The real parts of the nontrivial Riemann zeta function zeros

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to my love and wife Mary

ABSTRACT

This theorem is based on holomorphy of studied functions and the fact that near a singularity point the real part of some rational function can take an arbitrary preassigned value.

The colored markers are as follows:
- ● - assumption or a fact, which is not proven at present;
- ● - the statement, which requires additional attention;
- ● - statement, which is proved earlier or clearly undestandable.

THEOREM

● The real parts of all the nontrivial Riemann zeta function zeros ρ are equal $Re\left(\rho\right)=\dfrac{1}{2}$.

PROOF:

● In relation to $\zeta\left(s\right)$ - Zeta function of Riemann is known [8, p. 5] two equations each of which can serve as its definition:

$$\zeta\left(s\right)=\sum_{n=1}^{\infty}\frac{1}{n^{s}},\qquad \zeta\left(s\right)=\prod_{n=1}^{\infty}\left(1-\frac{1}{p_{n}^{s}}\right)^{-1},\qquad Re\left(s\right)>1, \qquad (1)$$

where $p_1,p_2,\ldots,p_n,\ldots$ is a series of primes.

● According to the functional equality [8, p. 22], [4, p. 8-11] by part $\Gamma\left(s\right)$ is the Gamma function:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta\left(s\right)=\Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta\left(1-s\right),\qquad Re\left(s\right)>0. \qquad (2)$$

- From [4, p. 8-11] $\zeta(\bar{s}) = \overline{\zeta(s)}$, it means that $\forall \rho = \sigma + it$: $\zeta(\rho) = 0$ and $0 \leqslant \sigma \leqslant 1$ we have:

$$\zeta(\bar{\rho}) = \zeta(1-\rho) = \zeta(1-\bar{\rho}) = 0 \tag{3}$$

- From [9], [7, p. 128], [8, p. 45] we know that $\zeta(s)$ has no nontrivial zeros on the line $\sigma = 1$ and consequently on the line $\sigma = 0$ also, in accordance with (3) they don't exist.

- Let's denote the set of nontrivial zeros $\zeta(s)$ through $\mathcal{P}$ (multiset with consideration of multiplicitiy):

$$\mathcal{P} \stackrel{\text{def}}{=} \{\rho:\ \zeta(\rho) = 0,\ \rho = \sigma + it,\ 0 < \sigma < 1\}.$$

$$\text{And: } \mathcal{P}_1 \stackrel{\text{def}}{=} \left\{\rho:\ \zeta(\rho) = 0,\ \rho = \sigma + it,\ 0 < \sigma < \frac{1}{2}\right\},$$

$$\mathcal{P}_2 \stackrel{\text{def}}{=} \left\{\rho:\ \zeta(\rho) = 0,\ \rho = \frac{1}{2} + it\right\},$$

$$\mathcal{P}_3 \stackrel{\text{def}}{=} \left\{\rho:\ \zeta(\rho) = 0,\ \rho = \sigma + it,\ \frac{1}{2} < \sigma < 1\right\}.$$

Then:

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \quad \text{and} \quad \mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P}_2 \cap \mathcal{P}_3 = \mathcal{P}_1 \cap \mathcal{P}_3 = \varnothing,$$

$$\mathcal{P}_1 = \varnothing \Leftrightarrow \mathcal{P}_3 = \varnothing.$$

- Hadamard's theorem (Weierstrass preparation theorem) about the decomposition of function through the roots gives us the following result [8, p. 30], [4, p. 31], [10]:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}} e^{as}}{s(s-1)\Gamma\left(\frac{s}{2}\right)} \prod_{\rho \in \mathcal{P}} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \qquad Re(s) > 0 \tag{4}$$

$$a = \ln 2\sqrt{\pi} - \frac{\gamma}{2} - 1,\ \gamma - \text{Euler's constant and}$$

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\ln\pi + a - \frac{1}{s} + \frac{1}{1-s} - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \sum_{\rho \in \mathcal{P}} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \tag{5}$$

- According to the fact that $\dfrac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$ - Digamma function of [8, p. 31], [4, p. 23] we have:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + \sum_{\rho\in\mathcal{P}}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n}-\frac{1}{2n}\right) + C, \tag{6}$$

$$C = const.$$

- From [3, p. 160], [6, p. 272], [2, p. 81]:

$$\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = 1 + \frac{\gamma}{2} - \ln 2\sqrt{\pi} = 0,0230957\ldots \tag{7}$$

- Indeed, from (3):

$$\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = \frac{1}{2}\sum_{\rho\in\mathcal{P}}\left(\frac{1}{1-\rho}+\frac{1}{\rho}\right).$$

- From (5):

$$2\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = \lim_{s\to 1}\left(\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{1-s} + \frac{1}{s} - a - \frac{1}{2}\ln\pi + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right).$$

- Also it's known, for example, from [8, p. 49], [2, p. 98] that the number of nontrivial zeros of $\rho = \sigma + it$ in strip $0 < \sigma < 1$, the imaginary parts of which t are less than some number $T > 0$ is limited, i.e.,

$$\|\{\rho:\ \rho\in\mathcal{P},\ \rho=\sigma+it,\ |t|<T\}\| < \infty.$$

- Indeed, it can be presented that on the contrary the sum of $\sum_{\rho\in\mathcal{P}}\dfrac{1}{\rho}$ would have been unlimited.

- Thus $\forall\, T > 0\ \exists\, \delta_x > 0,\ \delta_y > 0$ such that

  in area $0 < t \leqslant \delta_y, 0 < \sigma \leqslant \delta_x$ there are no zeros $\rho = \sigma + it \in \mathcal{P}$.

Let's consider random root $q \in \mathcal{P}$.

Let's denote $k(q)$ the multiplicity of the root q.

Let's examine the area $Q\left(R\right) \stackrel{\text{def}}{=} \{s : \|s - q\| \leqslant R, R > 0\}$.

- From the fact of finiteness of set of nontrivial zeros $\zeta(s)$ in the limited area follows $\exists\ R > 0$, such that $Q(R)$ does not contain any root from $\mathcal{P}$ except q and also does not intersect with the axes of coordinates.

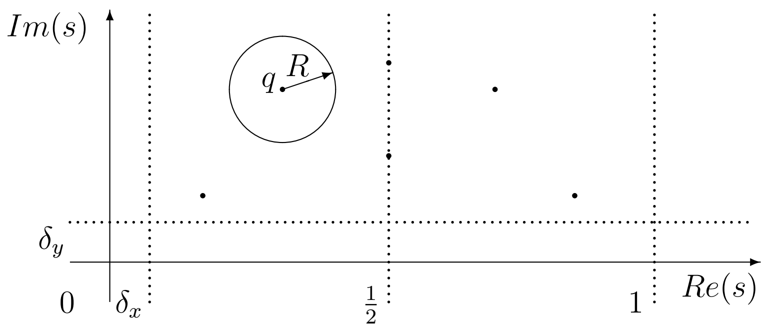

Fig. 1.

- From [1], [8, p. 31], [4, p. 23] we know that the Digamma function $\dfrac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$

in the area $Q(R)$ has no poles, i.e., $\forall s \in Q(R)$

$$\left\| \frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \right\| < \infty.$$

Let's denote:

$$I_{\mathcal{P}}(s) \stackrel{\text{def}}{=} -\frac{1}{s} + \frac{1}{1-s} + \sum_{\rho \in \mathcal{P}} \frac{1}{s - \rho}$$

and

$$I_{\mathcal{P}\setminus\{q\}}(s) \stackrel{\text{def}}{=} -\frac{1}{s} + \frac{1}{1-s} + \sum_{\rho \in \mathcal{P}\setminus\{q\}} \frac{1}{s - \rho}.$$

- Hereinafter $\mathcal{P} \setminus \{q\} \stackrel{\text{def}}{=} \mathcal{P} \setminus \{(q, k(q))\}$ (the difference in the multiset).

Also we shall consider the summation - $\sum_{\rho\in\mathcal{P}}\dfrac{1}{s-\rho}$ and $\sum_{\rho\in\mathcal{P}\setminus\{q\}}\dfrac{1}{s-\rho}$ further as the sum of pairs $\left(\dfrac{1}{s-\rho}+\dfrac{1}{s-(1-\rho)}\right)$ and $\sum_{\rho\in\mathcal{P}}\dfrac{1}{\rho}$ as the sum of pairs $\left(\dfrac{1}{\rho}+\dfrac{1}{1-\rho}\right)$ as a consequence of division of the sum from (6) $\sum_{\rho\in\mathcal{P}}\left(\dfrac{1}{s-\rho}+\dfrac{1}{\rho}\right)$ into $\sum_{\rho\in\mathcal{P}}\dfrac{1}{s-\rho}+\sum_{\rho\in\mathcal{P}}\dfrac{1}{\rho}$. As specifed in [3], [5], [6], [8].

- Let's note that $I_{\mathcal{P}\setminus\{q\}}(s)$ is holomorphic function $\forall\ s\in Q(R)$.

Then from (5) we have:

$$\frac{\zeta'(s)}{\zeta(s)}=\frac{1}{2}\ln\pi+a-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}+\sum_{\rho\in\mathcal{P}}\frac{1}{\rho}+I_{\mathcal{P}}(s).$$

And in view of (4), (7):

$$Re\frac{\zeta'(s)}{\zeta(s)}=\frac{1}{2}\ln\pi+Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}+I_{\mathcal{P}}(s)\right). \tag{8}$$

Let's note that from the equality of

$$\sum_{\rho\in\mathcal{P}}\frac{1}{1-s-\rho}=-\sum_{(1-\rho)\in\mathcal{P}}\frac{1}{s-(1-\rho)}=-\sum_{\rho\in\mathcal{P}}\frac{1}{s-\rho} \tag{9}$$

follows that:

$$I_{\mathcal{P}}(1-s)=-I_{\mathcal{P}}(s),\ I_{\mathcal{P}\setminus\{1-q\}}(1-s)=-I_{\mathcal{P}\setminus\{q\}}(s),\ Re\,(s)>0.$$

- Besides

$$I_{\mathcal{P}\setminus\{q\}}(s)=I_{\mathcal{P}}(s)-\frac{k(q)}{s-q}$$

and $I_{\mathcal{P}\setminus\{q\}}(s)$ is limited in the area of $s\in Q(R)$ as a result of absence of its poles in this area as well as its differentiability in each point of this area.

- If in (5) we replace s with $1-s$ that in view of (7), in a similar way if we take derivative of the principal logarithm (2):

$$\frac{\zeta'(s)}{\zeta(s)}+\frac{\zeta'(1-s)}{\zeta(1-s)}=-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}+\ln\pi,\ Re\,(s)>0. \quad (10)$$

- Let's examine a circle with the center in a point q and radius $r \leqslant R$, laying in the area of $Q(R)$:

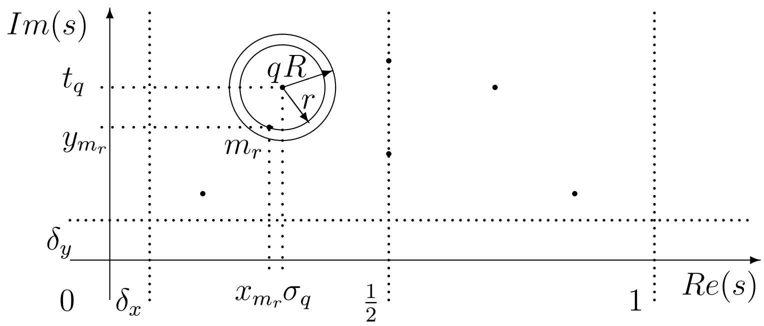

Fig. 2.

- For $s=x+iy,\ q=\sigma_q+it_q$

$$Re\frac{k(q)}{s-q}=Re\frac{k(q)}{x+iy-\sigma_q-it_q}=\frac{k(q)(x-\sigma_q)}{(x-\sigma_q)^2+(y-t_q)^2}=k(q)\frac{x-\sigma_q}{r^2}.$$

Let's prove a series of statements:

- STATEMENT A

In an arbitrarily small neighborhood of any nontrivial zero there is a point with the following properties:

$$\forall\, q\in\mathcal{P}$$

$$\exists\, 0<R_m\leqslant R:\ \ \forall\, 0<r\leqslant R_m\ \ \exists\, m_r:\|m_r-q\|=r,\ Re(m_r)\leqslant Re(q),$$

$$Re\frac{\zeta'(m_r)}{\zeta(m_r)}-Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}+Re\frac{\zeta'(Re(m_r))}{\zeta(Re(m_r))}-Re\frac{\zeta'(Re(1-m_r))}{\zeta(Re(1-m_r))}=0.$$

PROOF:

Let's define function for $s = x + iy \in Q(R)$:

$$T\left(s\right) \stackrel{\text{def}}{=}$$

$$= \frac{1}{2}\left(-\frac{1}{2}\frac{\Gamma'\left(\dfrac{s}{2}\right)}{\Gamma\left(\dfrac{s}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\dfrac{1-s}{2}\right)}{\Gamma\left(\dfrac{1-s}{2}\right)}\right) +$$

$$+\frac{1}{2}\left(-\frac{1}{2}\frac{\Gamma'\left(\dfrac{x}{2}\right)}{\Gamma\left(\dfrac{x}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\dfrac{1-x}{2}\right)}{\Gamma\left(\dfrac{1-x}{2}\right)}\right) + \ln\pi.$$

For $s = x + iy \in Q(R)$ consider the following function:

$$Re\left(\frac{\zeta'\left(s\right)}{\zeta\left(s\right)} - \frac{\zeta'\left(1-s\right)}{\zeta\left(1-s\right)} + \frac{\zeta'\left(x\right)}{\zeta\left(x\right)} - \frac{\zeta'\left(1-x\right)}{\zeta\left(1-x\right)} - 2\frac{k(q)}{s-q}\right)$$

- From (8) and (9) it is equal to:

$$Re\left(-\frac{1}{2}\frac{\Gamma'\left(\dfrac{s}{2}\right)}{\Gamma\left(\dfrac{s}{2}\right)} + \frac{1}{2}\frac{\Gamma'\left(\dfrac{1-s}{2}\right)}{\Gamma\left(\dfrac{1-s}{2}\right)} + 2I_{\mathcal{P}\setminus\{q\}}(s)\right) +$$

$$+Re\left(-\frac{1}{2}\frac{\Gamma'\left(\dfrac{x}{2}\right)}{\Gamma\left(\dfrac{x}{2}\right)} + \frac{1}{2}\frac{\Gamma'\left(\dfrac{1-x}{2}\right)}{\Gamma\left(\dfrac{1-x}{2}\right)} + 2I_{\mathcal{P}}(x)\right) =$$

$$= 2Re\left(T(s) + I_{\mathcal{P}\setminus\{q\}}(s) + I_{\mathcal{P}}(x)\right).$$

Since all the terms in parentheses are limited in the area of $Q(R)$, then

$\exists\ H_1(R) > 0,\ \ H_1(R) \in \mathbb{R},\ \forall s = x + iy \in Q(R) :$

$$\left| Re\left(\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(1-s)}{\zeta(1-s)} + \frac{\zeta'(x)}{\zeta(x)} - \frac{\zeta'(1-x)}{\zeta(1-x)} - 2\frac{k(q)}{s-q}\right)\right| < H_1(R).$$

- On each of the semicircles: the left -

  $\{s : \|s-q\| = r,\ \sigma_q - r \leqslant x \leqslant \sigma_q\}$ and right -

  $\{s : \|s-q\| = r,\ \sigma_q \leqslant x \leqslant \sigma_q + r\}$ the function $Re\dfrac{k(q)}{s-q}$ is continuous and takes values from $-\dfrac{k(q)}{r}$ to $\dfrac{k(q)}{r}$, $r > 0$.

  Consequently $\forall\ 0 < r < \dfrac{2k(q)}{H_1(R)}$, $\exists\ m_{min,r},\ m_{max,r}$:
  $\|m_{min,r} - q\| = r,\ \|m_{max,r} - q\| = r$:

$$Re\frac{2k(q)}{m_{min,r} - q} < -H_1(R),\ Re\frac{2k(q)}{m_{max,r} - q} > H_1(R)$$

  and the sum of two functions:

$$Re\left(\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(1-s)}{\zeta(1-s)} + \frac{\zeta'(x)}{\zeta(x)} - \frac{\zeta'(1-x)}{\zeta(1-x)} - 2\frac{k(q)}{s-q}\right)$$

  and

$$Re\frac{2k(q)}{s-q}$$

  at the points of $m_{min,r}$ and $m_{max,r}$ will have values with different signs.

- Properties of continuous functions on take all intermediate values between their extremes, it follows that $\exists\ R_m\ \in\ \mathbb{R}$,
  $R_m > 0$:

$$R_m \leqslant R,\ \frac{2k(q)}{R_m} > H_1(R)$$

and then $\forall\ 0 < r \leqslant R_m$
exists on the left semicircle point $m_r \stackrel{\text{def}}{=} x_{m_r} + iy_{m_r}$ such that:

$$Re\left(\frac{\zeta'(m_r)}{\zeta(m_r)} - \frac{\zeta'(1-m_r)}{\zeta(1-m_r)} + \frac{\zeta'(x_{m_r})}{\zeta(x_{m_r})} - \frac{\zeta'(1-x_{m_r})}{\zeta(1-x_{m_r})}\right) = 0.$$

- From this equality and (10), it follows that $\forall\ 0 < r \leqslant R_m$:

$$Re\frac{\zeta'(m_r)}{\zeta(m_r)} + Re\frac{\zeta'(x_{m_r})}{\zeta(x_{m_r})} = Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)} + Re\frac{\zeta'(1-x_{m_r})}{\zeta(1-x_{m_r})} =$$
$$= \frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{m_r}{2}\right)}{\Gamma\left(\frac{m_r}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-m_r}{2}\right)}{\Gamma\left(\frac{1-m_r}{2}\right)}\right) +$$
$$+\frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{x_{m_r}}{2}\right)}{\Gamma\left(\frac{x_{m_r}}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-x_{m_r}}{2}\right)}{\Gamma\left(\frac{1-x_{m_r}}{2}\right)}\right) + \ln\pi =$$
$$= ReT(m_r) = ReT(1-m_r) = O(1)_{r\to 0}. \tag{11}$$

□

- From (1) you can write:

$$\sum_{n=1}^{\infty}\frac{1}{n^s} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^s} = 2\sum_{n=2,4,\dots}\frac{1}{n^s} = 2\sum_{k=1}^{\infty}\frac{1}{(2k)^s} = 2^{1-s}\sum_{n=1}^{\infty}\frac{1}{n^s},$$

i.e.,

$$\zeta(s) = \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} = \frac{1}{1-2^{1-s}}\eta(s). \tag{12}$$

- The Dirichlet eta function is the function $\eta(s)$ defined by an alternating series:

$$\eta(s) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}, \qquad \forall\ s:\ Re(s) > 0.$$

This series in accordance with [8, §3, p. 29] converges $\forall\ s:\ Re\,(s) > 0$.

- And the formula (12) is true for $\forall\ s:\ Re\,(s) > 0,\ s \neq 1$.
- Lots of numbers type

$$p_1^{k_1} p_2^{k_2} * \cdots * p_{\pi(X)}^{k_{\pi(X)}}, \qquad 0 \leqslant k_i \leqslant \log_{p_i} X,\ \ 1 \leqslant i \leqslant \pi(X),$$

where $p_1, p_2, \ldots, p_n, \ldots$ - is a series of primes and $\pi(X)$ is the prime counting function:

$$\pi(X) = \sum_{p_n \leqslant X} 1,$$

in accordance with the main theorem of arithmetic on decomposition of natural numbers into the product of the powers of prime numbers contains all natural numbers less than or equal to $p_{\pi([X])+1} - 1$ exactly once.

- For arbitrary positive real numbers X, define a function $\forall\ s:\ Re\,(s) > 0$:

$$\eta_X\,(s) \stackrel{\text{def}}{=} \sum\nolimits_{n=1,\ n=p_1^{k_1} p_2^{k_2} * \cdots * p_{\pi(X)}^{k_{\pi(X)}},\ k_i \in \mathbb{N}_0}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

For $\forall\ s:\ Re(s) > 0$ is executed:

$$\eta_X\,(s) = \sum_{n=1,\ n=p_1^{k_1} p_2^{k_2} * \cdots * p_{\pi(X)}^{k_{\pi(X)}},\ k_i \in \mathbb{N}_0}^{\infty} \frac{1}{n^s} - \sum_{n=1,\ n=p_1^{k_1} p_2^{k_2} * \cdots * p_{\pi(X)}^{k_{\pi(X)}},\ k_i \in \mathbb{N}_0,\ k_1 \in \mathbb{N}_1}^{\infty} \frac{2}{n^s},$$

- I.e., the first sum of the cost components of type

$$\frac{1}{p_1^{k_1 s} p_2^{k_2 s} * \cdots * p_{\pi(X)}^{k_{\pi(X)} s}}, \qquad k_i \in \mathbb{N}_0,$$

and in the second - double composed with an even index n:

$$\frac{1}{p_1^{k_1 s} p_2^{k_2 s} * \cdots * p_{\pi(X)}^{k_{\pi(X)} s}}, \qquad k_2, \ldots, k_{\pi(X)} \in \mathbb{N}_0,\ \ k_1 \in \mathbb{N}_1.$$

That can be written as:

$$\eta_X(s) = \left(1 - \frac{2}{2^s}\right) \sum\nolimits_{n=1,\ n=p_1^{k_1} p_2^{k_2} * \cdots * p_{\pi(X)}^{k_{\pi(X)}},\ k_i \in \mathbb{N}_0}^{\infty} \frac{1}{n^s} =$$

$$= \left(1 - \frac{2}{2^s}\right) \prod\nolimits_{p_n \leqslant X} \left(1 - \frac{1}{p_n^s}\right)^{-1}. \qquad (13)$$

- For an arbitrary positive real number X define function $\forall\ s:\ Re(s) > 0,\ s \neq 1$:

$$\zeta_X(s) \stackrel{\text{def}}{=} \frac{1}{1 - 2^{1-s}} \eta_X(s).$$

- I.e., $\forall\ s:\ Re(s) > 0,\ s \neq 1$ and arbitrary fixed $X > 0$:

$$\zeta_X(s) = \prod\nolimits_{p_n \leqslant X} \left(1 - \frac{1}{p_n^s}\right)^{-1}. \qquad (14)$$

- STATEMENT B

For any value of the argument: $s:\ Re(s) > 0$ function $\eta_X(s)$ has a limit when $X \to \infty$ and it is:

$$\lim_{X \to \infty} \eta_X(s) = \eta(s), \quad \forall\ s:\ Re(s) > 0.$$

PROOF:

- For any $s:\ Re(s) > 1$ this statement follows from the definition of an infinite product, taking into account (1), (12), (13).

Let's consider $\forall\ s:\ Re(s) > 0$ a difference $\eta(s)$ and $\eta_X(s)$, denoting its:

$$\phi_X(s) \stackrel{\text{def}}{=} \eta(s) - \eta_X(s).$$

The function $\phi_X(s)$ is defined and analytic $\forall\ s:\ Re(s)>0$.

- Consequently $\forall\ s_0:\ Re(s_0)>0$ function $\phi_X(s)$ is displayed in Taylor's number:
$$\phi_X(s)=\sum_{k=0}^{\infty}\frac{\phi_X(s_0)^{(k)}}{k!}(s-s_0)^k.$$
Limit $\forall\ s:\ Re(s)>1$:
$$\lim_{X\to\infty}\phi_X(s)=0.$$
I.e., $\forall\ k\geqslant 0:$
$$\lim_{X\to\infty}\frac{\phi_X(s_0)^{(k)}}{k!}=0.$$
Consequently $\forall\ s:\ Re(s)>0$:
$$\lim_{X\to\infty}\phi_X(s)=0.$$
$\square$

- This in turn means that $\forall\ s:\ Re(s)>0,\ s\neq 1$:
$$\lim_{X\to\infty}\zeta_X(s)=\zeta(s). \tag{15}$$
And in particular, because $\forall\ 0<r\leqslant R_m:\ \ \zeta(m_r)\neq 0,\ \ \zeta(Re(m_r))\neq 0,$ $\zeta(1-m_r)\neq 0,\ \ \zeta(Re(1-m_r))\neq 0$:
$$\lim_{X\to\infty}\ln\|\zeta_X(m_r)\,\zeta_X(Re(m_r))\|=\ln\|\zeta(m_r)\,\zeta(Re(m_r))\|,$$
$$\lim_{X\to\infty}\ln\|\zeta_X(1-m_r)\,\zeta_X(Re(1-m_r))\|=\ln\|\zeta(1-m_r)\,\zeta(Re(1-m_r))\|.$$

- STATEMENT C

The limit of a private derivative on axis of ordinates of function

$$f_X(x,y) \stackrel{\text{def}}{=} \ln \|\zeta_X\,(x+iy)\,\zeta_X\,(x)\|$$

exists and is equal to a private derivative on a variable x to function

$$f(x,y) \stackrel{\text{def}}{=} \lim_{X\to\infty} f_X(x,y) = \ln \|\zeta\,(x+iy)\,\zeta\,(x)\|$$

in points (x_{m_r}, y_{m_r}) and $(1-x_{m_r}, -y_{m_r})$:

$$\lim_{X\to\infty} \frac{\partial}{\partial x} f_X(x, y_{m_r})\Big|_{x=x_{m_r}} = \frac{\partial}{\partial x} f(x, y_{m_r})\Big|_{x=x_{m_r}},$$

$$\lim_{X\to\infty} \frac{\partial}{\partial x} f_X(x, -y_{m_r})\Big|_{x=1-x_{m_r}} = \frac{\partial}{\partial x} f(x, -y_{m_r})\Big|_{x=1-x_{m_r}}.$$

PROOF:

- Since the function $\zeta\,(x+iy)$ is analytic, there are neighborhoods $U(x_{m_r})$ and $U(1-x_{m_r})$ of points x_{m_r} and $1-x_{m_r}$ for which is carried out:

$$\forall\; x \in U(x_{m_r}),\; x \in U(1-x_{m_r}),\; y = y_{m_r},\; y = -y_{m_r}:$$

$$\|\zeta\,(x+iy)\,\zeta\,(x)\| \neq 0.$$

And taking into account (15):

$$\forall\; x \in U(x_{m_r}),\; x \in U(1-x_{m_r}),\; y = y_{m_r},\; y = -y_{m_r},$$

$$\exists\; X_0 > 0: \quad \forall\; X > X_0:$$

$$\|\zeta_X\,(x+iy)\,\zeta_X\,(x)\| \neq 0.$$

Consequently all functions $f_X(x, y_{m_r})$, $f_X(x, -y_{m_r})$ at $X > X_0$ and $f(x, y_{m_r})$, $f(x, -y_{m_r})$ are correctly certain in neighborhoods $U(x_{m_r})$ and

$U(1-x_{m_r})$ accordingly.

From the fact that the derivative:

$$\frac{\partial}{\partial x} f(x_{m_r}, y_{m_r}) = \frac{\partial}{\partial x} \ln \|\zeta\left(x_{m_r} + iy_{m_r}\right) \zeta\left(x_{m_r}\right)\| =$$
$$= Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)} + Re\frac{\zeta'\left(x_{m_r}\right)}{\zeta\left(x_{m_r}\right)}, \tag{16}$$

in accordance with (11) limited for $\forall\ 0 < r \leqslant R_m$ should the existence of a neighborhood $U^*(x_{m_r}) \in U(x_{m_r})$ such that for $\forall\ x \in U^*(x_{m_r})$ will be limited to the derivative:

$$\left|\frac{\partial}{\partial x} f(x, y_{m_r})\right| < \infty.$$

- Based on the mean value theorem:

$$\forall\ \Delta x > 0 : x_{m_r} + \Delta x \in U^*(x_{m_r}),$$
$$\exists\ 0 < \theta_1 < 1,\ \ 0 < \theta_2 < 1 :$$

$$\frac{f_X(x_{m_r} + \Delta x, y_{m_r}) - f_X(x_{m_r}, y_{m_r})}{\Delta x} = \frac{\partial}{\partial x} f_X(x_{m_r} + \theta_1 \Delta x, y_{m_r})$$

and

$$\frac{f(x_{m_r} + \Delta x, y_{m_r}) - f(x_{m_r}, y_{m_r})}{\Delta x} = \frac{\partial}{\partial x} f(x_{m_r} + \theta_2 \Delta x, y_{m_r}).$$

- From the definition of the limit it follows that:

$$\forall\ \varepsilon > 0,\ \exists\ \ X_1 > X_0 > 0 :\ \forall\ X > X_1 :$$
$$|f(x_{m_r}, y_{m_r}) - f_X(x_{m_r}, y_{m_r})| < \frac{\varepsilon}{2}\Delta x,$$
$$|f(x_{m_r} + \Delta x, y_{m_r}) - f_X(x_{m_r} + \Delta x, y_{m_r})| < \frac{\varepsilon}{2}\Delta x.$$

I.e., $\exists\, X_1 \geqslant X_0 : \ \forall\, X > X_1$ the derivative of function $f_X(x, y_{m_r})$ also will be limited:

$$\left|\frac{\partial}{\partial x} f(x, y_{m_r})\right| < \infty, \quad \forall\, x \in U^*(x_{m_r})$$

and

$$\left|\frac{\partial}{\partial x} f(x_{m_r} + \theta_2 \Delta x, y_{m_r}) - \frac{\partial}{\partial x} f_X(x_{m_r} + \theta_1 \Delta x, y_{m_r})\right| < \varepsilon.$$

Because $\Delta x > 0$ can be chosen arbitrarily small, when $\Delta x \to 0$ have:

$$\left|\frac{\partial}{\partial x} f(x_{m_r}, y_{m_r}) - \frac{\partial}{\partial x} f_X(x_{m_r}, y_{m_r})\right| \leqslant \varepsilon,$$

this proves the statement for the point (x_{m_r}, y_{m_r}).

In a similar way it is possible to lead the same reasonings and for the point $(1 - x_{m_r}, -y_{m_r})$.

$\square$

- STATEMENT D

Since some instant, the sum of private derivatives on axis of ordinates of function $f_X(x, y)$ in points (x_{m_r}, y_{m_r}) and $(1 - x_{m_r}, -y_{m_r})$ slightly different from 0, i.e.:

$$\forall\, \varepsilon > 0, \ \exists \ \ X_\varepsilon > 0 : \ \forall\, X > X_\varepsilon :$$
$$\left|\frac{\partial}{\partial x} f_X(x_{m_r}, y_{m_r}) + \frac{\partial}{\partial x} f_X(1 - x_{m_r}, -y_{m_r})\right| < \varepsilon.$$

PROOF:

From the previous statement it follows that $\forall\ \varepsilon > 0,\ \exists\quad X_\varepsilon > 0$: $\forall\ X > X_\varepsilon$:

$$\left|\frac{\partial}{\partial x}f(x_{m_r}, y_{m_r}) - \frac{\partial}{\partial x}f_X(x_{m_r}, y_{m_r})\right| < \frac{\varepsilon}{2}$$

and

$$\left|\frac{\partial}{\partial x}f(1-x_{m_r}, -y_{m_r}) - \frac{\partial}{\partial x}f_X(1-x_{m_r}, -y_{m_r})\right| < \frac{\varepsilon}{2}.$$

And taking into account (16) and the same equality:

$$\frac{\partial}{\partial x}f(1-x_{m_r}, -y_{m_r}) = \frac{\partial}{\partial x}\ln\|\zeta\left(1-x_{m_r}-iy_{m_r}\right)\zeta\left(1-x_{m_r}\right)\| =$$
$$= -Re\frac{\zeta'\left(1-m_r\right)}{\zeta\left(1-m_r\right)} - Re\frac{\zeta'\left(1-x_{m_r}\right)}{\zeta\left(1-x_{m_r}\right)}.$$

it follows that:

$$\left|Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)} + Re\frac{\zeta'\left(x_{m_r}\right)}{\zeta\left(x_{m_r}\right)} - \frac{\partial}{\partial x}f_X(x_{m_r}, y_{m_r})\right| < \frac{\varepsilon}{2}$$

and

$$\left|Re\frac{\zeta'\left(1-m_r\right)}{\zeta\left(1-m_r\right)} + Re\frac{\zeta'\left(1-x_{m_r}\right)}{\zeta\left(1-x_{m_r}\right)} + \frac{\partial}{\partial x}f_X(1-x_{m_r}, -y_{m_r})\right| < \frac{\varepsilon}{2}.$$

And from (11):

$$\left|\frac{\partial}{\partial x}f_X(x_{m_r}, y_{m_r}) + \frac{\partial}{\partial x}f_X(1-x_{m_r}, -y_{m_r})\right| < \varepsilon.$$

□

- Note that:

$$\frac{\partial}{\partial x}f_X(x_{m_r}, y_{m_r}) = Re\frac{{\zeta_X}'\left(m_r\right)}{\zeta_X\left(m_r\right)} + Re\frac{{\zeta_X}'\left(x_{m_r}\right)}{\zeta_X\left(x_{m_r}\right)},$$

$$\frac{\partial}{\partial x} f_X(1-x_{m_r}, -y_{m_r}) = -Re\frac{\zeta_X{}'(1-m_r)}{\zeta_X(1-m_r)} - Re\frac{\zeta_X{}'(1-x_{m_r})}{\zeta_X(1-x_{m_r})}.$$

- And also from (14) for $s = m_r, \quad s = 1-m_r, \quad s = x_{m_r}, \quad s = 1-x_{m_r}$:

$$Re\frac{\zeta_X{}'(s)}{\zeta_X(s)} = Re\sum_{p_n \leqslant X} \frac{\dfrac{\ln p_n}{p_n^s}}{\left(1-\dfrac{1}{p_n^s}\right)} = Re\sum_{p_n \leqslant X}\sum_{k=1}^{\infty}\frac{\ln p_n}{p_n^{ks}}. \tag{17}$$

- STATEMENT E

In an arbitrarily small neighborhood of any nontrivial zero, there is a point with a real part equal to $\frac{1}{2}$.

$$\forall\, q \in \mathcal{P},$$
$$\exists\, 0 < R_m \leqslant R: \ \ \forall\, 0 < r \leqslant R_m \ \ \exists\, m_r: \|m_r - q\| = r, \ Re(m_r) \leqslant Re(q),$$
$$m_r = \frac{1}{2}.$$

PROOF:

From the previous statement, taking into account (17), we have:

$$\forall\, \varepsilon > 0, \ \exists \ X_\varepsilon > 0: \ \forall\, X > X_\varepsilon:$$

$$\left| Re\sum_{p_n \leqslant X}\sum_{k=1}^{\infty}\left(\frac{\ln p_n}{p_n^{km_r}} + \frac{\ln p_n}{p_n^{kx_{m_r}}} - \frac{\ln p_n}{p_n^{k(1-m_r)}} - \frac{\ln p_n}{p_n^{k(1-x_{m_r})}}\right)\right| < \varepsilon.$$

Or:

$$\sum_{p_n \leqslant X}\sum_{k=1}^{\infty}\ln p_n\,(1+\cos(ky_{m_r}\ln p_n))\left|\frac{1}{p_n^{kx_{m_r}}} - \frac{1}{p_n^{k(1-x_{m_r})}}\right| < \varepsilon.$$

Let's consider, that $X_\varepsilon > 3$, then at the same time two sums cannot be equal to 0:

$$1+\cos(y_{m_r}\ln 2),\quad 1+\cos(y_{m_r}\ln 3),$$

- because otherwise there would be two integers $m_1, m_2 \in \mathbb{Z}$:

$$y_{m_r}\ln 2 = \pi + 2\pi m_1,\quad y_{m_r}\ln 3 = \pi + 2\pi m_2.$$

And given the fact that $y_{m_r} \neq 0$:

$$\frac{\ln 3}{\ln 2} = \frac{1+2m_2}{1+2m_1}.$$

Since $\dfrac{\ln 3}{\ln 2} > 0$ should exist non-negative m_1 and m_2:

$$3^{1+2m_1} = 2^{1+2m_2}.$$

- That is impossible, since the left part of equality always odd, and right - even.

For definiteness, we assume that:

$$1+\cos(y_{m_r}\ln 2) > 0,$$

- then, assuming:

$$\frac{1}{2^{x_{m_r}}} - \frac{1}{2^{(1-x_{m_r})}} \neq 0,$$

as ε take:

$$\varepsilon = \frac{1}{2}\ln 2\,(1+\cos(y_{m_r}\ln 2))\left|\frac{1}{2^{x_{m_r}}} - \frac{1}{2^{(1-x_{m_r})}}\right| > 0.$$

- Let's come to the contradiction:

$$\sum_{p_n\leqslant X}\sum_{k=1}^{\infty}\ln p_n\,(1+\cos(ky_{m_r}\ln p_n))\left|\frac{1}{p_n^{kx_{m_r}}} - \frac{1}{p_n^{k(1-x_{m_r})}}\right| > \varepsilon,\quad \forall\ X > X_\varepsilon.$$

I.e.,

$$\frac{1}{2^{x_{m_r}}} = \frac{1}{2^{(1-x_{m_r})}},$$

that is equivalent to:

$$x_{m_r} = \frac{1}{2}.$$

□

- Thus, we took a random nontrivial root $q = \sigma_q + it_q \in \mathcal{P}$ and concluded that:

$$\sigma_q = \lim_{r \to 0} x_{m_r} = \frac{1}{2},$$

i.e., $\mathcal{P}_1 = \mathcal{P}_3 = \varnothing$ and

$$\mathcal{P} = \mathcal{P}_2,$$

that proves the basic statement and the assumption, which had been made by Bernhard Riemann about of the real parts of the nontrivial zeros of Zeta function.

Список литературы

[1] Abramowitz, M. and Stegun, I. A. (Eds.). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, 1972.

[2] Davenport, H. Multiplicative Number Theory, 2nd ed. New York: Springer-Verlag, 1980.

[3] Edwards, H. M. Riemann's Zeta Function. New York: Dover, 1974.

[4] Karatsuba, A. A. and Voronin, S. M. The Riemann Zeta-Function. Hawthorn, NY: de Gruyter, 1992.

[5] Keiper, J. B. "Power Series Expansions of Riemann's xi Function."Math. Comput. 58, 765-773, 1992.

[6] Lehmer, D. H. "The Sum of Like Powers of the Zeros of the Riemann Zeta Function."Math. Comput. 50, 265-273, 1988.

[7] Smith, D. E. A Source Book in Mathematics. New York: Dover, 1994.

[8] Titchmarsh, E. C. The Theory of the Riemann Zeta Function, 2nd ed. New York: Clarendon Press, 1987.

[9] Valle-Poussin C.J. De la Recherches analytiques sur la theorie des wombres partie I,II,III, Am. Soc. Sci. Bruxelles, Ser. A. - 1896-1897, t. 20-21.

[10] Voros, A. "Spectral Functions, Special Functions and the Selberg Zeta Function."Commun. Math. Phys. 110, 439-465, 1987.