Necessary conditions for any number to be expressed as difference of 4^{th} powers of 2 natural numbers

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Introduction :

We know how to find out the number of ways of expressing any number as difference of squares of 2 natural numbers.
Here, we would like to discuss about the conditions for the numbers which can be expressed as difference of 4^{th} powers of 2 natural numbers and number of solution.
$N=a^4-b^4=(a-b)(a+b)(a^2+b^2)$
We have to find the solution for (a,b) where both a and b are Natural numbers.

Main Results

Proposition 1: Number (N) should have at least 3 distinct factors.

$N=a^4-b^4=(a-b)(a+b)(a^2+b^2)$

Proposition 2:

Case I: N is even

N should be multiple of 16.

If N is multiple of 16 then there may be possibility but not surety.

As we know when a^4 is divided by 16, remainder is always 1(for odd values of a) or 0 (for even values of a).

If N is even then there are 2 possibilities. Either a,b both should be even or both should be odd.

Let a, b are even

$$N= \quad a^4 \quad - \quad b^4$$
$$16k \ - \ 16m$$

Hence N=16p.

Let a, b are odd

$$N= \quad a^4 \quad - \quad b^4$$
$$(16k+1) \ - \ (16m+1)$$

Hence N=16p.

Case II: N is odd

N should be either (16k+1) or (16k+15) i.e. when N is divided by 16 remainder should be 1 or 15.

If N is (16k+1) or (16k+15) then there is possibility but not surety.

If N is odd then out of a and b one should be odd and other should be even.

Let a is odd and b is even

Then

$$N= \quad a^4 \quad - \quad b^4$$
$$(16k+1) \ - \ (16m)$$

Hence N = 16p+1

Case 4: ‘a’ is even and b is odd

$$N= \quad a^4 \quad - \quad b^4$$
$$16k \ - \ (16m+1)$$

Hence N = 16p+15

Proposition 3:

If unit digit of N is 2,3,7 or 8 then there will be no solution but reverse is not true.

As we know,4^{th} power of any natural number ends with 0,1,5 or 6.

Therefore a^4-b^4 cannot end with 2,3,7 and 8.

Note: If all the conditions are satisfied then there is possibility of solution but not surety.

If any one condition is not satisfied there will be no solution.

Proposition 4: Any number can be expressed as difference of 4^{th} powers of 2natural numbers in not more than 1 way.

Let $N = p \times q = a^4-b^4=(a-b)(a+b)(a^2+b^2)$ Where a,b,p,q are natural nos.

$= p \times q =(a^2-b^2)(a^2+b^2)$

Let $p = a^2+b^2$ and $q = a^2-b^2$

Then $p+q =2a^2$ and $p-q = 2b^2$

Means N should have 2 factors p, q such that N=p x q and sum and difference of p and q should be double of perfect square otherwise no solution.

Eg. 15 =3x5

$3+5=8=2x2^2$
$5-3=2=2x1^2$
Hence 15 can be written as difference of 4^{th} powers of 2 natural numbers.
$15=2^4-1^4$
Here 15= 3x 5
If we multiply both the numbers (3 and 5) by any perfect square then we get another solution.
15x4x4 =(3x4)(5x4)
=12x20
$20+12=32=2x4^2$
$20-12=8=2x2^2$
Hence $240=4^4-2^4$

240 =1x240
=2x120
=3x80
=4x60
=5x48
=6x40
=8x30
=10x24
$\boxed{=12x\ 20}$
=15x16
Only one case (box) satisfies the condition.
Again if we multiply both numbers with some perfect square then again we get solution but then number is not same.
Let's take more example
$6^4-4^4=1040=1\ x1040$
=2 x520
=4 x260
=5 x204
=8 x130
=10 x 104
=13 x80
=16 x65
$\boxed{=20\ x52}$
=26 x40

Again only 1 case satisfies the condition.
If we try to adjust sum and difference as double of perfect square then number changes or we cannot adjust numbers again for which sum and difference is double of perfect square.
Hence, any natural number can be expressed as difference of 4^{th} powers of 2 natural numbers in at most 1 way.

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