# The real parts of the nontrivial Riemann zeta function zeros

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ABSTRACT

This theorem is based on holomorphy of studied functions and the fact that near a singularity point the real part of some rational function can take an arbitrary preassigned value.

The colored markers are as follows:

● - assumption or a fact which is not proven at present;

● - the statement which requires additional attention;

● - statement which is proved earlier or clearly undestandable.

THEOREM

● The real parts of all the nontrivial Riemann zeta function zeros $\rho$ are equal $Re\left(\rho\right) = \dfrac{1}{2}$.

PROOF:

● According to the functional equality [10, p. 22], [6, p. 8-11]:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta\left(s\right) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta\left(1-s\right), \qquad Re\left(s\right) > 0 \qquad (1)$$

$\zeta\left(s\right)$ - the Riemann zeta function, $\Gamma\left(s\right)$ - the Gamma function.

- From [6, p. 8-11] $\zeta(\bar{s}) = \overline{\zeta(s)}$, it means that $\forall \rho = \sigma + it$: $\zeta(\rho) = 0$ and $0 \leqslant \sigma \leqslant 1$ we have:

$$\zeta(\bar{\rho}) = \zeta(1-\rho) = \zeta(1-\bar{\rho}) = 0 \qquad (2)$$

- From [11], [9, p. 128], [10, p. 45] we know that $\zeta(s)$ has no nontrivial zeros on the line $\sigma = 1$ and consequently on the line $\sigma = 0$ also, in accordance with (2) they don't exist.

- Let's denote the set of nontrivial zeros $\zeta(s)$ through $\mathcal{P}$ (multiset with consideration of multiplicitiy):

$$\mathcal{P} \overset{\text{def}}{=} \{\rho : \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ 0 < \sigma < 1\}.$$

$$\text{And: } \mathcal{P}_1 \overset{\text{def}}{=} \left\{\rho : \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ 0 < \sigma < \frac{1}{2}\right\},$$

$$\mathcal{P}_2 \overset{\text{def}}{=} \left\{\rho : \ \zeta(\rho) = 0, \ \rho = \frac{1}{2} + it\right\},$$

$$\mathcal{P}_3 \overset{\text{def}}{=} \left\{\rho : \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ \frac{1}{2} < \sigma < 1\right\}.$$

Then:

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \quad \text{and} \quad \mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P}_2 \cap \mathcal{P}_3 = \mathcal{P}_1 \cap \mathcal{P}_3 = \varnothing,$$

$$\mathcal{P}_1 = \varnothing \Leftrightarrow \mathcal{P}_3 = \varnothing.$$

- Hadamard's theorem (Weierstrass preparation theorem) about the decomposition of function through the roots gives us the following result [10, p. 30], [6, p. 31], [12]:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}} e^{as}}{s(s-1)\Gamma\left(\frac{s}{2}\right)} \prod_{\rho \in \mathcal{P}} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \qquad Re(s) > 0 \qquad (3)$$

$$a = \ln 2\sqrt{\pi} - \frac{\gamma}{2} - 1, \ \gamma - \text{Euler's constant and}$$

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\ln\pi + a - \frac{1}{s} + \frac{1}{1-s} - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \sum_{\rho \in \mathcal{P}} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \qquad (4)$$

- According to the fact that $\dfrac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$ - Digamma function of [10, p. 31], [6, p. 23] we have:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + \sum_{\rho\in\mathcal{P}}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2n}\right) + C, \qquad (5)$$

$$C = const$$

.

- From [5, p. 160], [8, p. 272], [4, p. 81]:

$$\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = 1 + \frac{\gamma}{2} - \ln 2\sqrt{\pi} = 0,0230957\ldots \qquad (6)$$

- Indeed, from (2):

$$\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = \frac{1}{2}\sum_{\rho\in\mathcal{P}}\left(\frac{1}{1-\rho} + \frac{1}{\rho}\right).$$

- From (4):

$$2\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = \lim_{s\to 1}\left(\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{1-s} + \frac{1}{s} - a - \frac{1}{2}\ln\pi + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right).$$

- Also it's known, for example, from [10, p. 49], [4, p. 98] that the number of nontrivial zeros of $\rho = \sigma + it$ in strip $0 < \sigma < 1$, the imaginary parts of which $t$ are less than some number $T > 0$ is limited, i.e.

$$\|\{\rho:\ \rho\in\mathcal{P},\ \rho = \sigma + it,\ |t| < T\}\| < \infty.$$

- Indeed, it can be presented that on the contrary the sum of $\sum_{\rho\in\mathcal{P}}\frac{1}{\rho}$ would have been unlimited.

- Thus $\forall T > 0\ \exists\ \delta_x > 0,\ \delta_y > 0$ such that

  in area $0 < t \leqslant \delta_y, 0 < \sigma \leqslant \delta_x$ there are no zeros $\rho = \sigma + it \in \mathcal{P}$.

Let's consider random root $q \in \mathcal{P}_1 \cup \mathcal{P}_2$

Let's denote $k(q)$ the multiplicity of the root $q$.

Let's examine the area $Q\left(R\right) \overset{\text{def}}{=} \{s : \|s - q\| \leqslant R, R > 0\}$.

- From the fact of finiteness of set of nontrivial zeros $\zeta(s)$ in the limited area follows $\exists\ R > 0$, such that $Q(R)$ does not contain any root from $\mathcal{P}$ except $q$.

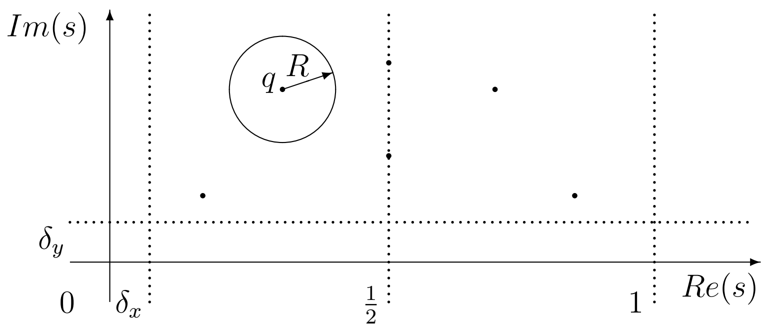


Fig. 1.

- From [1], [10, p. 31], [6, p. 23] we know that the Digamma function $\dfrac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$ in the area $Q(R)$ has no poles, i.e. $\forall s \in Q(R)$

$$\left\|\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right\| < \infty.$$

Let's denote:

$$I_{\mathcal{P}}(s) \overset{\text{def}}{=} -\frac{1}{s} + \frac{1}{1-s} + \sum_{\rho \in \mathcal{P}} \frac{1}{s-\rho}$$

and

$$I_{\mathcal{P}\setminus\{q\}}(s) \overset{\text{def}}{=} -\frac{1}{s} + \frac{1}{1-s} + \sum_{\rho \in \mathcal{P}\setminus\{q\}} \frac{1}{s-\rho}.$$

Hereinafter $\mathcal{P} \setminus \{q\} \overset{\text{def}}{=} \mathcal{P} \setminus \{(q, k(q))\}$ (the difference in the multiset).

Also we shall consider the summation - $\sum_{\rho\in\mathcal{P}}\dfrac{1}{s-\rho}$ and $\sum_{\rho\in\mathcal{P}\setminus\{q\}}\dfrac{1}{s-\rho}$ further as the sum of pairs $\left(\dfrac{1}{s-\rho}+\dfrac{1}{s-(1-\rho)}\right)$ and $\sum_{\rho\in\mathcal{P}}\dfrac{1}{\rho}$ as the sum of pairs $\left(\dfrac{1}{\rho}+\dfrac{1}{1-\rho}\right)$ as a consequence of division of the sum from (5) $\sum_{\rho\in\mathcal{P}}\left(\dfrac{1}{s-\rho}+\dfrac{1}{\rho}\right)$ into $\sum_{\rho\in\mathcal{P}}\dfrac{1}{s-\rho}+\sum_{\rho\in\mathcal{P}}\dfrac{1}{\rho}$. As specifed in [5], [7], [8], [10].

- Let's note that $I_{\mathcal{P}\setminus\{q\}}(s)$ is holomorphic function $\forall\ s\in Q(R)$.

Then from (4) we have:

$$\frac{\zeta'(s)}{\zeta(s)}=\frac{1}{2}\ln\pi+a-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}+\sum_{\rho\in\mathcal{P}}\frac{1}{\rho}+I_{\mathcal{P}}(s).$$

And in view of (3), (6):

$$Re\frac{\zeta'(s)}{\zeta(s)}=\frac{1}{2}\ln\pi+Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}+I_{\mathcal{P}}(s)\right). \tag{7}$$

Let's note that from the equality of

$$\sum_{\rho\in\mathcal{P}}\frac{1}{1-s-\rho}=-\sum_{(1-\rho)\in\mathcal{P}}\frac{1}{s-(1-\rho)}=-\sum_{\rho\in\mathcal{P}}\frac{1}{s-\rho} \tag{8}$$

follows that:

$$I_{\mathcal{P}}(1-s)=-I_{\mathcal{P}}(s),\ I_{\mathcal{P}\setminus\{q\}}(1-s)=-I_{\mathcal{P}\setminus\{1-q\}}(s),\ Re\,(s)>0.$$

- Besides

$$I_{\mathcal{P}\setminus\{q\}}(s)=I_{\mathcal{P}}(s)-\frac{k(q)}{s-q}$$

and $I_{\mathcal{P}\setminus\{q\}}(s)$ is limited in the area of $s\in Q(R)$ as a result of absence of its poles in this area as well as its differentiability in each point of this area.

- If in (4) we replace $s$ with $1-s$ that in view of (6), in a similar way if we take derivative of the basic logarithm (1):

$$\frac{\zeta'(s)}{\zeta(s)}+\frac{\zeta'(1-s)}{\zeta(1-s)}=-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}+\ln\pi,\ \ Re(s)>0. \quad (9)$$

- Let's examine a circle with the center in a point $q$ and radius $r \leqslant R$, laying in the area of $Q(R)$:

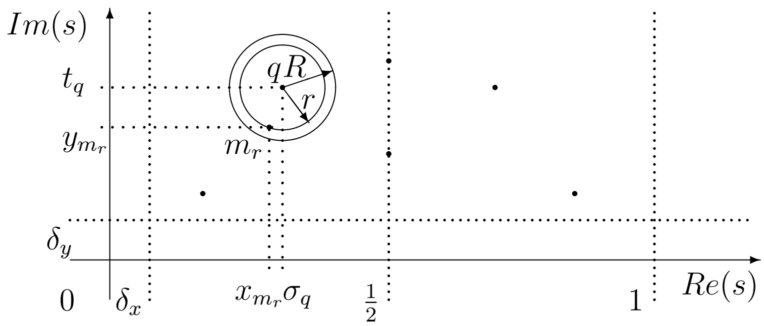


Fig. 2.

- For $s=x+iy,\ q=\sigma_q+it_q$

$$Re\frac{k(q)}{s-q}=Re\frac{k(q)}{x+iy-\sigma_q-it_q}=\frac{k(q)(x-\sigma_q)}{(x-\sigma_q)^2+(y-t_q)^2}=k(q)\frac{x-\sigma_q}{r^2}.$$

- Let us prove the following Lemma:

LEMMA 1

$$\forall\, q\in\mathcal{P}$$

$$\exists\, 0<R_q\leqslant R: \ \ \forall\, 0<r\leqslant R_q\ \ \exists\, m_r: \|m_r-q\|=r,\ Im(m_r)\leqslant Im(q),$$

- $$Re\frac{\zeta'(m_r)}{\zeta(m_r)}-Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}=0 \quad (10)$$

And for the angle $\beta_{m_r}$ between the ordinate axis and the straight line passing through the points $q$ and $m_r$, the following equality holds:

$$\operatorname{tg}\beta_{m_r}=O(r)_{r\to 0}. \quad (11)$$

PROOF:

For $s \in Q(R)$ we consider the function:

$$Re\frac{\zeta'(s)}{\zeta(s)} - Re\frac{\zeta'(1-s)}{\zeta(1-s)} - 2Re\frac{k(q)}{s-q}$$

From (7) and (8), it is equal to:

$$Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} + 2I_{\mathcal{P}\setminus\{q\}}(s)\right).$$

Since all components of the brace are limited in the area of $s \in Q(R)$, then $\exists\ H_1(R) > 0 : H_1(R) \in \mathbb{R}$:

$$\left|Re\frac{\zeta'(s)}{\zeta(s)} - Re\frac{\zeta'(1-s)}{\zeta(1-s)} - Re\frac{2k(q)}{s-q}\right| < H_1(R), \qquad \forall s \in Q(R).$$

- On each of the semicircles: the bottom semicircle -

  $\{s : \|s-q\| = r,\ t_q - r \leqslant y \leqslant t_q\}$ and the upper semicircle -

  $\{s : \|s-q\| = r,\ t_q \leqslant y \leqslant t_q + r\}$ the function $Re\dfrac{k(q)}{s-q}$ is continuous and takes values from $-\dfrac{k(q)}{r}$ to $\dfrac{k(q)}{r}$, $r > 0$.

  Consequently $\forall\ 0 < r < \dfrac{2k(q)}{H_1(R)}$, $\exists\ m_{min,r},\ m_{max,r}$ :

  $\|m_{min,r} - q\| = r,\ \|m_{max,r} - q\| = r$ :

$$Re\frac{2k(q)}{m_{min,r}-q} < -H_1(R),\ Re\frac{2k(q)}{m_{max,r}-q} > H_1(R)$$

and the sum of two functions:

$$Re\frac{\zeta'(s)}{\zeta(s)} - Re\frac{\zeta'(1-s)}{\zeta(1-s)} - Re\frac{2k(q)}{s-q} \qquad \text{and} \qquad Re\frac{2k(q)}{s-q}$$

in points $m_{min,r}$ and $m_{max,r}$ will have values with a different signs.

From the property of a continuous function on a segment taking all the intermediate values between its extrema, it follows that $\exists\ R_q\ \in\ \mathbb{R}$, $R_q > 0:$

$$R_q < R,\ \frac{2k(q)}{R_q} > H_1(R)$$

and then $\forall\ 0 < r \leqslant R_q$
exists on the lower semicircle point $m_r \stackrel{\text{def}}{=} x_{m_r} + iy_{m_r}$ such as that:

$$Re\frac{\zeta'(m_r)}{\zeta(m_r)} - Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)} = 0.$$

- From (9) and (10) it follows that $\forall\ 0 < r \leqslant R_q$:

$$Re\frac{\zeta'(m_r)}{\zeta(m_r)} = Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)} =$$
$$= \frac{1}{2}\ln\pi + \frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{m_r}{2}\right)}{\Gamma\left(\frac{m_r}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-m_r}{2}\right)}{\Gamma\left(\frac{1-m_r}{2}\right)}\right). \tag{12}$$

- I.e. taking into account the absence of singular points for $\Gamma(s)$, $\ \forall\ s \in Q(R)$ for $r \to 0$:

$$Re\frac{\zeta'(m_r)}{\zeta(m_r)} = Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)} = O(1). \tag{13}$$

Point $m_r$:

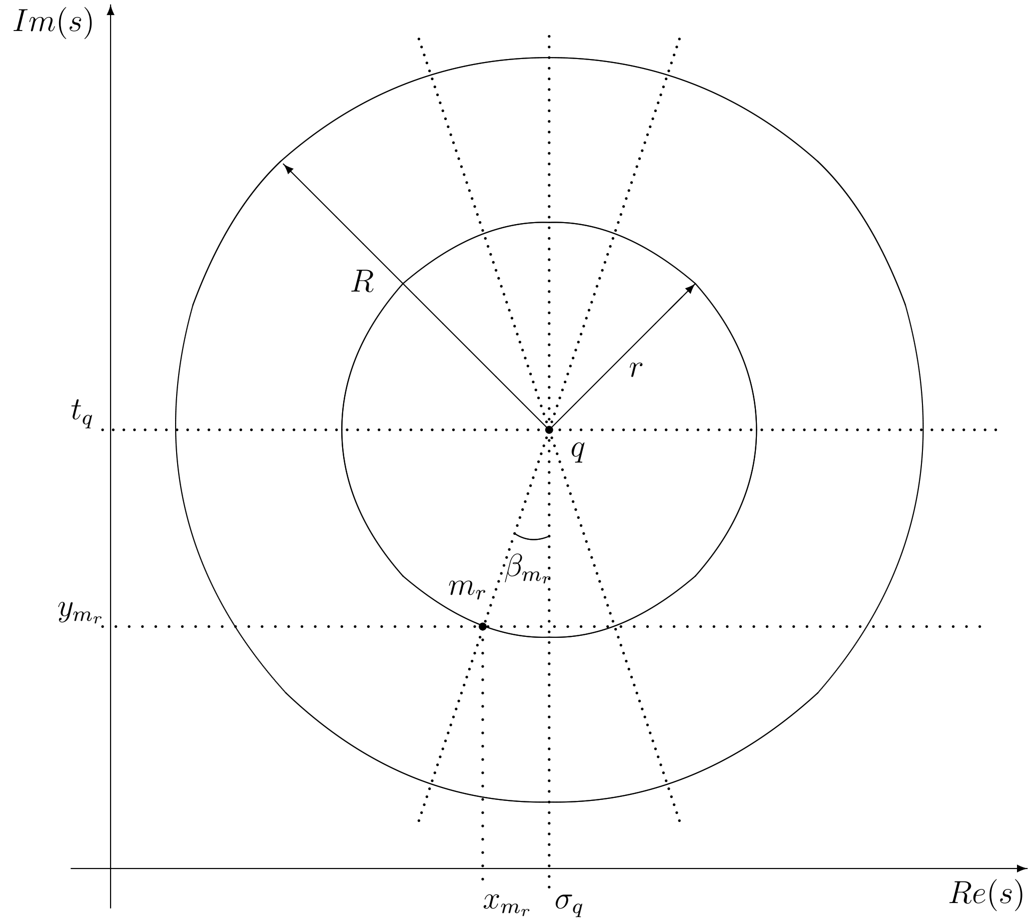


Fig. 3.

- In the case if $y_{m_r} \neq t_q$ the tangent modulus of the angle $\beta_{m_r}$ is equal to:

$$|\mathrm{tg}\,\beta_{m_r}| = \frac{|\sigma_q - x_{m_r}|}{t_q - y_{m_r}}.$$

From (7) it follows that:

$$k(q)\frac{x_{m_r} - \sigma_q}{r^2} = Re\frac{\zeta'(m_r)}{\zeta(m_r)} - \frac{1}{2}\ln\pi - Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{m_r}{2}\right)}{\Gamma\left(\frac{m_r}{2}\right)} + I_{\mathcal{P}\setminus\{q\}}(m_r)\right).$$

In view of (13) at $r \to 0$:

$$\frac{x_{m_r} - \sigma_q}{r^2} = O(1).$$

Then from equality:

$$(\sigma_q - x_{m_r})^2 + (t_q - y_{m_r})^2 = r^2$$

it follows that when $r \to 0$:

$$(t_q - y_{m_r})^2 = r^2 - O(r^4).$$

- I.e. $\exists\, 0 < R_1 \leqslant R_q : \;\; \forall\, 0 < r < R_1$

$$t_q - y_{m_r} \neq 0$$

and therefore $r \to 0$:

$$\operatorname{tg} \beta_{m_r} = \frac{O(r^2)}{\theta(r)} = O(r).$$

$\square$

- Let's prove the second Lemma:

LEMMA 2

$$\forall\, q \in \mathcal{P}$$

-

$$Re \left( \frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} \right)' = Re \left( \frac{\Gamma'\left(\dfrac{1-q}{2}\right)}{\Gamma\left(\dfrac{1-q}{2}\right)} \right)'.$$

PROOF:

- From the first Lemma $\forall \;\; 0 < r \leqslant R_q$, for $s = x + iy : \;\; \|s - q\| = r$ let's consider the function:

$$g(x, y) \stackrel{\text{def}}{=} Re \frac{\zeta'(s)}{\zeta(s)} Re \frac{\zeta'(1-s)}{\zeta(1-s)}.$$

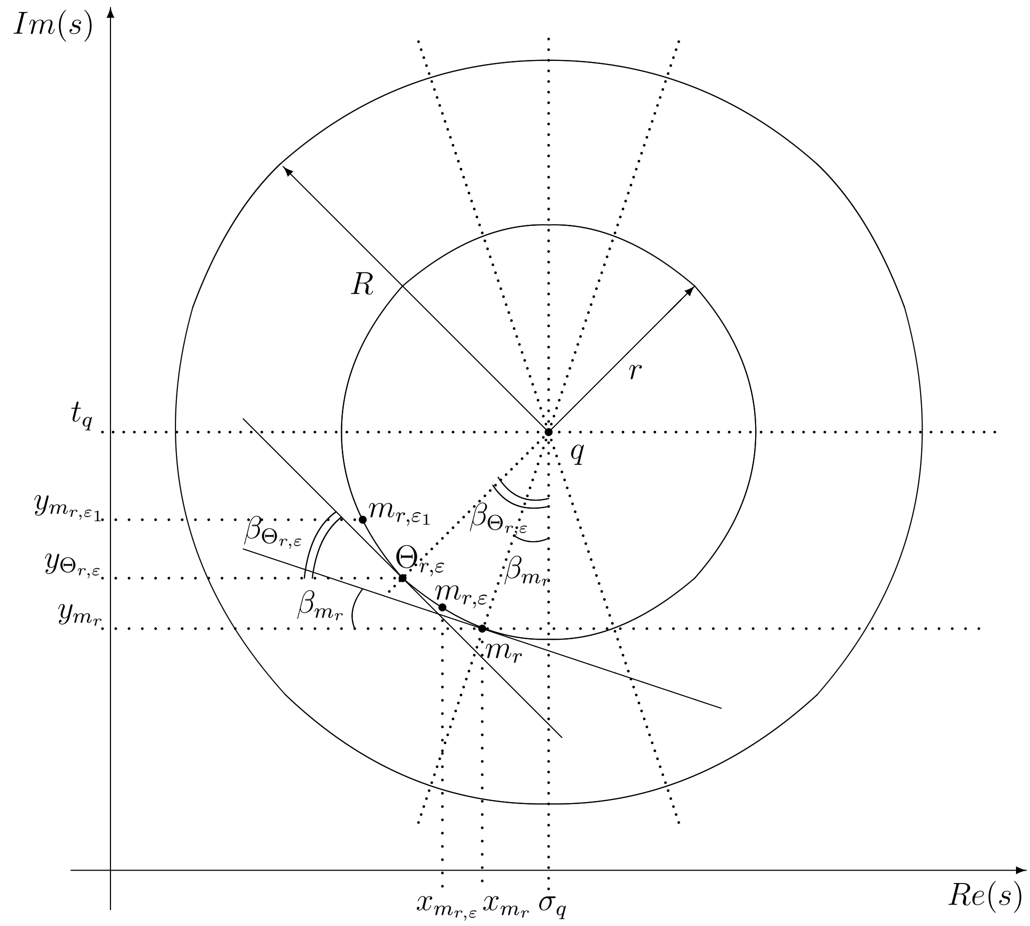


Fig. 4.

For arbitrary $\varepsilon,\ \varepsilon_1 > 0$, taking into account that the function $Re\dfrac{k(q)}{s-q}$ is continuous and takes values from $-\dfrac{k(q)}{r}$ to $\dfrac{k(q)}{r}$, there must exist a radius $0 < R_2 \leqslant R_1 : \ \forall\, 0 < r \leqslant R_2 : \ \exists\, m_{r,\varepsilon},\ m_{r,\varepsilon_1} :$

$$Re\frac{\zeta'\left(m_{r,\varepsilon}\right)}{\zeta\left(m_{r,\varepsilon}\right)} = Re\frac{\zeta'\left(m_{r}\right)}{\zeta\left(m_{r}\right)} - \varepsilon, \quad Re\frac{\zeta'\left(1-m_{r,\varepsilon_1}\right)}{\zeta\left(1-m_{r,\varepsilon_1}\right)} = Re\frac{\zeta'\left(m_{r}\right)}{\zeta\left(m_{r}\right)} + \varepsilon_1. \quad (14)$$

Let's designate $\forall\ s \in Q(R)$:

$$\alpha(s) \overset{\text{def}}{=} \frac{1}{2}\ln\pi + \frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\right).$$

- From (12) follows:

$$Re\frac{\zeta'(m_r)}{\zeta(m_r)} = Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)} = \frac{1}{2}\left(Re\frac{\zeta'(m_r)}{\zeta(m_r)} + Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}\right) = \alpha(m_r),$$

which means taking into account (9) and (14):

$$\begin{aligned} Re\frac{\zeta'(1-m_{r,\varepsilon})}{\zeta(1-m_{r,\varepsilon})} &= 2\alpha(m_{r,\varepsilon}) - Re\frac{\zeta'(m_r)}{\zeta(m_r)} + \varepsilon = \\ &= Re\frac{\zeta'(m_r)}{\zeta(m_r)} + \varepsilon + 2\alpha(m_{r,\varepsilon}) - 2\alpha(m_r), \\ Re\frac{\zeta'(m_{r,\varepsilon_1})}{\zeta(m_{r,\varepsilon_1})} &= 2\alpha(m_{r,\varepsilon_1}) - Re\frac{\zeta'(m_r)}{\zeta(m_r)} - \varepsilon_1 = \\ &= Re\frac{\zeta'(m_r)}{\zeta(m_r)} - \varepsilon_1 + 2\alpha(m_{r,\varepsilon_1}) - 2\alpha(m_r). \end{aligned} \tag{15}$$

Let's designate:

$$x_{m_{r,\varepsilon}} + iy_{m_{r,\varepsilon}} \stackrel{\text{def}}{=} m_{r,\varepsilon}, \quad x_{m_{r,\varepsilon_1}} + iy_{m_{r,\varepsilon_1}} \stackrel{\text{def}}{=} m_{r,\varepsilon_1}.$$

- The points $m_{r,\varepsilon}$ and $m_{r,\varepsilon_1}$ lie on a circle with center at the point $q$ and radius $r$, i.e. all the points $s = x + iy$ of the smallest of the arcs that connects them satisfy the equation:

$$y = f_r(x) \stackrel{\text{def}}{=} t_q - \sqrt{r^2 - (\sigma_q - x)^2}.$$

And:

$$f_r(x_{m_{r,\varepsilon}}) = y_{m_{r,\varepsilon}}, \; f_r(x_{m_{r,\varepsilon_1}}) = y_{m_{r,\varepsilon_1}}.$$

- Function $g(x, y)$ is differentiated, so function $g(x, f_r(x))$ is continuous and differentiated on $x$.

Let's designate $\forall\ s \in Q(R)$:

$$\omega(s) \stackrel{\text{def}}{=} \frac{1}{2}\ln\pi + Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + I_{\mathcal{P}\setminus\{q\}}(s)\right).$$

- From (7) and (14), because of the continuity of the function $Re\dfrac{\zeta'\left(x+if_r(x)\right)}{\zeta\left(x+if_r(x)\right)}$ for $\forall\ x \in (\sigma_q - R, \sigma_q + R)$, based on the mean value theorem for $r \to 0$:

$$
\begin{gathered}
-k(q)\frac{x_{m_r} - x_{m_{r,\varepsilon_1}}}{r^2} = -\varepsilon_1 - \omega(1-m_r) + \omega(1-m_{r,\varepsilon_1}) = \\
= -\varepsilon_1 + O(x_{m_r} - x_{m_{r,\varepsilon_1}}), \\
k(q)\frac{x_{m_r} - x_{m_{r,\varepsilon}}}{r^2} = \varepsilon - \omega(x_{m_r}) + \omega(x_{m_{r,\varepsilon}}) = \\
= \varepsilon + O(x_{m_r} - x_{m_{r,\varepsilon}}).
\end{gathered} \tag{16}
$$

- Thus $\forall\forall\ \varepsilon,\ \varepsilon_1 > 0,\ \ \exists\, 0 < R_3 \leqslant R_2 :\ \forall\, 0 < r \leqslant R_3$:

$$\lim_{\varepsilon\to 0} m_{r,\varepsilon} = m_r,\ \ \lim_{\varepsilon_1\to 0} m_{r,\varepsilon_1} = m_r.$$

- Thus, a real function that is continuous and differentiable on the inner interval takes on the values on its ends:

$$
\begin{gathered}
g(x_{m_{r,\varepsilon_1}}, f_r(x_{m_{r,\varepsilon_1}})) = \\
= \left(Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)} + \varepsilon_1\right)\left(Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)} - \varepsilon_1 + 2\alpha(m_{r,\varepsilon_1}) - 2\alpha(m_r)\right) = \\
= \left(Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)}\right)^2 - \varepsilon_1^2 + 2\left(Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)} + \varepsilon_1\right)(\alpha(m_{r,\varepsilon_1}) - \alpha(m_r)),
\end{gathered} \tag{17}
$$

$$
\begin{gathered}
g(x_{m_{r,\varepsilon}}, f_r(x_{m_{r,\varepsilon}})) = \\
= \left(Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)} - \varepsilon\right)\left(Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)} + \varepsilon + 2\alpha(m_{r,\varepsilon}) - 2\alpha(m_r)\right) = \\
= \left(Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)}\right)^2 - \varepsilon^2 + 2\left(Re\frac{\zeta'\left(m_r\right)}{\zeta\left(m_r\right)} - \varepsilon\right)(\alpha(m_{r,\varepsilon}) - \alpha(m_r)).
\end{gathered} \tag{18}
$$

- Let's consider equality:

$$g(x_{m_{r,\varepsilon}}, f_r(x_{m_{r,\varepsilon}})) = g(x_{m_{r,\varepsilon_1}}, f_r(x_{m_{r,\varepsilon_1}})). \tag{19}$$

- On the basis of the Lagrange theorem about the average value $\forall\forall\ \varepsilon, \varepsilon_1 > 0$, $\forall\ 0 < r \leqslant R_3$ on the arc described by $f_r(x)$ in interval $(x_{m_{r,\varepsilon_1}}, x_{m_r})$, there is a point $\Upsilon_{r,\varepsilon_1} \overset{\text{def}}{=} x_{\Upsilon_{r,\varepsilon_1}} + if_r(x_{\Upsilon_{r,\varepsilon_1}})$, for which it is true:

$$\alpha(m_{r,\varepsilon_1}) - \alpha(m_r) = \alpha'_x(x + if_r(x))_{x=x_{\Upsilon_{r,\varepsilon_1}}}(x_{m_{r,\varepsilon_1}} - x_{m_r}),$$

similarly in the interval $(x_{m_{r,\varepsilon}}, x_{m_r})\ \exists\ \Upsilon_{r,\varepsilon} \overset{\text{def}}{=} x_{\Upsilon_{r,\varepsilon}} + if_r(x_{\Upsilon_{r,\varepsilon}}):$

$$\alpha(m_{r,\varepsilon}) - \alpha(m_r) = \alpha'_x(x + if_r(x))_{x=x_{\Upsilon_{r,\varepsilon}}}(x_{m_{r,\varepsilon}} - x_{m_r}).$$

Also in the same intervals there are two more points:
$\varkappa_{r,\varepsilon_1} \overset{\text{def}}{=} x_{\varkappa_{r,\varepsilon_1}} + if_r(x_{\varkappa_{r,\varepsilon_1}})$ and $\varkappa_{r,\varepsilon} \overset{\text{def}}{=} x_{\varkappa_{r,\varepsilon}} + if_r(x_{\varkappa_{r,\varepsilon}})$,
$x_{\varkappa_{r,\varepsilon_1}} \in (x_{m_{r,\varepsilon_1}}, x_{m_r})$, $\ x_{\varkappa_{r,\varepsilon}} \in (x_{m_{r,\varepsilon}}, x_{m_r}):$

$$\omega(1 - m_r) - \omega(1 - m_{r,\varepsilon_1}) = \omega'_x(1 - x - if_r(x))_{x=x_{\varkappa_{r,\varepsilon_1}}}(x_{m_r} - x_{m_{r,\varepsilon_1}}),$$
$$\omega(m_r) - \omega(m_{r,\varepsilon}) = \omega'_x(x + if_r(x))_{x=x_{\varkappa_{r,\varepsilon}}}(x_{m_r} - x_{m_{r,\varepsilon}}).$$

- And (16) will be as follows:

$$(x_{m_r} - x_{m_{r,\varepsilon_1}})\left(\frac{k(q)}{r^2} + \omega'_x(1 - x - if_r(x))_{x=x_{\varkappa_{r,\varepsilon_1}}}\right) = \varepsilon_1,$$

$$(x_{m_r} - x_{m_{r,\varepsilon}})\left(\frac{k(q)}{r^2} + \omega'_x(x + if_r(x))_{x=x_{\varkappa_{r,\varepsilon}}}\right) = \varepsilon.$$

Or:

$$\begin{aligned} x_{m_r} - x_{m_{r,\varepsilon_1}} &= \frac{\varepsilon_1 r^2}{k(q) + r^2\omega'_x(1 - x - if_r(x))_{x=x_{\varkappa_{r,\varepsilon_1}}}}, \\ x_{m_r} - x_{m_{r,\varepsilon}} &= \frac{\varepsilon r^2}{k(q) + r^2\omega'_x(x + if_r(x))_{x=x_{\varkappa_{r,\varepsilon}}}}. \end{aligned} \tag{20}$$

- And then the equality (19) will look like (17) and (18) as follows:

$$-\varepsilon^2+\varepsilon\frac{2r^2\left(Re\dfrac{\zeta'(m_r)}{\zeta(m_r)}-\varepsilon\right)\alpha'_x(x+if_r(x))_{x=x_{\Upsilon_{r,\varepsilon}}}}{k(q)+r^2\omega'_x(x+if_r(x))_{x=x_{\varkappa r,\varepsilon}}}=$$

$$=-\varepsilon_1^2+\varepsilon_1\frac{2r^2\left(Re\dfrac{\zeta'(m_r)}{\zeta(m_r)}+\varepsilon_1\right)\alpha'_x(x+if_r(x))_{x=x_{\Upsilon_{r,\varepsilon_1}}}}{k(q)+r^2\omega'_x(1-x-if_r(x))_{x=x_{\varkappa r,\varepsilon_1}}}.$$

- Or:

$$A_{r,\varepsilon}\varepsilon^2-B_{r,\varepsilon}\varepsilon=A_{r,\varepsilon_1}\varepsilon_1^2-B_{r,\varepsilon_1}\varepsilon_1. \tag{21}$$

So from (21) it is visible, that $\exists\, 0<R_4\leqslant R_3:\ \forall\, 0<r\leqslant R_4$:

$$A_{r,\varepsilon}\overset{\text{def}}{=}1+\frac{2r^2\alpha'_x(x+if_r(x))_{x=x_{\Upsilon_{r,\varepsilon}}}}{k(q)+r^2\omega'_x(x+if_r(x))_{x=x_{\varkappa r,\varepsilon}}}>0$$

and

$$A_{r,\varepsilon_1}\overset{\text{def}}{=}1-\frac{2r^2\alpha'_x(x+if_r(x))_{x=x_{\Upsilon_{r,\varepsilon_1}}}}{k(q)+r^2\omega'_x(1-x-if_r(x))_{x=x_{\varkappa r,\varepsilon_1}}}>0,$$

as well as:

$$k(q)+r^2\omega'_x(x+if_r(x))_{x=x_{\varkappa r,\varepsilon}}>0$$

and

$$k(q)+r^2\omega'_x(1-x-if_r(x))_{x=x_{\varkappa r,\varepsilon_1}}>0.$$

Where:

$$B_{r,\varepsilon}\overset{\text{def}}{=}\frac{2r^2Re\dfrac{\zeta'(m_r)}{\zeta(m_r)}\alpha'_x(x+if_r(x))_{x=x_{\Upsilon_{r,\varepsilon}}}}{k(q)+r^2\omega'_x(x+if_r(x))_{x=x_{\varkappa r,\varepsilon}}},$$

$$B_{r,\varepsilon_1}\overset{\text{def}}{=}\frac{2r^2Re\dfrac{\zeta'(m_r)}{\zeta(m_r)}\alpha'_x(x+if_r(x))_{x=x_{\Upsilon_{r,\varepsilon_1}}}}{k(q)+r^2\omega'_x(1-x-if_r(x))_{x=x_{\varkappa r,\varepsilon_1}}}.$$

- Let's assume that:

$$\alpha(q)\frac{\partial}{\partial x}\alpha(q) \neq 0. \tag{22}$$

- Then, taking into account the existence of a two-dimensional neighborhood of the point $q$ in which the continuous function of two variables $\alpha(x+iy)\frac{\partial}{\partial x}\alpha(x+iy)$ preserves the sign, and also that :

$$\frac{d}{dx}\alpha(x+if_r(x)) = \frac{\partial}{\partial x}\alpha(x+if_r(x)) + \frac{d}{dx}f_r(x)\frac{\partial}{\partial y}\alpha(x+iy)_{y=f_r(x)}$$

and in accordance with (20) $\forall\, x \in (\min(x_{m_{r,\varepsilon_1}}, x_{m_{r,\varepsilon}}), \max(x_{m_{r,\varepsilon_1}}, x_{m_{r,\varepsilon}}))$:

$$\frac{d}{dx}f_r(x) = \frac{\sigma_q - x}{\sqrt{r^2 - (\sigma_q - x)^2}} = O(r)_{r\to 0}.$$

- We have: $\exists\, 0 < R_5 \leqslant R_4 :\ \forall\, 0 < r \leqslant R_5$:

$$\alpha(x+if_r(x)) \neq 0,\ \ \frac{d}{dx}\alpha(x+if_r(x)) \neq 0,\ \ \forall\, x \in [\sigma_q - R_5, \sigma_q + R_5]. \tag{23}$$

- Let's notice, that in the assumption (22), $\forall\, 0 < r \leqslant R_5$ factors $B_{r,\varepsilon_1}$ and $B_{r,\varepsilon}$ are not equal 0 and have one sign.

For resolvability of the equation (21) it is enough to us to show a continuity $\alpha(m_{r,\varepsilon})$ on $\varepsilon$. Really in this case, in view of (18), the left part of equality (21) will be continuous on $\varepsilon$.

And then $\forall\ \varepsilon_1 > 0$ at $\varepsilon \to 0$ there would be a value $\varepsilon$ such, that the left part (21) is less on the module of value in the right part, as well as $\exists\ \Delta_1 > 0 :\ \ \forall\, 0 < \varepsilon_1 \leqslant \Delta_1$ at $\varepsilon \to \infty$, there would be a value of a variable at which the module of the left part is more the than module right and both parts have one sign.

Consequently, in view of a continuity, between the specified values of parameter there should be a point which is a root of the equation (21) concerning a variable $\varepsilon$, for fixed $0 < \varepsilon_1 \leqslant \Delta_1$.

The continuity of $\alpha(m_{r,\varepsilon})$ with respect to $\varepsilon$ follows from the continuity of the function $\alpha(s)$ for $\forall\ s \in Q(R)$ and the continuity of $m_{r,\varepsilon}$ in $\varepsilon$ because the equation (16) can be written as follows:

$$k(q)\frac{x_{m_r} - \sigma_q - h_r(\tau)}{r^2} = -\tau - \omega(m_r) + \omega(\sigma_q + h_r(\tau) + i(t_q - \sqrt{r^2 - h_r(\tau)^2})),$$

where $h_r(\tau)$ it is defined from equality:

$$h_r(\varepsilon) \stackrel{\text{def}}{=} x_{m_{r,\varepsilon}} - \sigma_q. \tag{24}$$

Or:

$$k(q)\frac{\sigma_q + h_r(\tau) - x_{m_r}}{r^2} - \omega(m_r) + \omega(\sigma_q + h_r(\tau) + i(t_q - \sqrt{r^2 - h_r(\tau)^2})) = \tau.$$

- Those the function $h_r(\tau)$ is the inverse of the function:

$$k(q)\frac{\sigma_q + \tau - x_{m_r}}{r^2} - \omega(m_r) + \omega(\sigma_q + \tau + i(t_q - \sqrt{r^2 - \tau^2})). \tag{25}$$

- It follows from the inverse function theorem that if a function is defined, continuous and strictly monotone on some interval, then it has an inverse function that is continuous and strictly monotone on the corresponding interval, the image of the initial interval.

Let's look at a derivative of function from (25):

$$\frac{k(q)}{r^2} + \omega'_x(\sigma_q + \tau + i(t_q - \sqrt{r^2 - \tau^2})) + \frac{\tau}{\sqrt{r^2 - \tau^2}}\omega'_y(\sigma_q + \tau + i(t_q - \sqrt{r^2 - \tau^2})).$$

From (20) follows, that $\forall\ \tau > 0,\ r \to 0$:

$$h_r(\tau) = O(r^2).$$

I.e. area of values of argument $\tau$ in (25) at $r \to 0$:

$$\tau = O(r^2).$$

Then $\exists\ 0 < R_6 \leqslant R_5 :\ \forall\ 0 < r \leqslant R_6$ and for any argument $\tau$ possible in (25) the derivative of this function will be strictly positive.

I.e. function from (25) is continuous and strictly increasing on all interval of definition of the argument.

- This means that $\forall\ 0 < r \leqslant R_6,\ \exists\ \Delta_1 > 0 :\ \ \forall\ 0 < \varepsilon_1 \leqslant \Delta_1,\ \exists\ \varepsilon > 0$ is executed (19):

$$g(x_{m_{r,\varepsilon}}, f_r(x_{m_{r,\varepsilon}})) = g(x_{m_{r,\varepsilon_1}}, f_r(x_{m_{r,\varepsilon_1}})).$$

- Let's assume, that:

$$x_{m_{r,\varepsilon_1}} = x_{m_{r,\varepsilon}}. \tag{26}$$

Then from (17), (18), (19), in view of equality $\alpha(m_{r,\varepsilon_1}) = \alpha(m_{r,\varepsilon})$:

$$\varepsilon^2 - \varepsilon_1^2 + 2\,(\varepsilon + \varepsilon_1)\,(\alpha(m_{r,\varepsilon_1}) - \alpha(m_r)) = 0.$$

I.e.

$$\varepsilon - \varepsilon_1 = 2(\alpha(m_r) - \alpha(m_{r,\varepsilon_1})). \tag{27}$$

And from (15):

$$Re\frac{\zeta'\,(m_{r,\varepsilon_1})}{\zeta\,(m_{r,\varepsilon_1})} = Re\frac{\zeta'\,(m_r)}{\zeta\,(m_r)} - \varepsilon_1 + 2\alpha(m_{r,\varepsilon_1}) - 2\alpha(m_r) = Re\frac{\zeta'\,(m_r)}{\zeta\,(m_r)} - \varepsilon.$$

I.e.

$$Re\frac{\zeta'\,(m_{r,\varepsilon_1})}{\zeta\,(m_{r,\varepsilon_1})} = Re\frac{\zeta'\,(m_{r,\varepsilon})}{\zeta\,(m_{r,\varepsilon})}.$$

- That means according to (24), (25), (26) that:

$$\sigma_q + h_r(\varepsilon) = \sigma_q + h_r(\varepsilon_1) \Leftrightarrow h_r(\varepsilon) = h_r(\varepsilon_1).$$

And in view of a continuity and strict monotony $h_r(\varepsilon)$:

$$\varepsilon = \varepsilon_1.$$

And then from (27):

$$\alpha(m_r) - \alpha(m_{r,\varepsilon_1}) = 0,$$

that contradicts strict monotony of function $\alpha(s)$ according to (20) and (23).

- Hence the assumption of (26) is false, that is:

$$\forall\, 0 < r \leqslant R_6, \;\; \exists\, 0 < \varDelta \leqslant \varDelta_1 : \; \forall\, 0 < \varepsilon_1 < \varDelta, \; \exists\, \varepsilon > 0 : \quad m_{r,\varepsilon_1} \neq m_{r,\varepsilon}$$

and

$$g(x_{m_{r,\varepsilon}}, f_r(x_{m_{r,\varepsilon}})) = g(x_{m_{r,\varepsilon_1}}, f_r(x_{m_{r,\varepsilon_1}})).$$

I.e. continuous and differentiable on the inner interval, the real function takes on its ends the same values.

- By Rolle's theorem about the extremum of a differentiable function on an interval, we have:

$$\exists\, x_{\Theta_{r,\varepsilon_1}} : \quad (g(x, f_r(x)))'_{x = x_{\Theta_{r,\varepsilon_1}}} = 0. \tag{28}$$

Where:

$$\Theta_{r,\varepsilon_1} \overset{\text{def}}{=} (x_{\Theta_{r,\varepsilon_1}}, f_r(x_{\Theta_{r,\varepsilon_1}})).$$

From (21) follows:

$$\varepsilon = o(1)_{\varepsilon_1 \to 0}.$$

And taking into account that the value of $x_{\Theta_{r,\varepsilon_1}}$ lies between $x_{m_{r,\varepsilon_1}}$ and $x_{m_{r,\varepsilon}}$, we have:

$$\lim_{\varepsilon_1 \to 0} \Theta_{r,\varepsilon_1} = m_r.$$

Let $\beta_{\Theta_{r,\varepsilon_1}}$ be the angle between the ordinate axis and the line passing through the points $q$ and $\Theta_{r,\varepsilon_1}$.

- Also:

$$\lim_{\varepsilon_1 \to 0} \beta_{\Theta_{r,\varepsilon_1}} = \beta_{m_r}$$

and in view of infinite differentiability of function $Re\dfrac{\zeta\prime\,(x+if_r(x))}{\zeta\,(x+if_r(x))}$ for $\forall\ x$ between $x_{m_{r,\varepsilon_1}}$ and $x_{m_{r,\varepsilon}}$, i.e. to appropriating continuity of derivative function $g(x, f_r(x))$, from equality (28) follows:

$$(g(x, f_r(x)))'_{x=x_{m_r}} = 0.$$

- This equality, taking into account the fact that the angle $\beta_{m_r}$ between the axis of ordinates and the line passing through the points $q$ and $m_r$ coincides with the angle of inclination of the tangent passing through the point $m_r$, can be written as follows:

$$(g(x, f_r(x)))'_{x=x_{m_r}} =$$

$$= \frac{d}{dx}\left(Re\frac{\zeta'\,(x+if_r(x))}{\zeta\,(x+if_r(x))}Re\frac{\zeta'\,(1-x-if_r(x))}{\zeta\,(1-x-if_r(x))}\right)_{x=x_{m_r}} =$$

$$\begin{aligned} = Re\frac{\zeta'\,(m_r)}{\zeta\,(m_r)}\left(Re\frac{\zeta'\,(m_r)}{\zeta\,(m_r)}\right)'_x - Re\frac{\zeta'\,(1-m_r)}{\zeta\,(1-m_r)}\left(Re\frac{\zeta'\,(1-m_r)}{\zeta\,(1-m_r)}\right)'_x - \\ - \operatorname{tg}\beta_{m_r}\left(Re\frac{\zeta'\,(m_r)}{\zeta\,(m_r)}\left(Re\frac{\zeta'\,(m_r)}{\zeta\,(m_r)}\right)'_y - \right. \\ \left. -Re\frac{\zeta'\,(1-m_r)}{\zeta\,(1-m_r)}\left(Re\frac{\zeta'\,(1-m_r)}{\zeta\,(1-m_r)}\right)'_y\right) = 0. \end{aligned} \tag{29}$$

- And taking into account (10), (9):

$$Re\frac{\zeta'(m_r)}{\zeta(m_r)} = Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)},$$

$$\left(Re\frac{\zeta'(m_r)}{\zeta(m_r)}\right)'_x - \left(Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}\right)'_x =$$
$$= \left(Re\frac{\zeta'(m_r)}{\zeta(m_r)} + Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}\right)'_x =$$
$$= \frac{\partial}{\partial x}\left(Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} + \ln\pi\right)\right)_{s=m_r} =$$
$$= Re\frac{d}{ds}\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{m_r}{2}\right)}{\Gamma\left(\frac{m_r}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-m_r}{2}\right)}{\Gamma\left(\frac{1-m_r}{2}\right)}\right),$$

$$\left(Re\frac{\zeta'(m_r)}{\zeta(m_r)}\right)'_y - \left(Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}\right)'_y =$$
$$= \left(Re\frac{\zeta'(m_r)}{\zeta(m_r)} + Re\frac{\zeta'(1-m_r)}{\zeta(1-m_r)}\right)'_y =$$
$$= \frac{\partial}{\partial y}\left(Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} + \ln\pi\right)\right)_{s=m_r} =$$
$$= Im\frac{d}{ds}\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{m_r}{2}\right)}{\Gamma\left(\frac{m_r}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-m_r}{2}\right)}{\Gamma\left(\frac{1-m_r}{2}\right)}\right).$$

- Thus, the equality (29) can be written as follows:

$$(g(x, f_r(x)))'_{x=x_{m_r}} =$$
$$= Re\frac{\zeta'(m_r)}{\zeta(m_r)}\left(Re\frac{d}{ds}\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{m_r}{2}\right)}{\Gamma\left(\frac{m_r}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-m_r}{2}\right)}{\Gamma\left(\frac{1-m_r}{2}\right)}\right)-\right.$$
$$\left.-\operatorname{tg}\beta_{m_r}Im\frac{d}{ds}\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{m_r}{2}\right)}{\Gamma\left(\frac{m_r}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-m_r}{2}\right)}{\Gamma\left(\frac{1-m_r}{2}\right)}\right)\right)=0.$$

- And taking into account (11), (12) as well as the presence of the last equality of finite limits for all the terms at $r \to 0$ we get:

$$0 = \lim_{r\to 0}(g(x, f_r(x)))'_{x=x_{m_r}} =$$
$$= \left(\frac{1}{2}\ln\pi + \frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)}\right)\right)*$$
$$*\left(Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)}\right)'\right) = \frac{1}{2}\alpha(q)\frac{\partial}{\partial x}\alpha(q).$$

- This contradicts the assumption that (22), i.e.:

$$\alpha(q)\frac{\partial}{\partial x}\alpha(q) = 0.$$

What is equivalent:

$$\left(\frac{1}{2}\ln\pi+\frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)}\right)\right)*$$
$$*\left(Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)}\right)'\right)=0. \qquad (30)$$

Taking into account (4), (5) and the formula of the Digamma function from [1, p.259 §6.3.16] we estimate the first factor:

$$\frac{1}{2}\ln\pi+\frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)}\right)=$$
$$=\frac{1}{2}Re\left(\ln\pi+\frac{\gamma}{2}+\frac{1}{q}+\sum_{n=1}^{\infty}\left(\frac{1}{q+2n}-\frac{1}{2n}\right)+\right.$$
$$\left.+\frac{\gamma}{2}+\frac{1}{1-q}+\sum_{n=1}^{\infty}\left(\frac{1}{1-q+2n}-\frac{1}{2n}\right)\right)=$$

$$=\frac{1}{2}\left(\ln\pi+\gamma+\frac{\sigma_q}{\sigma_q^2+t_q^2}+\sum_{n=1}^{\infty}\left(\frac{2n+\sigma_q}{(2n+\sigma_q)^2+t_q^2}-\frac{1}{2n}\right)+\right.$$
$$\left.+\frac{1-\sigma_q}{(1-\sigma_q)^2+t_q^2}+\sum_{n=1}^{\infty}\left(\frac{2n+1-\sigma_q}{(2n+1-\sigma_q)^2+t_q^2}-\frac{1}{2n}\right)\right). \qquad (31)$$

Let's note that the derivative of the function:

$$\frac{1}{2}\ln\pi + \frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{x+iy}{2}\right)}{\Gamma\left(\frac{x+iy}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-x-iy}{2}\right)}{\Gamma\left(\frac{1-x-iy}{2}\right)}\right)$$

along the ordinate axis for any fixed $0 < x \leqslant \frac{1}{2}$ and $y > 0$ is negative:

$$\frac{\partial}{\partial y}\left(\frac{1}{2}\ln\pi + \frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{x+iy}{2}\right)}{\Gamma\left(\frac{x+iy}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-x-iy}{2}\right)}{\Gamma\left(\frac{1-x-iy}{2}\right)}\right)\right) =$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{\partial}{\partial y}\left(\frac{2n+x}{(2n+x)^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{2n+1-x}{(2n+1-x)^2+y^2}\right)\right) =$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{2(2n+x)y}{((2n+x)^2+y^2)^2} + \frac{2(2n+1-x)y}{((2n+1-x)^2+y^2)^2}\right) < 0.$$

- Therefore, if the left-hand side of the equality (31) is negative for numbers of the form $q_0 = \sigma_0 + it_0$, where $t_0 > 0$ is fixed and $0 < \sigma_0 \leqslant \frac{1}{2}$ is arbitrarily chosen, then it will be negative for any $q = \sigma_q + it_q : \; t_q \geqslant t_0, \;\; 0 < \sigma_q \leqslant \frac{1}{2}$.

Consider $q_0 = \sigma_0 + 8i,\ \ 0 < \sigma_0 \leqslant \frac{1}{2}$, then from (31) will follow:

$$\frac{1}{2}\ln\pi + \frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q_0}{2}\right)}{\Gamma\left(\frac{q_0}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-q_0}{2}\right)}{\Gamma\left(\frac{1-q_0}{2}\right)}\right) =$$

$$= \frac{1}{2}\Bigg(\ln\pi + \gamma + \frac{1-\sigma_0}{(1-\sigma_0)^2+8^2} + \frac{\sigma_0}{\sigma_0^2+8^2} +$$

$$+ \textstyle\sum_{n=1}^{\infty}\left(\frac{2n+\sigma_0}{(2n+\sigma_0)^2+8^2} - \frac{1}{2n}\right) +$$

$$+ \textstyle\sum_{n=1}^{\infty}\left(\frac{2n+1-\sigma_0}{(2n+1-\sigma_0)^2+8^2} - \frac{1}{2n}\right)\Bigg) <$$

$$< \frac{1}{2}\left(\ln\pi + \gamma + \frac{1}{8^2} + \textstyle\sum_{n=1}^{\infty}\left(\frac{2n+\frac{1}{2}}{(2n)^2+8^2} - \frac{1}{2n}\right) +\right.$$

$$\left. + \textstyle\sum_{n=1}^{\infty}\left(\frac{2n+1}{(2n)^2+8^2} - \frac{1}{2n}\right)\right) =$$

$$= \frac{1}{2}\left(\ln\pi + \gamma + \frac{1}{8^2} + \textstyle\sum_{n=1}^{\infty}\left(\frac{n-8^2}{2n((2n)^2+8^2)} + \frac{2n-8^2}{2n((2n)^2+8^2)}\right)\right) =$$

$$= \frac{1}{2}\left(\ln\pi + \gamma + \frac{1}{64} + \textstyle\sum_{n=1}^{\infty}\left(\frac{3}{8(n^2+16)} - \frac{16}{n(n^2+16)}\right)\right). \qquad (32)$$

From [1, p.259], [2, § 6.495] :

$$y\sum_{n=1}^{\infty}\frac{1}{n^2+y^2} = -\frac{1}{2y} + \frac{\pi}{2}\coth\pi y.$$

Consequently:

$$\sum_{n=1}^{\infty}\frac{1}{n^2+16} = -\frac{1}{32} + \frac{\pi}{8}\coth 4\pi = 0,3614490... \qquad (33)$$

- The remaining amount in the (32) is estimated for the first nine terms:

$$\sum_{n=1}^{\infty}\frac{16}{n(n^2+16)} > \sum_{n=1}^{9}\frac{16}{n(n^2+16)} > 1,8873330. \qquad (34)$$

Thus, taking into account (33) and (34) the inequality (32) can be continued:

$$\frac{1}{2}\left(\ln\pi+\gamma+\frac{1}{64}+\sum_{n=1}^{\infty}\left(\frac{3}{8(n^2+16)}-\frac{16}{n(n^2+16)}\right)\right)<$$
$$<\frac{1}{2}\left(1,1447299+0,5772157+0,015625+\frac{3}{8}0,3614491-1,8873330\right)<$$
$$<\frac{1}{2}(1,8731141-1,8873330)<0.$$

I.e. for $\forall\ q=\sigma_q+it_q:\ t_q\geqslant 8,\quad 0<\sigma_q\leqslant\dfrac{1}{2}$ the first multiplier of work from (30) is not equal 0.
And taking into account the symmetry of the values of this factor relative to the line $\sigma_q=\dfrac{1}{2}$ it is not equal to 0 for $\forall\ q=\sigma_q+it_q:\ t_q\geqslant 8,\quad 0<\sigma_q<1.$

Let's estimate the minimal value $t_q>0$.

For $\forall\ \rho=\sigma+it:$

$$\frac{1}{\rho}+\frac{1}{\bar\rho}+\frac{1}{1-\rho}+\frac{1}{1-\bar\rho}=$$
$$=\frac{\sigma}{\sigma^2+t^2}+\frac{\sigma}{\sigma^2+t^2}+\frac{1-\sigma}{(1-\sigma)^2+t^2}+\frac{1-\sigma}{(1-\sigma)^2+t^2}=$$
$$=\frac{2\sigma}{\sigma^2+t^2}+\frac{2(1-\sigma)}{(1-\sigma)^2+t^2}>\frac{2\sigma}{1+t^2}+\frac{2(1-\sigma)}{1+t^2}=\frac{2}{1+t^2}.$$

Let's designate through $t_1\overset{\text{def}}{=}\min_{\rho\in\mathcal{P}}|Im(\rho)|$ then in view of (6):

$$\frac{2}{1+t_1^2}<\sum_{\rho\in\mathcal{P}}\frac{1}{\rho}<0,0230958,$$

i.e.

$$t_1>9,2518015.$$

Thus $\forall\ q \in \mathcal{P}$ multiplier:

$$\frac{1}{2}\ln\pi + \frac{1}{2}Re\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)}\right) \neq 0.$$

Hence the second factor of (30) must be equal to 0, which is equivalent to:

$$Re\left(\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}\right)' = Re\left(\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)}\right)'.$$

$\square$

Let's prove the third Lemma:

LEMMA 3

$$\forall\, s = x + iy,\ 0 < x \leqslant \frac{1}{2},\ y \geqslant 4 :$$

- $$Re\left(\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right)' = Re\left(\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\right)' \Leftrightarrow \tag{35}$$

$$\Leftrightarrow x = \frac{1}{2}.$$

PROOF:

- From (31), the equality (35) can be written as follows:

$$\sum_{n=0}^{\infty}\left(\frac{(2n+x)^2-y^2}{((2n+x)^2+y^2)^2} - \frac{(2n+1-x)^2-y^2}{((2n+1-x)^2+y^2)^2}\right) = 0 \tag{36}$$

In its turn:

$$\textstyle\sum_{n=0}^{\infty}\left(\frac{(2n+x)^2-y^2}{((2n+x)^2+y^2)^2} - \frac{(2n+1-x)^2-y^2}{((2n+1-x)^2+y^2)^2}\right) =$$
$$= \textstyle\sum_{n=0}^{\infty}\left(\frac{1}{(2n+x)^2+y^2} - \frac{1}{(2n+1-x)^2+y^2}\right) -$$
$$-2y^2 \textstyle\sum_{n=0}^{\infty}\left(\frac{1}{((2n+x)^2+y^2)^2} - \frac{1}{((2n+1-x)^2+y^2)^2}\right) =$$

$$= \sum_{n=0}^{\infty} \frac{(1-2x)(4n+1)}{((2n+x)^2+y^2)((2n+1-x)^2+y^2)} -$$
$$-2y^2 \sum_{n=0}^{\infty} \frac{(1-2x)(4n+1)((2n+x)^2+(2n+1-x)^2+2y^2)}{((2n+x)^2+y^2)^2((2n+1-x)^2+y^2)^2} =$$
$$= (1-2x) \left( \sum_{n=0}^{\infty} \frac{4n+1}{((2n+x)^2+y^2)((2n+1-x)^2+y^2)} - \right.$$
$$\left. -2y^2 \sum_{n=0}^{\infty} \frac{(4n+1)((2n+x)^2+(2n+1-x)^2+2y^2)}{((2n+x)^2+y^2)^2((2n+1-x)^2+y^2)^2} \right). \qquad (37)$$

Let's estimate the sum of the general brackets of equality (37):

$$\sum_{n=0}^{\infty} \frac{4n+1}{((2n+x)^2+y^2)((2n+1-x)^2+y^2)} -$$
$$-2y^2 \sum_{n=0}^{\infty} \frac{(4n+1)((2n+x)^2+(2n+1-x)^2+2y^2)}{((2n+x)^2+y^2)^2((2n+1-x)^2+y^2)^2}.$$

From [1, p.259], [2, § 6.495] :

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2+y^2} = \frac{\pi}{4y} \tanh \frac{\pi y}{2},$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2+y^2} = -\frac{1}{2y^2} + \frac{\pi}{4y} \coth \frac{\pi y}{2}.$$

And then the first composed in the considered sum:

$$\sum_{n=0}^{\infty} \frac{4n+1}{((2n+x)^2+y^2)((2n+1-x)^2+y^2)} <$$
$$< \frac{1}{(x^2+y^2)((1-x)^2+y^2)} + \sum_{n=1}^{\infty} \frac{4n+1}{((2n-1)^2+y^2)((2n)^2+y^2)} <$$
$$< \frac{1}{y^4} + \sum_{n=1}^{\infty} \left( \frac{1}{(2n-1)^2+y^2} - \frac{1}{(2n)^2+y^2} \right) +$$
$$+ \sum_{n=1}^{\infty} \frac{2}{((2n-1)^2+y^2)^2}.$$

Here:

$$\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)^2+y^2}-\frac{1}{(2n)^2+y^2}\right)=\frac{\pi}{4y}\tanh\frac{\pi y}{2}-\frac{\pi}{4y}\coth\frac{\pi y}{2}+\frac{1}{2y^2},$$

$$\sum_{n=1}^{\infty}\frac{2}{((2n-1)^2+y^2)^2}=-\frac{1}{y}\frac{d}{dy}\left(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2+y^2}\right)=$$
$$=-\frac{1}{y}\frac{d}{dy}\left(\frac{\pi}{4y}\tanh\frac{\pi y}{2}\right)=\frac{\pi}{4y^3}\tanh\frac{\pi y}{2}-\frac{\pi^2}{8y^2}\frac{1}{\cosh^2\frac{\pi y}{2}}$$

I.e.

$$\sum_{n=0}^{\infty}\frac{4n+1}{((2n+x)^2+y^2)((2n+1-x)^2+y^2)}<$$
$$<\frac{1}{2y^2}+\frac{\pi}{4y^3}\tanh\frac{\pi y}{2}+\frac{1}{y^4}-$$
$$-\frac{\pi}{4y}\left(\coth\frac{\pi y}{2}-\tanh\frac{\pi y}{2}\right)-\frac{\pi^2}{8y^2}\frac{1}{\cosh^2\frac{\pi y}{2}}.$$

The second composed:

$$\sum_{n=0}^{\infty}\frac{(4n+1)((2n+x)^2+(2n+1-x)^2+2y^2)}{((2n+x)^2+y^2)^2((2n+1-x)^2+y^2)^2}=$$
$$=\sum_{n=1}^{\infty}\frac{4n-3}{((2n-2+x)^2+y^2)((2n-1-x)^2+y^2)}*$$
$$*\left(\frac{1}{(2n-2+x)^2+y^2}+\frac{1}{(2n-1-x)^2+y^2}\right)>$$
$$>\sum_{n=1}^{\infty}\frac{4n-1}{((2n-1)^2+y^2)((2n)^2+y^2)}\left(\frac{1}{(2n-1)^2+y^2}+\frac{1}{(2n)^2+y^2}\right)-$$
$$-\sum_{n=1}^{\infty}\frac{2}{((2n-1)^2+y^2)((2n)^2+y^2)}\left(\frac{1}{(2n-1)^2+y^2}+\frac{1}{(2n)^2+y^2}\right)>$$
$$>\sum_{n=1}^{\infty}\left(\frac{1}{((2n-1)^2+y^2)^2}-\frac{1}{((2n)^2+y^2)^2}\right)-$$
$$-\sum_{n=1}^{\infty}\frac{4}{((2n-1)^2+y^2)^3}.$$

Here:

$$\textstyle\sum_{n=1}^{\infty}\left(\frac{1}{((2n-1)^2+y^2)^2}-\frac{1}{((2n)^2+y^2)^2}\right)=$$

$$=-\frac{1}{2y}\frac{d}{dy}\left(\textstyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2+y^2}-\frac{1}{(2n)^2+y^2}\right)=$$

$$=-\frac{1}{2y}\frac{d}{dy}\left(\frac{\pi}{4y}\tanh\frac{\pi y}{2}+\frac{1}{2y^2}-\frac{\pi}{4y}\coth\frac{\pi y}{2}\right)=$$

$$=\frac{\pi}{8y^3}\tanh\frac{\pi y}{2}-\frac{\pi^2}{16y^2}\frac{1}{\cosh^2\frac{\pi y}{2}}+\frac{1}{2y^4}-\frac{\pi}{8y^3}\coth\frac{\pi y}{2}-\frac{\pi^2}{16y^2}\frac{1}{\sinh^2\frac{\pi y}{2}}.$$

And

$$\textstyle\sum_{n=1}^{\infty}\frac{4}{((2n-1)^2+y^2)^3}=$$

$$=-\frac{1}{2y}\frac{d}{dy}\left(-\frac{1}{y}\frac{d}{dy}\left(\textstyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2+y^2}\right)\right)=$$

$$=-\frac{1}{2y}\frac{d}{dy}\left(-\frac{1}{y}\frac{d}{dy}\left(\frac{\pi}{4y}\tanh\frac{\pi y}{2}\right)\right)=$$

$$=-\frac{1}{2y}\frac{d}{dy}\left(\frac{\pi}{4y^3}\tanh\frac{\pi y}{2}-\frac{\pi^2}{8y^2}\frac{1}{\cosh^2\frac{\pi y}{2}}\right)=$$

$$=\frac{3\pi}{8y^5}\tanh\frac{\pi y}{2}-\frac{\pi^2}{16y^4}\frac{1}{\cosh^2\frac{\pi y}{2}}-\frac{\pi^2}{8y^4}\frac{1}{\cosh^2\frac{\pi y}{2}}-\frac{\pi^3}{16y^3}\frac{\tanh\frac{\pi y}{2}}{\cosh^2\frac{\pi y}{2}}=$$

$$=-\frac{\pi^2}{8y^3\cosh^2\frac{\pi y}{2}}\left(\frac{3}{2y}+\frac{\pi}{2}\tanh\frac{\pi y}{2}\right)+\frac{3\pi}{8y^5}\tanh\frac{\pi y}{2}.$$

- Hence:

$$
\begin{gathered}
\sum_{n=0}^{\infty}\frac{4n+1}{((2n+x)^2+y^2)((2n+1-x)^2+y^2)}-\\
-2y^2\sum_{n=0}^{\infty}\frac{(4n+1)((2n+x)^2+(2n+1-x)^2+2y^2)}{((2n+x)^2+y^2)^2((2n+1-x)^2+y^2)^2}<\\
<\frac{1}{2y^2}+\frac{\pi}{4y^3}\tanh\frac{\pi y}{2}+\frac{1}{y^4}-\\
-\frac{\pi}{4y}\left(\coth\frac{\pi y}{2}-\tanh\frac{\pi y}{2}\right)-\frac{\pi^2}{8y^2}\frac{1}{\cosh^2\frac{\pi y}{2}}-\\
-2y^2\left(\frac{\pi}{8y^3}\tanh\frac{\pi y}{2}-\frac{\pi^2}{16y^2}\frac{1}{\cosh^2\frac{\pi y}{2}}+\frac{1}{2y^4}-\right.\\
\left.-\frac{\pi}{8y^3}\coth\frac{\pi y}{2}-\frac{\pi^2}{16y^2}\frac{1}{\sinh^2\frac{\pi y}{2}}\right)-\\
-\left(-\frac{\pi^2}{8y^3\cosh^2\frac{\pi y}{2}}\left(\frac{3}{2y}+\frac{\pi}{2}\tanh\frac{\pi y}{2}\right)+\frac{3\pi}{8y^5}\tanh\frac{\pi y}{2}\right)=
\end{gathered}
$$

$$
\begin{gathered}
=\frac{1}{y^2}\left(-\frac{1}{2}-\frac{\pi^2}{8}\frac{1}{\cosh^2\frac{\pi y}{2}}-\frac{3\pi}{8y^3}\tanh\frac{\pi y}{2}+\right.\\
+\frac{\pi^2}{8}\frac{y^2}{\cosh^2\frac{\pi y}{2}}+\frac{\pi^2}{8}\frac{y^2}{\sinh^2\frac{\pi y}{2}}+\\
\left.+\frac{\pi}{4y}\tanh\frac{\pi y}{2}+\frac{1}{y^2}+\frac{\pi^2}{8y\cosh^2\frac{\pi y}{2}}\left(\frac{3}{2y}+\frac{\pi}{2}\tanh\frac{\pi y}{2}\right)\right). \qquad (38)
\end{gathered}
$$

Let's consider positive composed inside of the general bracket of the right part of an inequality (38) at $y \geqslant 4$:

Derivative:

$$\left(\frac{y^2}{\cosh^2\dfrac{\pi y}{2}}\right)' = y\frac{2\cosh\dfrac{\pi y}{2} - \pi y\sinh\dfrac{\pi y}{2}}{\cosh^3\dfrac{\pi y}{2}} < 0,$$

since for $\forall\, y \geqslant 4$

$$\frac{2}{\pi}\coth\frac{\pi y}{2} < y.$$

Similarly, the derivative:

$$\left(\frac{y^2}{\sinh^2\dfrac{\pi y}{2}}\right)' = y\frac{2\sinh\dfrac{\pi y}{2} - \pi y\cosh\dfrac{\pi y}{2}}{\sinh^3\dfrac{\pi y}{2}} < 0,$$

since for $\forall\, y \geqslant 4$

$$\frac{2}{\pi}\tanh\frac{\pi y}{2} < y.$$

Hence $\forall\, y \geqslant 4$:

$$\frac{\pi^2}{8}\frac{y^2}{\cosh^2\dfrac{\pi y}{2}} \leqslant \frac{\pi^2}{8}\frac{16}{\cosh^2 2\pi} < 0,0002754,$$

$$\frac{\pi^2}{8}\frac{y^2}{\sinh^2\dfrac{\pi y}{2}} \leqslant \frac{\pi^2}{8}\frac{16}{\sinh^2 2\pi} < 0,0002754.$$

Further $\forall\, y \geqslant 4$:

$$\frac{\pi}{4y}\tanh\frac{\pi y}{2} < \frac{\pi}{16} < 0,1963496,$$

$$\frac{1}{y^2} \leqslant 0,0625,$$

$$\frac{\pi^2}{8y\cosh^2\dfrac{\pi y}{2}}\left(\frac{3}{2y} + \frac{\pi}{2}\tanh\frac{\pi y}{2}\right) < \frac{\pi^2}{32\cosh^2 2\pi}\left(\frac{3}{8} + \frac{\pi}{2}\right) < 0,0000084.$$

- Hence $\forall\ y \geqslant 4$ the total sum of positive composed in the general bracket does not exceed $\frac{1}{2}$:

$$\frac{\pi^2}{8}\frac{y^2}{\cosh^2\frac{\pi y}{2}}+\frac{\pi^2}{8}\frac{y^2}{\sinh^2\frac{\pi y}{2}}+$$
$$+\frac{\pi}{4y}\tanh\frac{\pi y}{2}+\frac{1}{y^2}+\frac{\pi^2}{8y\cosh^2\frac{\pi y}{2}}\left(\frac{3}{2y}+\frac{\pi}{2}\tanh\frac{\pi y}{2}\right)<0,2594088.$$

This means that for $\forall\ y \geqslant 4,\ \ 0 < x \leqslant \frac{1}{2}$ the second factor of the right side of the equality (37) does not turn into 0,
hence from (36) and (37):

$$x=\frac{1}{2}.$$

- In a underside the validity of the statement of the Lemma 3 is obvious.

$\square$

So, assuming that an arbitrary nontrivial root $q$ of zeta functions belongs to the union $\mathcal{P}_1 \cup \mathcal{P}_2$ we found that it belongs only to $\mathcal{P}_2$, i.e. $\mathcal{P}_1 = \varnothing$.

And according to the fact that $\mathcal{P}_1 = \varnothing \Leftrightarrow \mathcal{P}_3 = \varnothing$ we have:

$$\mathcal{P}_3 = \mathcal{P}_1 = \varnothing,\ \ \mathcal{P} = \mathcal{P}_2.$$

This proves the basic statement and the assumption which had been made by Bernhard Riemann about of the real parts of the nontrivial zeros of zeta function.

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