shown in Figures 4 through 7, i.e., there is an asymmetry between the two hemispheres, more pronounced as R approaches D.

4. Conclusion

The explanation for the Earth's tides is not intuitively obvious, but appears to suggest an expectation of symmetry on the two hemispheres, i.e., equally-high high tides and equally-low low tides, diametrically opposite. The analysis performed here suggests that this symmetry is the result of an approximation, usually quite good when the radius of the affected body is much less than the distance between its center and that of the affecting body (e.g., Earth-Moon). However, exact solution of the differential tidal force equations demonstrates that there always is an asymmetry, more pronounced as the affected body radius approaches the center-to-center distance from the affecting body. This peaks at approximately 10 percent in terms of magnitude and direction for $R/D = 0.0166$ for the Earth-Moon system (Figures 6 and 7).

[3] Also shown in this figure is the trend for the Earth-Moon system with the actual values of the gravitational constant ($6.674x10^{-11}m^3/kg-s$) and Moon's mass ($7.348x10^{22}kg$) included. The actual center-to-center distance between the Earth and Moon and the Earth's actual radius are already accounted for by $R/D = 0.0166$.
[4] The inflection point is impossible to see until R/D reaches 0.1 due to the scale of the axes.

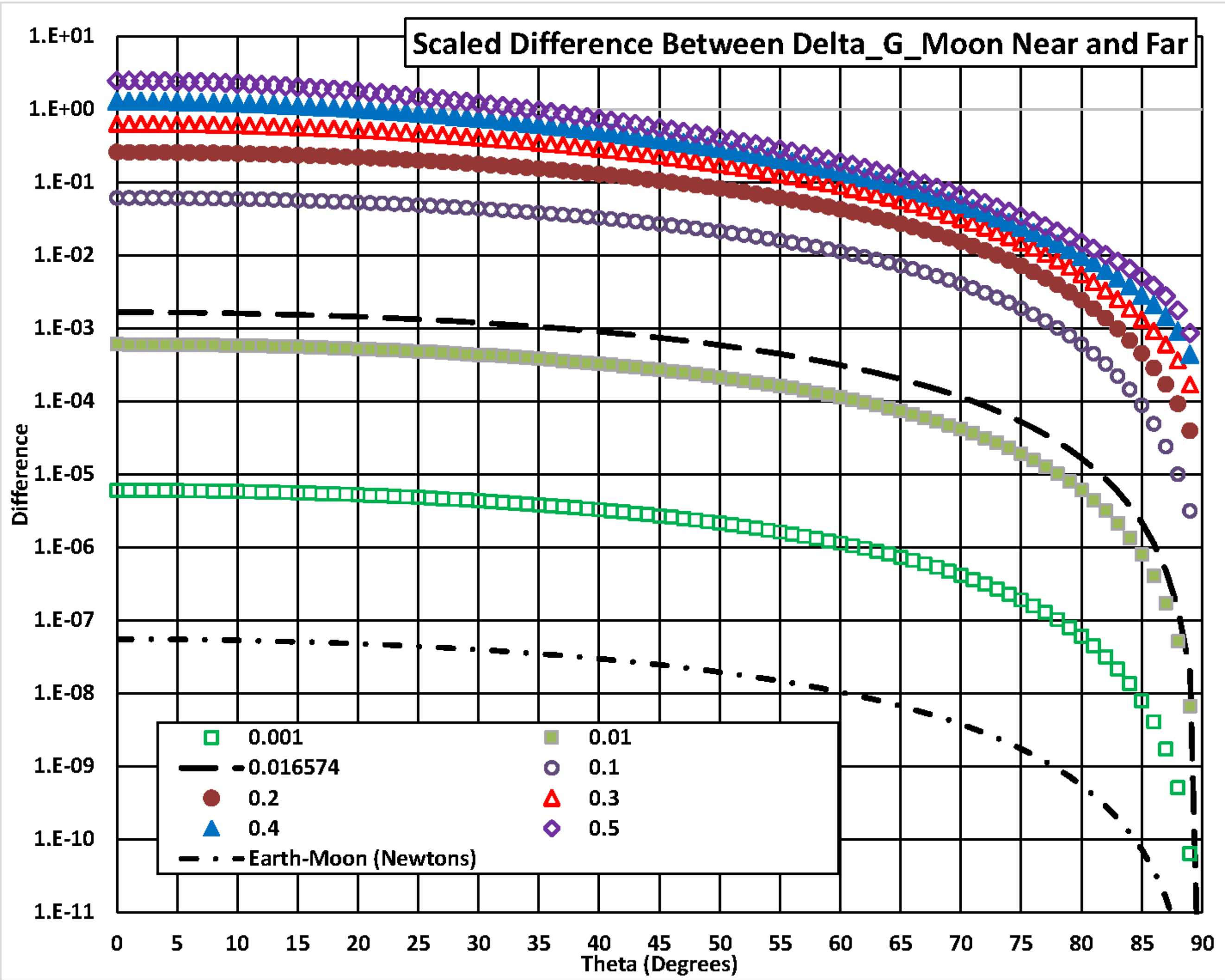


Figure 4. Differences between Moon's Differential Force for Corresponding Position on Near and Far Hemisphere

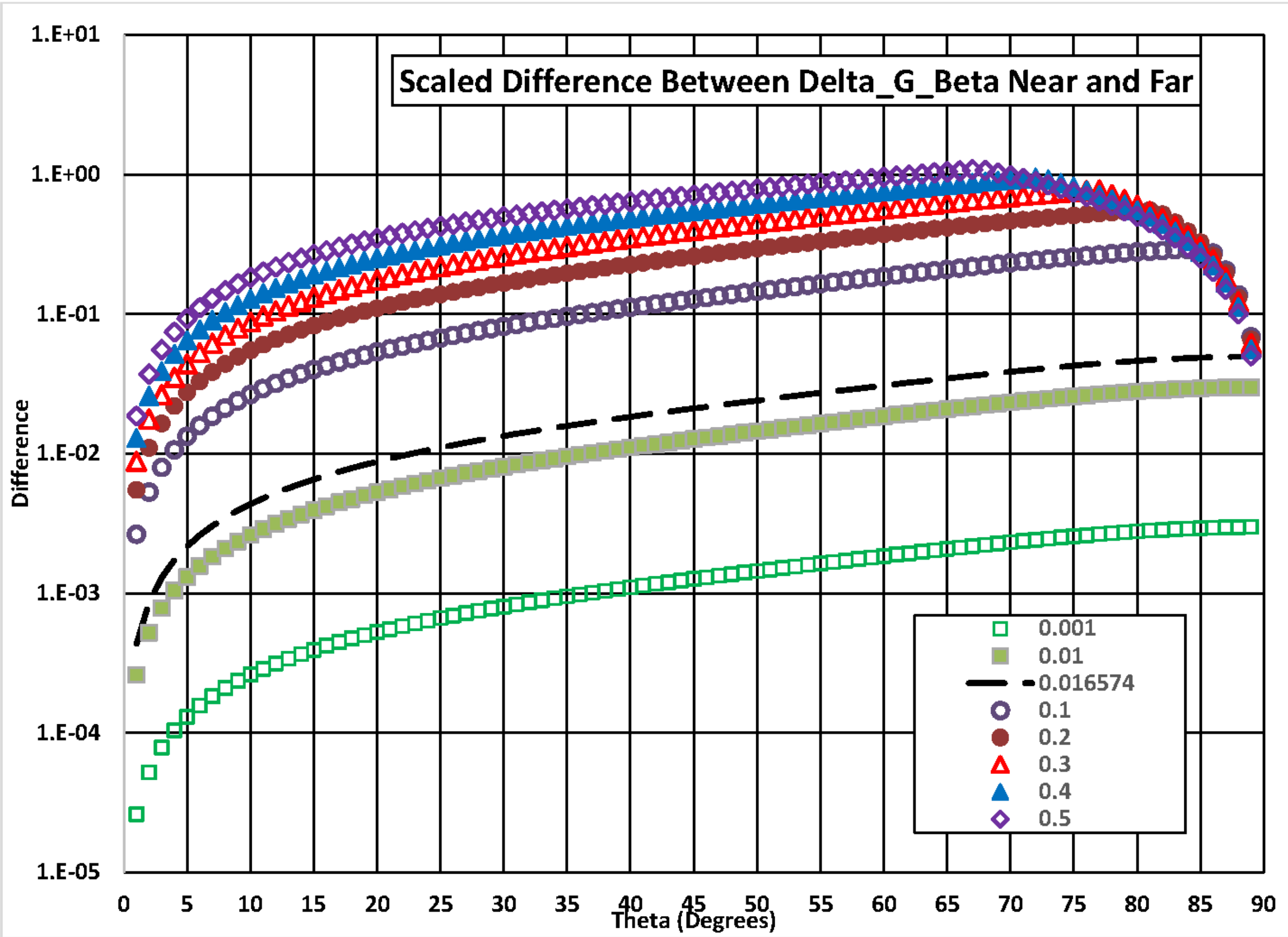


Figure 5. Differences between Angles of Moon's Differential Force for Corresponding Position on Near and Far Hemisphere

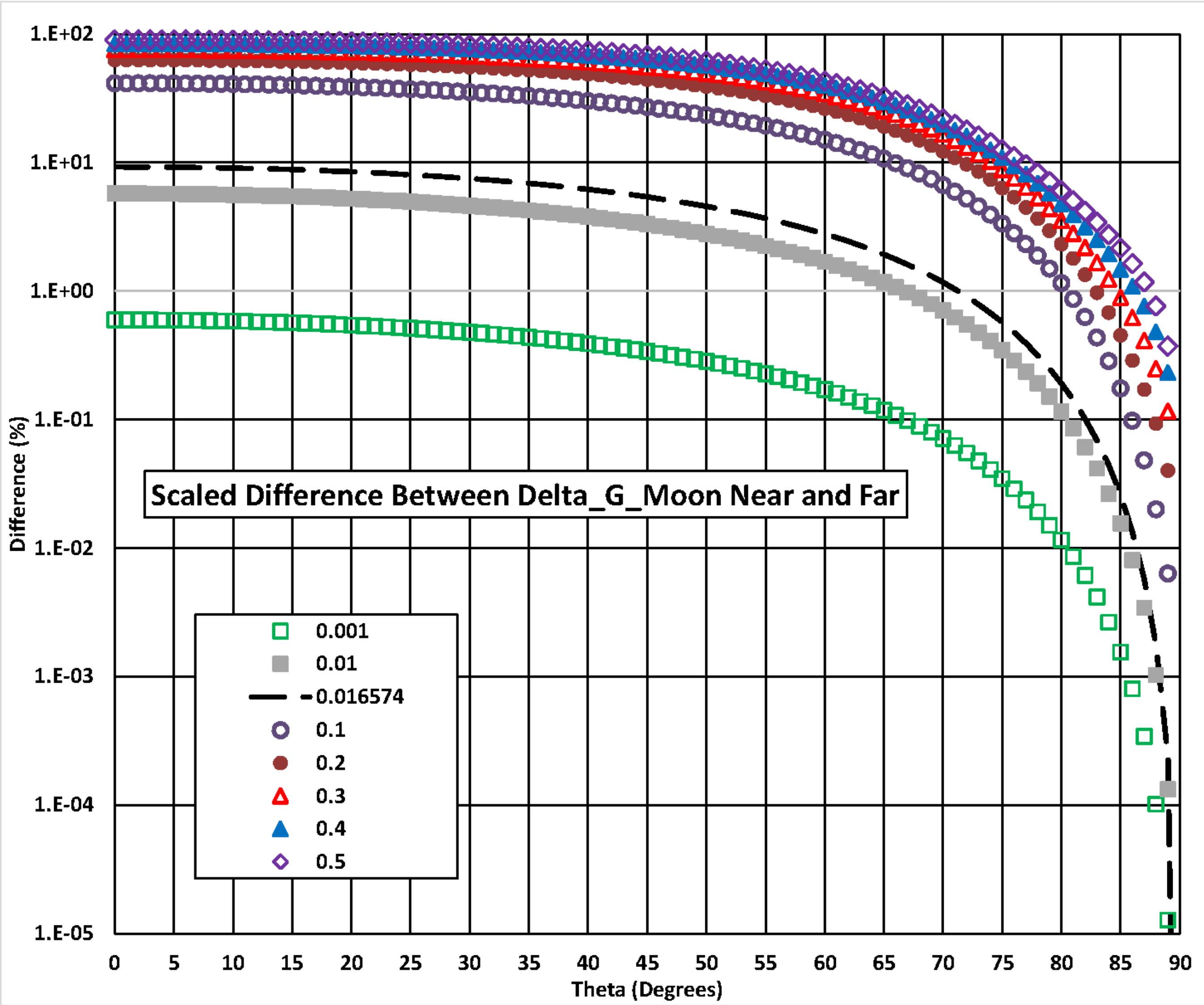


Figure 6. Figure 4 Differences Measured as Percents

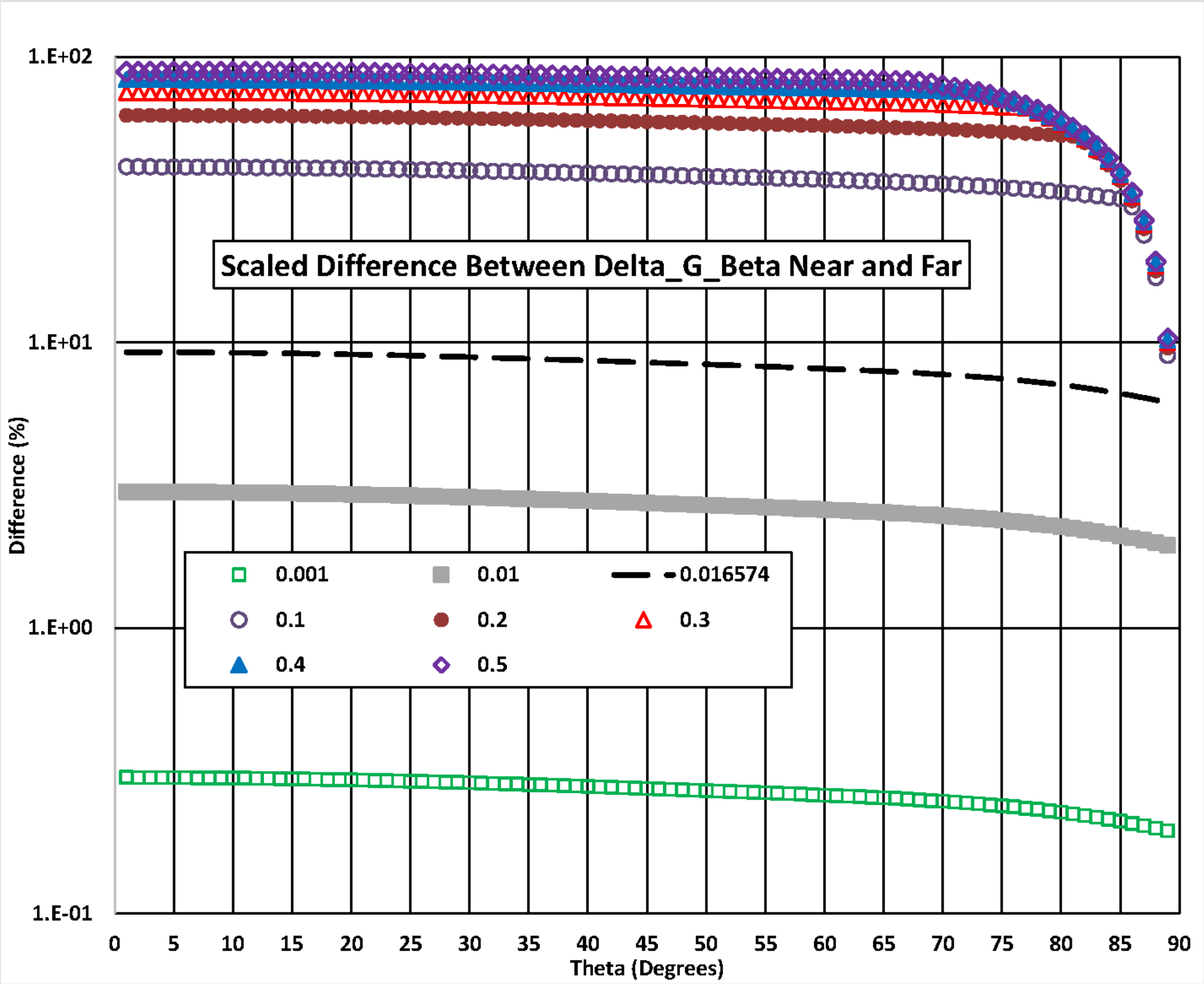


Figure 7. Figure 5 Differences Measured as Percents

θ_n (degrees) for $\beta_n - \beta_f$ maximum →	R/D							
	0.001	0.01	0.0166	0.1	0.2	0.3	0.4	0.5
	89.97	89.71	89.53	87.13	84.26	81.37	78.46	75.52

TABLE 1. Angle where Difference between Near and Far Hemisphere Tidal Forces is Maximum

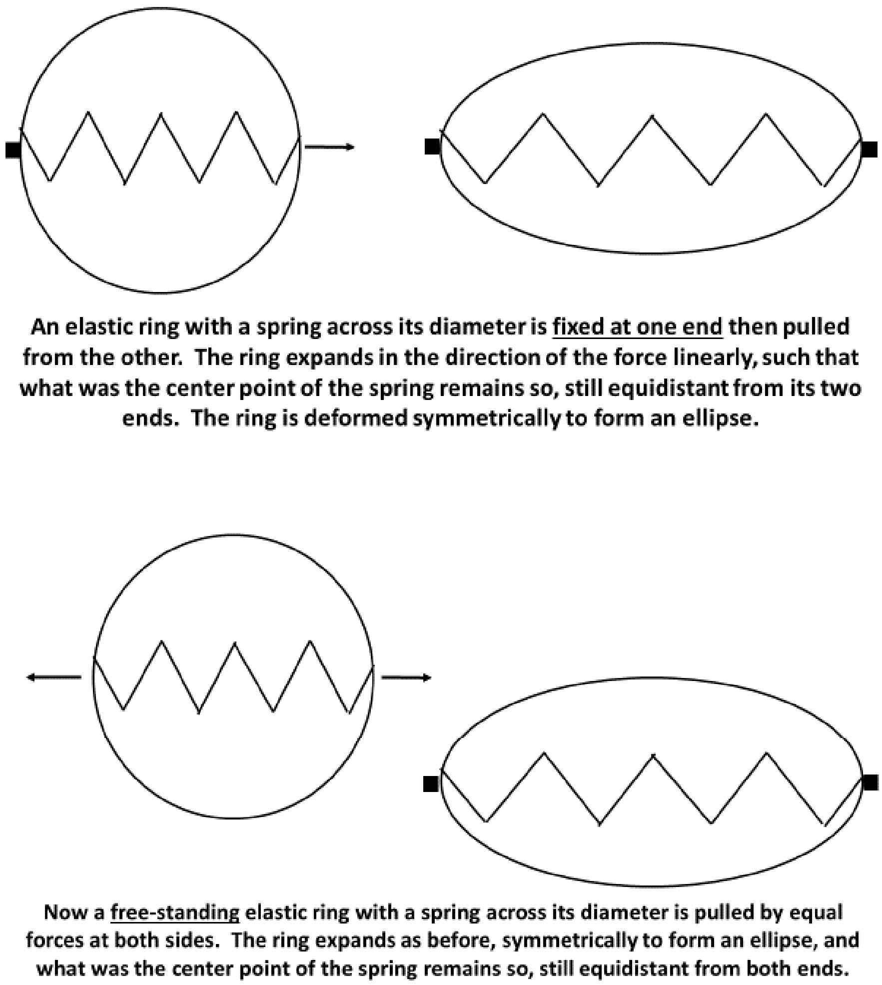


Figure 8. Ring-Spring Analogy for Asymptotic Deformation $R << D$

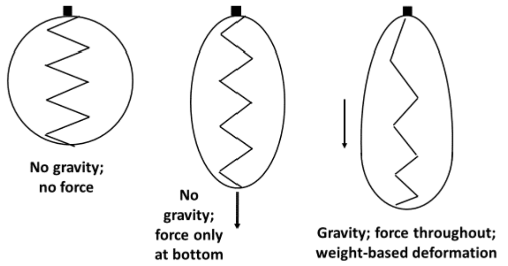


Figure 9. Ring-Spring without and with Gravity

5. Addendum I: Effect of Earth's Rotation about the Earth-Moon Barycenter

The Earth's monthly rotation about the Earth-Moon barycenter can affect the tides. Referring to Figure 10, the Earth-Moon barycenter (B) is located $4671km$ $(4.671x10^6m)$ from the Earth's center, within the Earth itself. With a rotational period (p) about this point of 27.32 d (sidereal month [5]), the tangential speed at $\theta_f = 0^o$ (along Earth-Moon axis on the 'far' side) is $2(R+B)/p = 29.4m/s$ for R (earth) = $6371km$ $(6.371x10^6m)$. Compared to the daily rotational speed at the equator, $2\pi R(86400s) = 463m/s$, this is small (approximately 6 percent) but not negligible. The centrifugal force on $1kg$ of ocean due to this barycentric rotation is $(1kg)(29.4m/s)^2/(R+B) = 7.82x10^{-5}N$. The differential gravitational force on $1kg$ of ocean from the Moon at this point is $\frac{Gm(1kg)}{1/D^2 - 1/(R+D)^2} = 1.07x10^{-6}N$ where m (moon) = $7.348x10^{22}kg$ and D = $384400km$ $(3.844x10^8m)$. Therefore the centrifugal force from the barycentric rotation is approximately 70 times that from the differential gravitational force at this point.

Even at the 'near' side ($\theta_n = 0^o$), the barycentric centrifugal force dominates that from the differential gravitational force. At this point (along Earth-Moon axis on the 'near' side), the tangential speed is $2\pi(R-B)/([27.32d][86400s/d]) = 4.52m/s$. The centrifugal force on $1kg$ of ocean due to this barycentric rotation is $(1kg)(4.52m/s)^2/(R-B) = 1.20x10^{-5}N$. The differential gravitational force on $1kg$ of ocean from the Moon at this point is $\frac{Gm(1kg)}{1/(D-R)^2 - 1/D^2} = 1.07x10^{-6}N$. Therefore the centrifugal force from the barycentric rotation is still approximately 10 times that from the differential gravitational force at this point. Clearly, the dynamic effects from the rotation of the Earth-Moon system about its barycenter dominates over the static effect from the differential gravitational force from the Moon.

If one examines the variation of the radial (outward from center of Earth) component of the barycentric centrifugal force over each hemisphere ($C_{f,r}$ *and* $C_{n,r}$), one finds the difference between these forces $(C_{f,r} - C_{n,r})$ decreasing from a maximum of $6.62x10^{-5}N$ *at* $\theta_f = 0^o$ *vs.* $\theta_n = 0^o$ *to* $1.15x10^{-6}N$ *at* $\theta_f = 89^o$ *vs.* $\theta_n = 89^o$ (the difference is naturally zero when both θ_f *and* $\theta_n = 90^o$). Therefore, there is a strong asymmetry between the two hemispheres, as one would expect given $R+B \approx 7(R-B)$. Scaled to $1x10^{-5}N$, this asymmetry is shown in Figure 11. Note from the plot also the ratios of the radial component of the barycentric centrifugal force to the magnitude of the differential gravitational force on the two hemispheres. It remains between approximately 70 and 90 for the far hemisphere, peaking around $\theta_f = 75^o$, while rising from approximately 10 to 82 from $\theta_n = 0^o$ *to* $\theta_n = 90^o$. Clearly there is strong asymmetry predicted due to the barycentric centrifugal force, with the tides on the far hemisphere exceeding those on the near, opposite to the trend for the differential gravitational force. However, due to the dominance of the former, the latter does not come close to an offset, so asymmetric tides are predicted.

6. Addendum II: An Intriguing 'Coincidence?'

Neither the exact solution to the differential gravitational force approach nor incorporating the effect of the Earth's barycentric centrifugal force was able to establish the alleged symmetry of the tides across the Earth's hemispheres. However, an interesting anomaly that might bear further examination is revealed if one combines the barycentric effect with the Moon's gravitational force directly, i.e., without the differential effect. In Addendum I, the barycentric centrifugal on $1kg$ of ocean at $\theta_f = 0^o$ (along Earth-Moon axis on the 'far' side) was calculated as $7.82x10^{-5}N$. At $\theta_n = 0^o$ (along Earth-Moon axis on the 'near' side), the corresponding force is $1.20x10^{-5}N$. What are the gravitational forces (direct, not differential) of the Moon on the same $1kg$ of ocean water at these points?

At the 'far' side ($\theta_f = 0^o$), this is $Gm/(R+D)^2 = \frac{(6.674x10^{-11}m^3/kg-s)(7.348x10^{22}kg)}{(6.371x10^6m + 3.844x10^8m)^2} = 3.21x10^{-5}N$. At the 'near' side ($\theta_n = 0^o$), it is $Gm/(D-R)^2 = \frac{(6.674x10^{-11}m^3/kg-s)(7.348x10^{22}kg)}{(3.844x10^8m - 6.371x10^6m)^2} = 3.43x10^{-5}N$. At the 'far' side ($\theta_f = 0^o$), the barycentric centrifugal and Moon's gravitational forces act in opposite directions, yielding a net force radially outward of $7.82x10^{-5}N - 3.21x10^{-5}N = 4.61x10^{-5}N$. At the

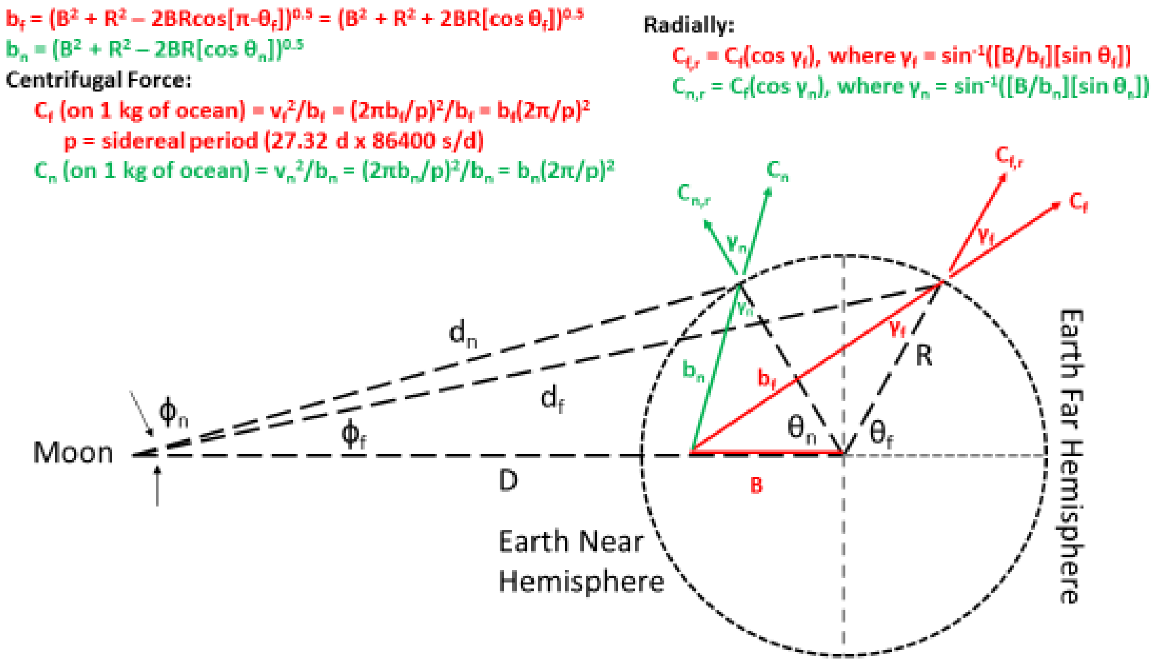


Figure 10. Earth-Moon System Showing Barycenter

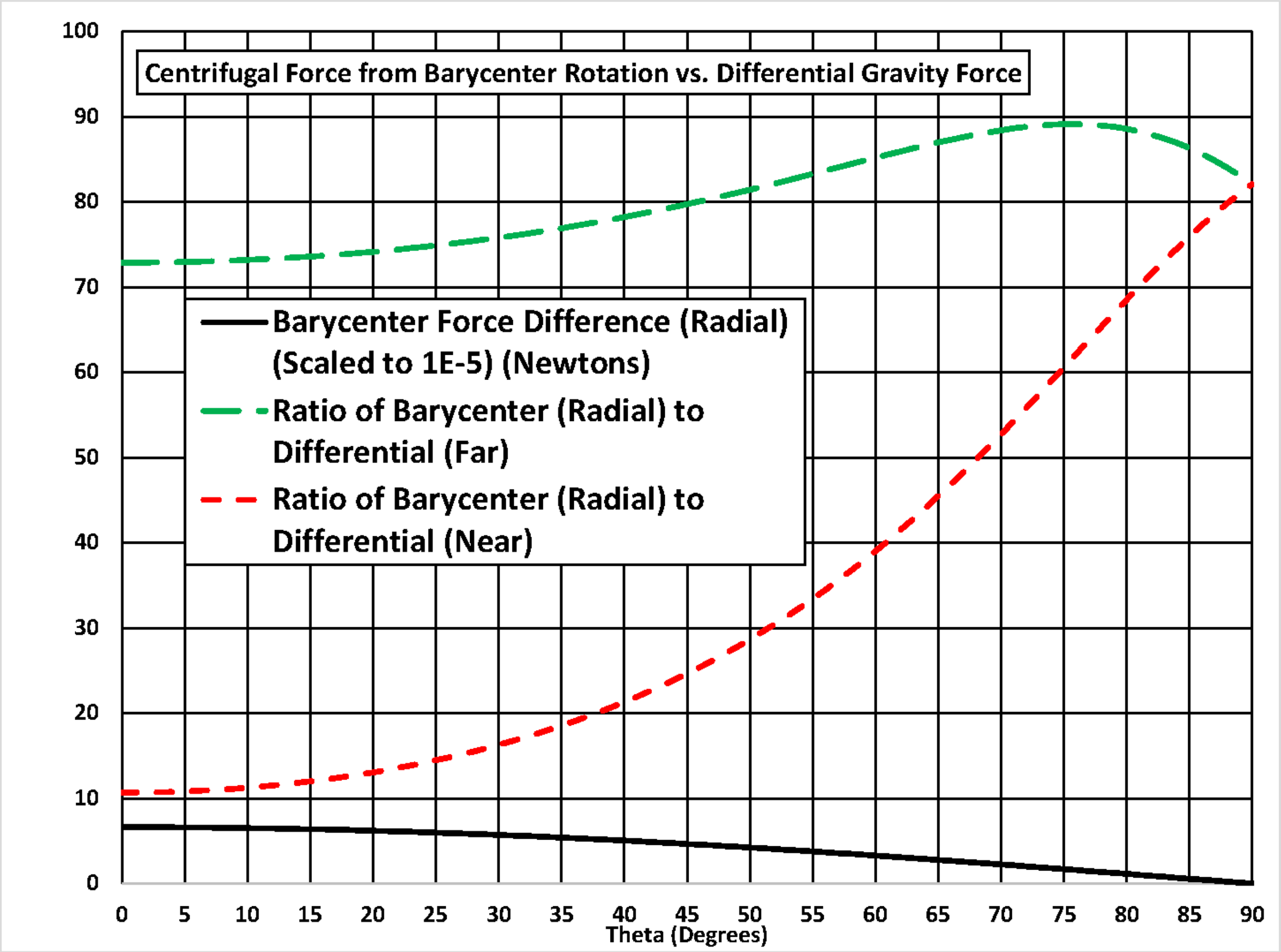


Figure 11. Centrifugal Force from Barycenter Rotation vs. Differential Gravity Force

'near' side ($\theta_n = 0^o$), these two forces act in the same directions, yielding a net force radially outward of $1.20x10^{-5}N + 3.43x10^{-5}N = 4.63x10^{-5}N$. These are essentially equal, both radially outward, inferring symmetry of the tides at these highest points. Is this just a coincidence, or might an explanation for tidal symmetry rely on this combination of forces, i.e., including the Moon's direct gravitational rather than its differential gravitational effect? To examine this conjecture over the entire pair of hemispheres, we employ the geometry as shown in Figure 12.

The barycentric centrifugal and Moon's gravitational (direct) forces are calculated over both hemispheres from the preceding formulas. These are then combined vectorially to yield the net forces along with the directions relative to the x-axis, also as shown in the figures. The differences between the net forces on the 'far' and 'near' sides and the differences between the angles of these forces are shown in Figure 13. Also shown are the ratio of these differences to their average values for the corresponding locations in each hemisphere.

The results are as follows. For the net forces themselves, the differences between corresponding locations in each hemisphere are quite small, on the order of $1x10^{-7}N$ or less, or < 1 percent of their average value. Similarly, the differences between the angles for these net forces at corresponding locations is quite small, on the order of 0.01 radians or less, again < 1 percent of their average value. What this suggests is that combining the barycentric centrifugal force and the Moon's gravitational (direct, not differential) forces vectorially produces the alleged symmetry between the tides on the opposite hemispheres. Might this, and not just the one differential gravitational force, be the reason for the symmetry of the alleged tides?

However, now that we are considering the direct gravitational effect of the Moon, what about that of the Sun, which is $(M/m)(D/S)^2 = 180$ times stronger, where M = mass of the Sun ($1.989x10^{30}kg$) and S = distance from Sun's center to Earth's ($1.496x10^8km$)? The Moon's direct gravitational force on the near vs. far side differs by $3.43x10^{-5}N - 3.21x10^{-5}N = 2.20x10^{-6}N$, approximately 7 percent relative to the force at the Earth's center. For the Sun, this difference is $\frac{GM}{1/[S-R]^2 - 1/[S+R]^2} = 1.01x10^{-6}N$, approximately 0.02 percent relative to the force at the Earth's center ($GM/S^2 = 0.00593N$). When considering the effect of the differential gravitational forces, this is an important contributor, nearly half the value of the Moon's. However, for direct gravitational force, the variation across the Earth due to the Sun's gravity is negligible, i.e., it affects the Earth gravitationally on essentially an equal basis everywhere (0.00593 N). Therefore, as with the Earth's own daily rotational centrifugal and direct gravitational forces, the Sun's direct gravitational force is essentially uniform over the entire planet, thereby introducing no asymmetry.

We are left to ponder whether there is an alternative explanation for the alleged symmetry of the tides other than accepting the approximation employed when deriving the differential gravitational effect as in [1]. If the essentially uniform effects from the Sun's and Earth's own direct gravitational forces, as well as that from the Earth's daily rotational centrifugal force, introduce no asymmetry, might the combination of the barycentric centrifugal and Moon's direct gravitational forces explain what has so far been attributed to an approximation in the differential gravitational force derivation?

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$\theta_{n,c}$ is the angle ('near side') at which the barycentric centrifugal force is exactly aligned with the y axis, i.e., $\theta_{n,c} = \cos^{-1}(B/R) = 0.748 = 42.85°$.

Far side:
Using the symbols from the previous figures, the angle relative to the x-axis at which the barycentric centrifugal force (C_f) occurs is $\pi - (\pi - \theta_f) - \gamma_f = \theta_f - \gamma_f$. The Moon's gravitational (direct) force ($g_{m,f}$) occurs at the angle ϕ_f relative to the x-axis. The net force (N_f) is the vector sum of these, as shown. Its components in the x and y directions are as follows:

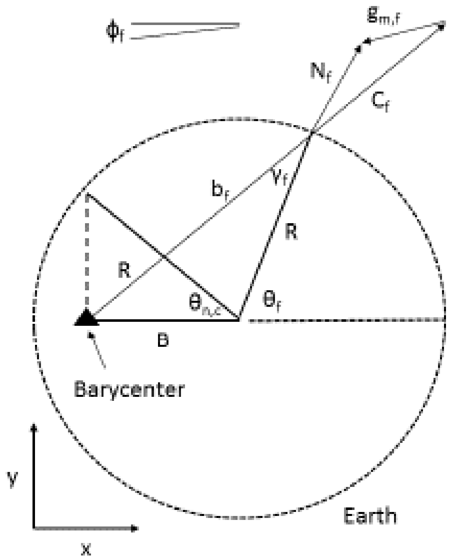
$N_{f,x} = C_{f,x} - g_{m,f,x}$ where $C_{f,x} = C_f(\cos[\theta_f - \gamma_f])$ and $g_{m,f,x} = g_{m,f}(\cos \phi_f)$
$N_{f,y} = C_{f,y} - g_{m,f,y}$ where $C_{f,y} = C_f(\sin[\theta_f - \gamma_f])$ and $g_{m,f,y} = g_{m,f}(\sin \phi_f)$

The total force $N_f = (N_{f,x}^2 + N_{f,y}^2)^{0.5}$; relative to the x-axis, it occurs at an angle = $\cos^{-1}(N_{f,x}/N_f)$.

Near Side ($\theta_{n,c} \le \theta_n \le \pi/2$):
Although not shown, lest the diagram become too crowded, analogous formulas as above also apply here, now using the 'n(ear)' subscript, as follows:

$N_{n,x} = C_{n,x} - g_{m,n,x}$ where $C_{n,x} = C_n(\cos[\pi - \theta_n - \gamma_n])$ and $g_{m,n,x} = g_{m,n}(\cos \phi_n)$
$N_{n,y} = C_{ny} - g_{m,n,y}$ where $C_{n,y} = C_f(\sin[\pi - \theta_n - \gamma_n])$ and $g_{m,n,y} = g_{m,n}(\sin \phi_n)$

The total force $N_n = (N_{n,x}^2 + N_{n,y}^2)^{0.5}$; relative to the x-axis (in the negative direction), it occurs at an angle = $\cos^{-1}(N_{n,x}/N_n)$.



Near Side ($0 \le \theta_n \le \theta_{n,c}$):
Once $\theta_n \le \theta_{n,c}$, both the barycentric centrifugal and Moon's gravitational (direct) forces reinforce each other in the negative x direction. This leads to the following set of equations:

$N_{n,x} = C_{n,x} + g_{m,n,x}$
where $C_{n,x} = C_n(\cos[\theta_n + \gamma_n])$ and $g_{m,n,x} = g_{m,n}(\cos \phi_n)$
$N_{n,y} = C_{ny} - g_{m,n,y}$
where $C_{n,y} = C_f(\sin[\theta_n + \gamma_n])$ and $g_{m,n,y} = g_{m,n}(\sin \phi_n)$

The total force $N_n = (N_{n,x}^2 + N_{n,y}^2)^{0.5}$; relative to the x-axis (in the negative direction), it occurs at an angle = $\cos^{-1}(N_{n,x}/N_n)$.

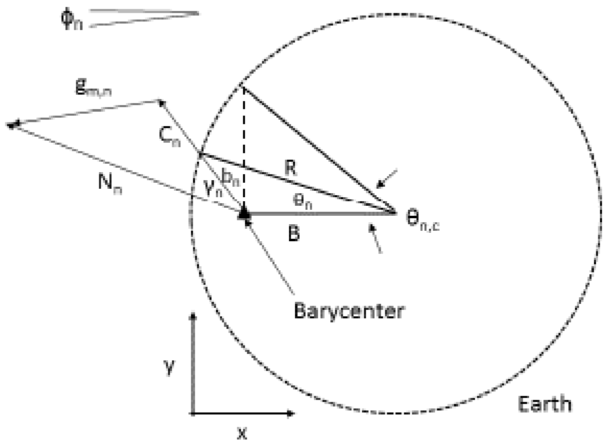


Figure 12. Moon's Direct Gravitational and Barycentric Centrifugal Forces

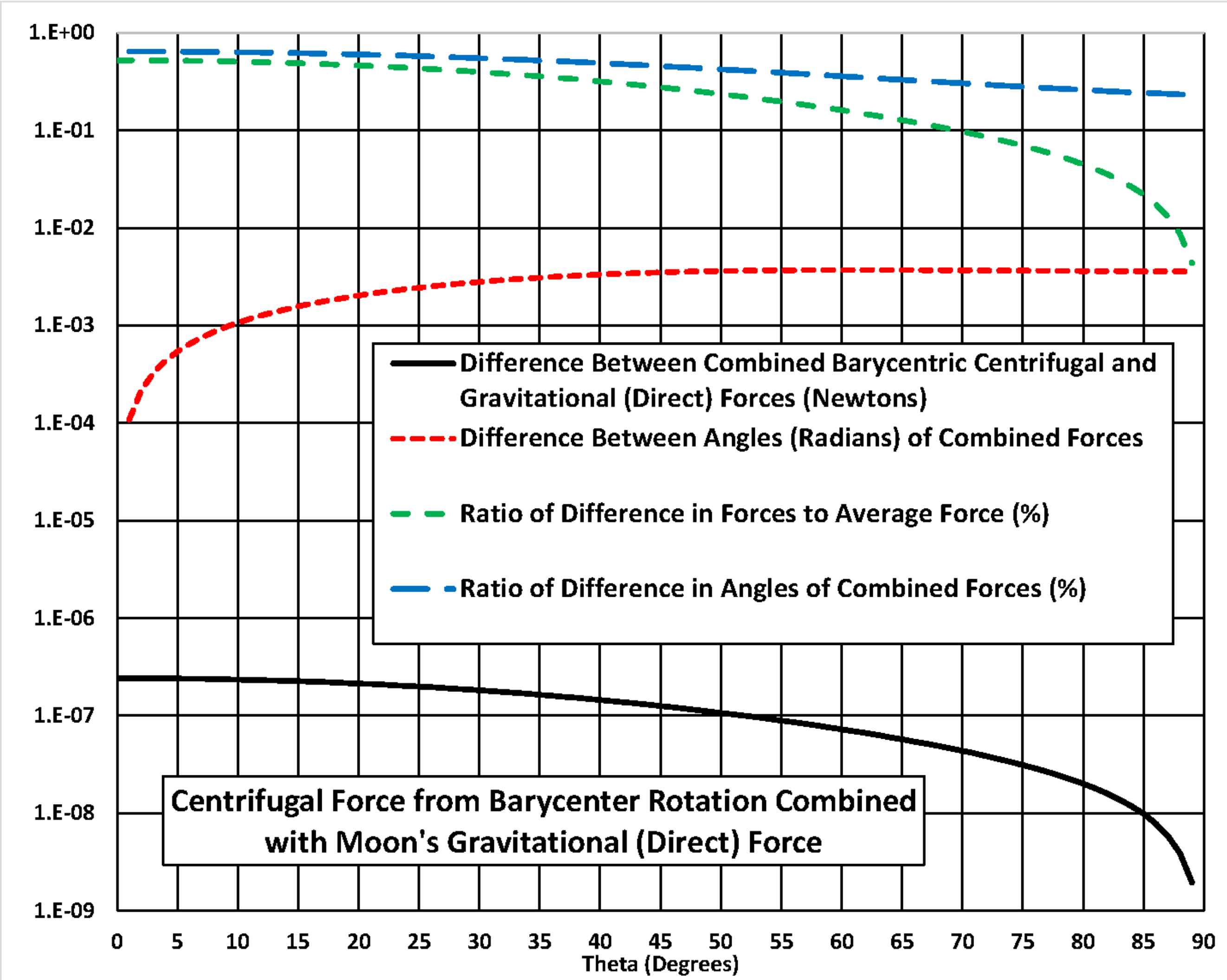


Figure 13. Centrifugal Force from Barycenter Rotation Combined with Moon's Gravitational (Direct) Force