The real parts of the nontrivial Riemann zeta function zeros

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ABSTRACT

This theorem is based on holomorphy of studied functions and the fact that nearby of a singularity point the imaginary part of the specific function can accept zero value.

The colored markers are:
- ● - assumption or a fact which is not proven at present;
- ● - the statement which requires additional attention;
- ● - statement which is proved earlier or clearly undestandable.

THEOREM

● The real parts of all the nontrivial Riemann zeta function zeros ρ are equal $Re\left(\rho\right) = \frac{1}{2}$.

PROOF:

● According to the functional equality [10, p. 22], [5, p. 8-11]:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta\left(s\right) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta\left(1-s\right), \qquad Re\left(s\right) > 0 \qquad (1)$$

$\zeta\left(s\right)$ - the Riemann zeta function, $\Gamma\left(s\right)$ - the Gamma function.

- From [5, p. 8-11] $\zeta(\bar{s}) = \overline{\zeta(s)}$, it means that $\forall \rho = \sigma + it$: $\zeta(\rho) = 0$ and $0 \leqslant \sigma \leqslant 1$ we have:

$$\zeta(\bar{\rho}) = \zeta(1-\rho) = \zeta(1-\bar{\rho}) = 0 \tag{2}$$

- From [11], [9, p. 128], [10, p. 45] we know that $\zeta(s)$ has no nontrivial zeros on the line $\sigma = 1$ and consequently on the line $\sigma = 0$ also, in accordance with (2) they don't exist.

- Let's denote the set of nontrivial zeros $\zeta(s)$ through $\mathcal{P}$ (multiset with consideration of multiplicitiy):

$$\mathcal{P} \stackrel{\text{def}}{=} \{\rho : \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ 0 < \sigma < 1\}.$$

$$\text{And: } \mathcal{P}_1 \stackrel{\text{def}}{=} \left\{\rho : \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ 0 < \sigma < \frac{1}{2}\right\}, \tag{3}$$

$$\mathcal{P}_2 \stackrel{\text{def}}{=} \left\{\rho : \ \zeta(\rho) = 0, \ \rho = \frac{1}{2} + it\right\},$$

$$\mathcal{P}_3 \stackrel{\text{def}}{=} \left\{\rho : \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ \frac{1}{2} < \sigma < 1\right\}.$$

Then:

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \quad \text{and} \quad \mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P}_2 \cap \mathcal{P}_3 = \mathcal{P}_1 \cap \mathcal{P}_3 = \varnothing,$$

$$\mathcal{P}_1 = \varnothing \Leftrightarrow \mathcal{P}_3 = \varnothing.$$

- Hadamard's theorem (Weierstrass preparation theorem) on the decomposition of function through the roots gives us the following result [10, p. 30], [5, p. 31], [12]:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}} e^{as}}{s(s-1)\Gamma\left(\frac{s}{2}\right)} \prod_{\rho \in \mathcal{P}} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \qquad Re(s) > 0 \tag{4}$$

$$a = \ln 2\sqrt{\pi} - \frac{\gamma}{2} - 1, \ \gamma - \text{Euler's constant and}$$

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\ln\pi + a - \frac{1}{s} + \frac{1}{1-s} - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \sum_{\rho\in\mathcal{P}}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \tag{5}$$

- According to the fact that $\dfrac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$ - Digamma function of [10, p. 31], [5, p. 23] we have:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + \sum_{\rho\in\mathcal{P}}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2n}\right) + C, \tag{6}$$

$$C = const$$

.

- From [4, p. 160], [8, p. 272], [3, p. 81]:

$$\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = 1 + \frac{\gamma}{2} - \ln 2\sqrt{\pi} = 0,0230957\ldots \tag{7}$$

- Indeed, from (2):

$$\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = \frac{1}{2}\sum_{\rho\in\mathcal{P}}\left(\frac{1}{1-\rho} + \frac{1}{\rho}\right)$$

- From (5):

$$2\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = \lim_{s\to 1}\left(\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{1-s} + \frac{1}{s} - a - \frac{1}{2}\ln\pi + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right).$$

- Also it's known, for example, from [10, p. 49], [3, p. 98] that the number of nontrivial zeros of $\rho = \sigma + it$ in strip $0 < \sigma < 1$, the imaginary parts of which t are less than some number $T > 0$ is limited, i.e.

$$\|\{\rho :\ \rho\in\mathcal{P},\ \rho = \sigma + it,\ |t| < T\}\| < \infty.$$

- Indeed, it can be presented that on the contrary the sum of $\sum_{\rho\in\mathcal{P}}\dfrac{1}{\rho}$ would have been unlimited.

- Thus $\forall T > 0\ \exists\ \delta_x > 0,\ \delta_y > 0$ such that

$$\text{in area } 0 < t \leqslant \delta_y, 0 < \sigma \leqslant \delta_x \text{ there are no zeros } \rho = \sigma + it \in \mathcal{P}. \qquad (8)$$

Let's consider random root $q \in \mathcal{P}_1 \cup \mathcal{P}_2$
Let's denote $k(q)$ the multiplicity of the root q.
Let's examine the area $Q\,(R) \overset{\text{def}}{=} \{s : \|s - q\| \leqslant R, R > 0\}$.

- From the fact of finiteness of set of nontrivial zeros $\zeta(s)$ in the limited area follows $\exists\ R > 0$, such that $Q(R)$ does not contain any root from $\mathcal{P}$ except q.

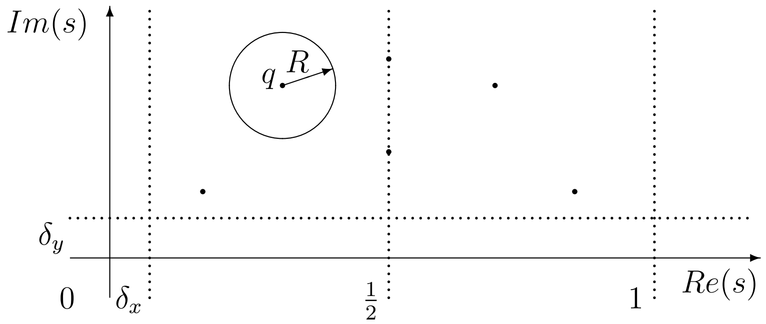

Fig. 1.

- From [1], [10, p. 31], [5, p. 23] we know that the Digamma function $\dfrac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$ in the area $Q(R)$ has no poles, i.e. $\forall s \in Q(R)$

$$\left\| \frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \right\| < \infty.$$

Let's denote:

$$I_{\mathcal{P}}(s) \overset{\text{def}}{=} -\frac{1}{s} + \frac{1}{1-s} + \sum_{\rho \in \mathcal{P}} \frac{1}{s - \rho}$$

and

$$I_{\mathcal{P}\setminus\{q\}}(s) = -\frac{1}{s} + \frac{1}{1-s} + \sum_{\rho \in \mathcal{P}\setminus\{q\}} \frac{1}{s - \rho}. \qquad (9)$$

Hereinafter $\mathcal{P} \setminus \{q\} \stackrel{\text{def}}{=} \mathcal{P} \setminus \{(q, k(q))\}$ (the difference in the multiset).

Summation - $\sum_{\rho\in\mathcal{P}} \dfrac{1}{s-\rho}$ and $\sum_{\rho\in\mathcal{P}\setminus\{q\}} \dfrac{1}{s-\rho}$ further we shall consider as the sum of pairs $\left(\dfrac{1}{s-\rho} + \dfrac{1}{s-(1-\rho)}\right)$ and $\sum_{\rho\in\mathcal{P}} \dfrac{1}{\rho}$ as the sum of pairs $\left(\dfrac{1}{\rho} + \dfrac{1}{1-\rho}\right)$ as a consequence of division of the sum from (6) $\sum_{\rho\in\mathcal{P}} \left(\dfrac{1}{s-\rho} + \dfrac{1}{\rho}\right)$ into $\sum_{\rho\in\mathcal{P}} \dfrac{1}{s-\rho} + \sum_{\rho\in\mathcal{P}} \dfrac{1}{\rho}$. As specifed in [4], [6], [8], [10].

- Let's note that $I_{\mathcal{P}\setminus\{q\}}(s)$ is holomorphic function $\forall\ s \in Q(R)$.

Then from (5) we have:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\ln\pi + a - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \sum_{\rho\in\mathcal{P}} \frac{1}{\rho} + I_{\mathcal{P}}(s). \tag{10}$$

And in view of (7):

$$Im\frac{\zeta'(s)}{\zeta(s)} = Im\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + I_{\mathcal{P}}(s)\right). \tag{11}$$

Let's note that from the equality of

$$\sum_{\rho\in\mathcal{P}} \frac{1}{1-s-\rho} = -\sum_{(1-\rho)\in\mathcal{P}} \frac{1}{s-(1-\rho)} = -\sum_{\rho\in\mathcal{P}} \frac{1}{s-\rho} \tag{12}$$

follows that:

$$I_{\mathcal{P}}(1-s) = -I_{\mathcal{P}}(s),\ I_{\mathcal{P}\setminus\{1-q\}}(1-s) = -I_{\mathcal{P}\setminus\{q\}}(s),\ Re\,(s) > 0.$$

- Besides

$$I_{\mathcal{P}\setminus\{q\}}(s) = I_{\mathcal{P}}(s) - \frac{k(q)}{s-q}$$

and $I_{\mathcal{P}\setminus\{q\}}(s)$ is limited in the area of $s \in Q(R)$ as a result of absence of its poles in this area as well as its differentiability in each point of this area.

- If in (5) to replace s with $1-s$ that in view of (7):

$$\frac{\zeta'(s)}{\zeta(s)}+\frac{\zeta'(1-s)}{\zeta(1-s)}=-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}-\frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}+\ln\pi,\ Re\,(s)>0. \quad (13)$$

- Let's examine a circle with the center in a point q and radius $r \leqslant R$, laying in the area of $Q(R)$:

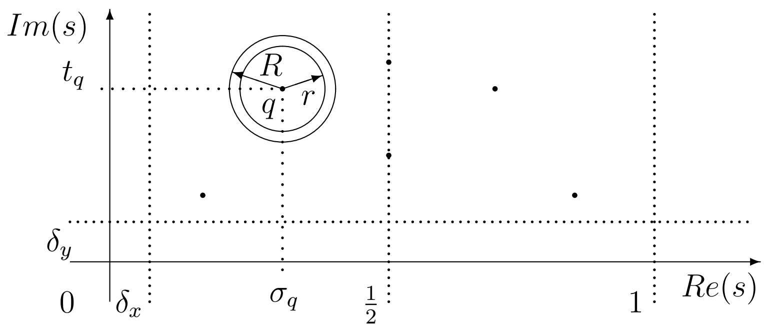

Fig. 2.

- For $s=x+iy,\ q=\sigma_q+it_q$

$$Im\frac{k(q)}{s-q}=Im\frac{k(q)}{x+iy-\sigma_q-it_q}=\frac{k(q)(t_q-y)}{(x-\sigma_q)^2+(y-t_q)^2}=k(q)\frac{t_q-y}{r^2},$$

- Consider also the function $\ln\zeta(s)$ - principal branch function $Ln\zeta(s)$ for which of the (4), in view of (7) $\forall\ s\in Q(R)$ is true:

$$\ln\zeta(s)=\frac{\ln\pi}{2}s-\ln\left(s(1-s)\right)-\ln\Gamma\left(\frac{s}{2}\right)+\sum_{\rho\in\mathcal{P}}\ln\left(1-\frac{s}{\rho}\right). \quad (14)$$

The sum as stipulated earlier, is taken in pairs:

$$\ln\left(1-\frac{s}{\rho}\right)+\ln\left(1-\frac{s}{1-\rho}\right).$$

Let's designate the real function of two variables for $s = x + iy$:

$$\omega(x,y) \stackrel{\text{def}}{=} Im \left(-\ln\left(s(1-s)\right) - \ln\Gamma\left(\frac{s}{2}\right) + \sum_{\rho\in\mathcal{P}\setminus\{q\}} \ln\left(1-\frac{s}{\rho}\right)\right).$$

- Note that the function $\omega(x,y)$ and its partial derivatives on both variables exist and are limited to $\forall\ s = x + iy \in Q(R)$, since

$$\omega(x,y) = Im\ln\zeta(x+iy) - \frac{\ln\pi}{2}y - k(q)Im\ln\left(1-\frac{x+iy}{q}\right).$$

Inside of area $Q(R)$ we take a point $M \stackrel{\text{def}}{=} (x_M, y_M)$, does not coincide which the point q.

- Let's draw a looped curve from the point M so that it doesn't pass through the point q and is described by the function which has a continuous derivative at each point.

Let's designate: $f_M(\tau) \stackrel{\text{def}}{=} f_x(\tau) + if_y(\tau)$:

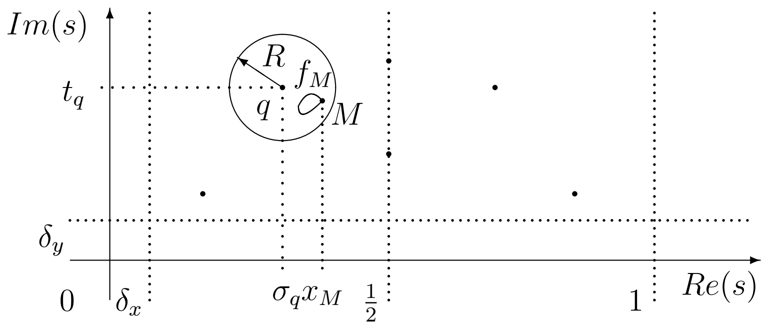

Fig. 3.

In accordance with the construction let's denote $\tau_{M,1}$ and $\tau_{M,2}$ such that:

$$f_M(\tau_{M,1}) = x_M + iy_M, \quad f_M(\tau_{M,2}) = x_M + iy_M.$$

- Function $Im \ln \zeta(s) - \dfrac{\ln \pi}{2} Im(s)$ is differentiable and therefore continuous and differentiable in τ function $Im \ln \zeta(f_M(\tau)) - \dfrac{\ln \pi}{2} Im(f_M(\tau))$.

  It means that continuous on the segment and differentiable in the domestic range of this segment the real function gets at its ends the same values:

$$Im \ln \zeta(f_M(\tau_{M,1})) - \frac{\ln \pi}{2} Im(f_M(\tau_{M,1})) =$$
$$= Im \ln \zeta(f_M(\tau_{M,2})) - \frac{\ln \pi}{2} Im(f_M(\tau_{M,2})) =$$
$$= Im \ln \zeta(f_M(x_M + iy_M)) - \frac{\ln \pi}{2} Im(f_M(x_M + iy_M)).$$

- By Rolle's theorem on the extremum of a differentiable function on the interval we have:

$$\exists \tau_1 \in (\tau_{M,1}, \tau_{M,2}) : \quad \left(Im \ln \zeta(f_M(\tau)) - \frac{\ln \pi}{2} Im(f_M(\tau)) \right)'_{\tau=\tau_1} = 0. \quad (15)$$

  I.e. on a curve described by function $f_M(\tau)$, $\tau \in (\tau_{M,1}, \tau_{M,2})$ there is a point $\Theta_w = \Theta_w(M) \stackrel{\text{def}}{=} f_M(\tau_1)$ for which it is true (15).

- Let's consider the following option line, as a closed curve which passes through a point of M.

  For any $0 < r < R$ construct a circle centered at the point q and the radius r.

  The point of intersection of the left semicircle of the circle and the line $y = t_q$ let's denote as $J \stackrel{\text{def}}{=} (x_J, y_J)$.

  Let's construct a circle with the center in a point J, with radius:

$$0 < r_\delta < \min(r, R - r).$$

  As a point of M let's take more distant from q point of intersection of the circle with the line $y = t_q$.

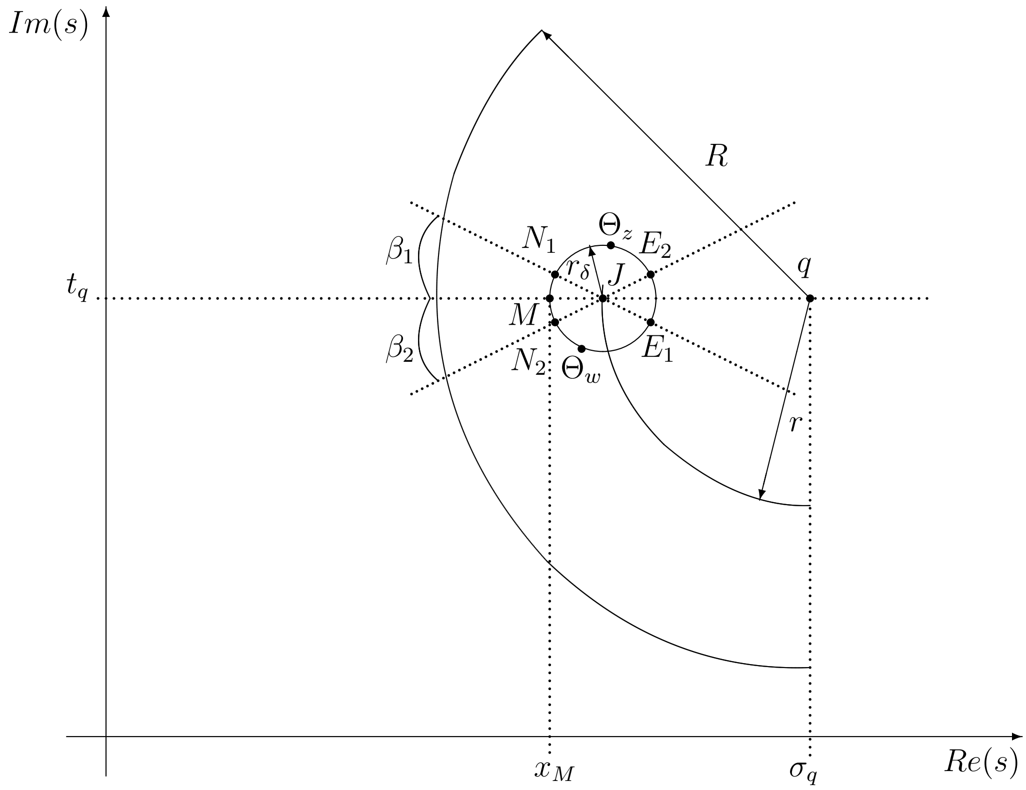

Fig. 4.

- As the desired curve, we consider the circle with center at J and the radius r_δ.

Let's notice that the received curve satisfies to all declared properties for any as much as small radius since $\min(r, R - r)$.

Let's lead from a point J two straight lines under angles $-\beta_1$ and β_2 :

$$\mathrm{tg}(\beta_1) > 0, \quad \mathrm{tg}(\beta_2) > 0.$$

Pairs of the intersection points of these lines with the circle of radius r_δ is denoted by $N_1,\ E_1$ and $N_2,\ E_2$ accordingly.

- Let's assign:

$$0 < r_\delta = O(r^3)_{r\to 0}, \quad \mathrm{tg}(\beta_1) = \mathrm{tg}(\beta_2) = \frac{1}{r^{\frac{1}{2}}}. \tag{16}$$

Then $\exists\, 0 < R_1 \leqslant R : \ \forall\, 0 < r < R_1$ the condition is satisfied:

$$0 < r_\delta < \min(r, R - r).$$

- Let's assume, that the point Θ_w lays on any of the opened arches: MN_2, E_1E_2 or N_1M.

  As τ take $Im(s) = y$, then:

$$f_y(y) = y, \ \ x = f_x(y) = x_J \pm \sqrt{r_\delta^2 - (y - y_J)^2}.$$

- In view of (16) for any point $(x(\tau), \tau)$, laying on the considered opened arches:

$$0 \leqslant |x(\tau)'| < \frac{1}{r^{\frac{1}{2}}}. \tag{17}$$

- According to [1, p. 67, 82]:

$$Im \ln\left(1 - \frac{x + iy}{q}\right) = Im \ln\left(\sigma_q - x + i(t_q - y)\right) - Im \ln\left(q\right) =$$
$$= \arctan\left(\frac{t_q - y}{\sigma_q - x}\right) - \arctan\left(\frac{t_q}{\sigma_q}\right),$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1 + x^2}.$$

- Then:

$$\frac{d}{d\tau} Im \ln\left(1-\frac{x(\tau)+i\tau}{q}\right)_{\tau=\tau_1} = \frac{d}{d\tau}\arctan\left(\frac{t_q-\tau}{\sigma_q-x(\tau)}\right)_{\tau=\tau_1} =$$
$$= \frac{1}{1+\left(\frac{t_q-\tau_1}{\sigma_q-x(\tau_1)}\right)^2}\frac{\partial}{\partial\tau}\left(\frac{t_q-\tau}{\sigma_q-x(\tau_1)}\right)_{\tau=\tau_1} +$$
$$+\frac{1}{1+\left(\frac{t_q-\tau_1}{\sigma_q-x(\tau_1)}\right)^2}\frac{\partial}{\partial x}\left(\frac{t_q-\tau_1}{\sigma_q-x}\right)_{x=x(\tau_1)} x'(\tau_1) =$$
$$= -\frac{\sigma_q-x(\tau_1)}{(\sigma_q-x(\tau_1))^2+(t_q-\tau_1)^2} + \frac{x'(\tau_1)(t_q-\tau_1)}{(\sigma_q-x(\tau_1))^2+(t_q-\tau_1)^2}. \qquad (18)$$

- And for the $s = x+iy$ equation (15) at the point $\Theta_w = (x(\tau_1),\tau_1)$ can be written as follows:

$$\frac{\sigma_q-x(\tau_1)}{(\sigma_q-x(\tau_1))^2+(t_q-\tau_1)^2} - \frac{x'(\tau_1)(t_q-\tau_1)}{(\sigma_q-x(\tau_1))^2+(t_q-\tau_1)^2} =$$
$$= \omega(x(\tau_1),y)'_{y=\tau_1} + x(\tau_1)'\omega(x,\tau_1)'_{x=x(\tau_1)}. \qquad (19)$$

- At $r\to 0$ and (16):

$$(\sigma_q-x(\tau_1))^2+(t_q-\tau_1)^2 \leqslant (r+r_\delta)^2+r_\delta^2 = \theta(r^2),$$
$$(\sigma_q-x(\tau_1))^2+(t_q-\tau_1)^2 \geqslant (r-r_\delta)^2 = \theta(r^2),$$

i.e.

$$(\sigma_q-x(\tau_1))^2+(t_q-\tau_1)^2 = \theta(r^2).$$

- Hence the equation (19) at $r\to 0$ looks like this:

$$\theta\left(\frac{1}{r}\right) + O\left(\frac{1}{r^{\frac{1}{2}}}\right) = O\left(\frac{1}{r^{\frac{1}{2}}}\right).$$

That is impossible starting from some moment, i.e. $\exists\ 0 < R_2 \leqslant R_1$: $\forall\ 0 < r < R_2$ the assumption that the point Θ_w lays on the opened arches MN_2, E_1E_2 or N_1M is false.

Therefore $\forall\ 0 < r < R_2$ on closed arcs N_2E_1 and N_1E_2 should be the point Θ_w for which the equality (15) is true.

- Indeed, if to assume that the point Θ_w lays on the closed arches N_2E_1 and N_1E_2 then the equation (15) in a point $\Theta_w = (\tau_1, y(\tau_1))$ where $f_x(\tau) = \tau,\ f_y(\tau) = y(\tau)$ could be written as follows:

$$y(\tau_1)' \frac{\sigma_q - \tau_1}{(\sigma_q - \tau_1)^2 + (t_q - y(\tau_1))^2} - \frac{t_q - y(\tau_1)}{(\sigma_q - \tau_1)^2 + (t_q - y(\tau_1))^2} = \tag{20}$$
$$= y(\tau_1)'\omega(\tau_1, y)'_{y=y(\tau_1)} + \omega(x, y(\tau_1))'_{x=\tau_1}.$$

- Hence the equation (20) at $r \to 0$ looks like this:

$$O\left(\frac{1}{r^{\frac{1}{2}}}\right) + O\left(1\right) = O\left(1\right).$$

I.e. always has a solution, and $y(\tau_1)' = O(r)_{r \to 0}$.

More precisely:

$$y(\tau_1)' \left(1 - \omega(\tau_1, y)'_{y=y(\tau_1)} \frac{(\sigma_q - \tau_1)^2 + (t_q - y(\tau_1))^2}{\sigma_q - \tau_1}\right) =$$
$$= \frac{t_q - y(\tau_1)}{\sigma_q - \tau_1} + \omega(x, y(\tau_1))'_{x=\tau_1} \frac{(\sigma_q - \tau_1)^2 + (t_q - y(\tau_1))^2}{\sigma_q - \tau_1}.$$

And given the (16), where $r \to 0$:

$$y(\tau_1)' = \omega(x, y(\tau_1))'_{x=\tau_1} r + O(r^2). \tag{21}$$

The same reasonings for $s = x + iy$ in the same area $s \in Q(R)$ points q for function $Im \ln \zeta(1-s) - \dfrac{\ln \pi}{2} Im(1-s)$ we shall come to conclusion that $\exists\ 0 < R_3 \leqslant R_2 :\ \forall\ 0 < r < R_3$ on the closed arches N_2E_1 and N_1E_2 there should be a point $\Theta_z = \Theta_z(M) \stackrel{\text{def}}{=} f_M(\tau_2)$ for some $\tau_2 \in (\tau_{M,1}, \tau_{M,2})$ for which equality is true:

$$\left(Im \ln \zeta\left(1 - f_M(\tau)\right) - \frac{\ln \pi}{2} Im(1 - f_M(\tau))\right)'_{\tau=\tau_2} = 0. \tag{22}$$

- Indeed, if we apply the (14) value is $1-s$ instead of s:

$$\ln \zeta(1-s) - \frac{\ln \pi}{2}(1-s) =$$
$$-\ln\left((1-s)s\right) - \ln \Gamma\left(\frac{1-s}{2}\right) + \textstyle\sum_{\rho\in\mathcal{P}} \ln\left(1 - \dfrac{1-s}{\rho}\right)$$

that in view of that $\forall\ \ \rho \in \mathcal{P}$:

$$\left(1 - \frac{1-s}{\rho}\right)\left(1 - \frac{1-s}{1-\rho}\right) = \left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{1-\rho}\right),$$

we have:

$$\ln \zeta(1-s) - \frac{\ln \pi}{2}(1-s) = -\ln\left(s(1-s)\right) - \ln \Gamma\left(\frac{1-s}{2}\right) + \sum_{\rho\in\mathcal{P}} \ln\left(1 - \frac{s}{\rho}\right). \tag{23}$$

Let's designate the real function of two variables for $s = x + iy$:

$$\omega_-(x,y) \stackrel{\text{def}}{=} Im\left(-\ln\left(s(1-s)\right) - \ln \Gamma\left(\frac{1-s}{2}\right) + \sum_{\rho\in\mathcal{P}\setminus\{q\}} \ln\left(1 - \frac{s}{\rho}\right)\right).$$

- Note that the function $\omega_-(x,y)$ and its partial derivatives on both variables exist and are limited to $\forall\ s = x+iy \in Q(R)$, since

$$\omega_-(x,y) = Im\ln\zeta(1-x-iy) + \frac{\ln\pi}{2}y - k(q)Im\ln\left(1-\frac{x+iy}{q}\right).$$

And all reasonings for functions $\ln\zeta(1-s) - \dfrac{\ln\pi}{2}(1-s)$ and $\omega_-(x,y)$ are similar to reasonings for appropriate functions $\ln\zeta(s) - \dfrac{\ln\pi}{2}(s)$ and $\omega(x,y)$.

Hence at $r \to 0$:

$$y(\tau_2)' = \omega_-(x, y(\tau_2))'_{x=\tau_2} r + O(r^2). \tag{24}$$

- And note that from (21), (24) at $r \to 0$:

$$(y(\tau_1)' - y(\tau_2)')ReI_{\mathcal{P}}(x_J + iy_J) =$$

$$= (y(\tau_1)' - y(\tau_2)')\left(ReI_{\mathcal{P}\setminus\{q\}}(x_J+iy_J) - \frac{1}{r}\right) =$$

$$= -\omega(x, y(\tau_1))'_{x=\tau_1} + \omega_-(x, y(\tau_2))'_{x=\tau_2} + O(r) =$$

$$= -\frac{\partial}{\partial x}Im\left(-\ln\Gamma\left(\frac{x+iy_J}{2}\right) + \ln\Gamma\left(\frac{1-x-iy_J}{2}\right)\right)_{x=x_J} + O(r) =$$

$$= -Im\frac{d}{ds}\left(-\ln\Gamma\left(\frac{s}{2}\right) + \ln\Gamma\left(\frac{1-s}{2}\right)\right)_{s=x_J+iy_J} + O(r) =$$

$$= -\frac{1}{2}Im\left(-\frac{\Gamma'\left(\dfrac{x_J+iy_J}{2}\right)}{\Gamma\left(\dfrac{x_J+iy_J}{2}\right)} - \frac{\Gamma'\left(\dfrac{1-x_J-iy_J}{2}\right)}{\Gamma\left(\dfrac{1-x_J-iy_J}{2}\right)}\right) + O(r). \tag{25}$$

- So, for $\forall\ 0 < r < R_3$ in closed arcs N_2E_1 and N_1E_2 are two points of Θ_w and Θ_z, for which the equalities (15) and (22) respectively.

For $\forall\ s : \|s-(x_J+iy_J)\| < r_\delta$ derivative of function $-\ln\left(s(1-s)\right)+\sum_{\rho\in\mathcal{P}}\ln\left(1-\dfrac{s}{\rho}\right)$ is continuous and equal to:

$$\frac{d}{ds}\left(-\ln\left(s(1-s)\right)+\sum_{\rho\in\mathcal{P}}\ln\left(1-\frac{s}{\rho}\right)\right)=I_{\mathcal{P}}(s).$$

Hence for fixed $0<r<R_3$ from the continuity of the given above function at the point J:
$\forall\ \varepsilon_r>0,\ \exists\ \delta_{\varepsilon_r}>0 : \forall\ s : \|s-(x_J+iy_J)\| < \delta_{\varepsilon_r}$ follows:

$$|ReI_{\mathcal{P}}(s)-ReI_{\mathcal{P}}(x_J+iy_J)|<\varepsilon_r,$$
$$|ImI_{\mathcal{P}}(s)-ImI_{\mathcal{P}}(x_J+iy_J)|<\varepsilon_r.$$

- Let's assign $\varepsilon_r=r^2$ then $\forall\ 0<r_\delta<\delta_{\varepsilon_r}$ at $r\to 0$ with the condition (16) on the closed arches N_2E_1 and N_1E_2 where $f_x(\tau)=\tau,\ f_y(\tau)=y(\tau)$ and $y(\tau_1)'=O(r)$, $y(\tau_2)'=O(r)$ it is carried out in a similar way (20):

$$\frac{d}{d\tau}Im\left(-\ln\left(f_M(\tau)(1-f_M(\tau))\right)+\textstyle\sum_{\rho\in\mathcal{P}}\ln\left(1-\dfrac{f_M(\tau)}{\rho}\right)\right)_{\tau=\tau_1}-$$
$$-\frac{d}{d\tau}Im\left(-\ln\left(f_M(\tau)(1-f_M(\tau))\right)+\textstyle\sum_{\rho\in\mathcal{P}}\ln\left(1-\dfrac{f_M(\tau)}{\rho}\right)\right)_{\tau=\tau_2}=$$
$$=(-y(\tau_1)'ReI_{\mathcal{P}}(\Theta_w)+ImI_{\mathcal{P}}(\Theta_w))-$$
$$-(-y(\tau_2)'ReI_{\mathcal{P}}(\Theta_z)+ImI_{\mathcal{P}}(\Theta_z))=$$

$$=-y(\tau_1)'ReI_{\mathcal{P}}(\Theta_w)+y(\tau_1)'ReI_{\mathcal{P}}(x_J+iy_J)-$$
$$-y(\tau_2)'ReI_{\mathcal{P}}(x_J+iy_J)+y(\tau_2)'ReI_{\mathcal{P}}(\Theta_z)+$$
$$-y(\tau_1)'ReI_{\mathcal{P}}(x_J+iy_J)+y(\tau_2)'ReI_{\mathcal{P}}(x_J+iy_J)+$$
$$+ImI_{\mathcal{P}}(\Theta_w)-ImI_{\mathcal{P}}(x_J+iy_J)+$$
$$+ImI_{\mathcal{P}}(x_J+iy_J)-ImI_{\mathcal{P}}(\Theta_z)=$$

$$=O(r^3)+O(r^3)-(y(\tau_1)'-y(\tau_2)')ReI_{\mathcal{P}}(x_J+iy_J)+O(r^2)+O(r^2)=$$
$$=Im\frac{d}{ds}\left(-\ln\Gamma\left(\frac{s}{2}\right)+\ln\Gamma\left(\frac{1-s}{2}\right)\right)_{s=x_J+iy_J}+O(r). \tag{26}$$

- Let's consider a difference of the equations (15) and (22) which on construction is equal 0 for $\forall\ 0 < r < R_3$:

$$\lim_{r\to 0} 0 = \lim_{r\to 0}\left(\left(Im\ln\zeta\left(f_M(\tau)\right) - \frac{\ln\pi}{2}Im(f_M(\tau))\right)'_{\tau=\tau_1} - \right.$$

$$\left. - \left(Im\ln\zeta\left(1-f_M(\tau)\right) - \frac{\ln\pi}{2}Im(1-f_M(\tau))\right)'_{\tau=\tau_2}\right) =$$

$$= \lim_{r\to 0}\left(-\left(Im\ln\Gamma\left(\frac{f_M(\tau)}{2}\right)\right)'_{\tau=\tau_1} + \left(Im\ln\Gamma\left(\frac{1-f_M(\tau)}{2}\right)\right)'_{\tau=\tau_2} + \right.$$

$$+\frac{d}{d\tau}Im\left(-\ln\left(f_M(\tau)(1-f_M(\tau))\right) + \textstyle\sum_{\rho\in\mathcal{P}}\ln\left(1-\frac{f_M(\tau)}{\rho}\right)\right)_{\tau=\tau_1} -$$

$$\left. - \left(\frac{d}{d\tau}Im\left(-\ln\left(f_M(\tau)(1-f_M(\tau))\right) + \textstyle\sum_{\rho\in\mathcal{P}}\ln\left(1-\frac{f_M(\tau)}{\rho}\right)\right)_{\tau=\tau_2}\right)\right) =$$

And then from (26) follows:

$$= \lim_{r\to 0}\left(-\left(Im\ln\Gamma\left(\frac{f_M(\tau)}{2}\right)\right)'_{\tau=\tau_1} + \left(Im\ln\Gamma\left(\frac{1-f_M(\tau)}{2}\right)\right)'_{\tau=\tau_2} + \right.$$

$$\left. + Im\frac{d}{ds}\left(-\ln\Gamma\left(\frac{s}{2}\right) + \ln\Gamma\left(\frac{1-s}{2}\right)\right)_{s=x_J+iy_J}\right) =$$

$$= \lim_{r\to 0}\left(-\frac{\partial}{\partial x}\left(Im\ln\Gamma\left(\frac{x+iy(\tau_1)}{2}\right)\right)_{x=\tau_1} - \right.$$

$$-y(\tau_1)'\frac{\partial}{\partial y}\left(Im\ln\Gamma\left(\frac{\tau_1+iy}{2}\right)\right)_{y=y(\tau_1)} +$$

$$+\frac{\partial}{\partial x}\left(Im\ln\Gamma\left(\frac{1-x-iy(\tau_2)}{2}\right)\right)_{x=\tau_2} +$$

$$+y(\tau_2)'\frac{\partial}{\partial y}\left(Im\ln\Gamma\left(\frac{1-\tau_2-iy}{2}\right)\right)_{y=y(\tau_2)} +$$

$$\left. + Im\frac{d}{ds}\left(-\ln\Gamma\left(\frac{s}{2}\right) + \ln\Gamma\left(\frac{1-s}{2}\right)\right)_{s=x_J+iy_J}\right) =$$

$$= \lim_{r\to 0}\left(-\frac{\partial}{\partial x}\left(Im\ln\Gamma\left(\frac{x+iy(\tau_1)}{2}\right)\right)_{x=\tau_1}+\right.$$

$$+\frac{\partial}{\partial x}\left(Im\ln\Gamma\left(\frac{1-x-iy(\tau_2)}{2}\right)\right)_{x=\tau_2}+$$

$$\left.+Im\frac{d}{ds}\left(-\ln\Gamma\left(\frac{s}{2}\right)+\ln\Gamma\left(\frac{1-s}{2}\right)\right)_{s=x_J+iy_J}\right)=$$

$$=2Im\left(-\frac{d}{ds}\ln\Gamma\left(\frac{s}{2}\right)_{s=q}+\frac{d}{ds}\ln\Gamma\left(\frac{1-s}{2}\right)_{s=q}\right)=$$

$$=Im\left(-\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}-\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)}\right)=0.$$

- Thus for the selected root q is:

$$Im\left(-\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}-\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)}\right)=0,\ \ \forall q\in\mathcal{P}_1\cup\mathcal{P}_2. \tag{27}$$

- From (6) equality (27) can be rewritten as follows:

$$\sum_{n=0}^{\infty}\left(\frac{t_q}{(2n+\sigma_q)^2+t_q^2}-\frac{t_q}{(2n+1-\sigma_q)^2+t_q^2}\right)=0.$$

I.e.

$$\textstyle\sum_{n=0}^{\infty} \frac{t_q((2n+1-\sigma_q)^2-(2n+\sigma_q)^2)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)} =$$
$$= \textstyle\sum_{n=0}^{\infty} \frac{t_q(1-2\sigma_q)(4n+1)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)} =$$
$$= (1-2\sigma_q) \textstyle\sum_{n=0}^{\infty} \frac{t_q(4n+1)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)} = 0.$$

Sum

$$\sum_{n=0}^{\infty} \frac{t_q(4n+1)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)}$$

exists and is not equal to 0 when $t_q \neq 0$ so the equality (27) is performed exclusively at

$$\sigma_q = \frac{1}{2}.$$

So, assuming that an arbitrary nontrivial root q of zeta functions belongs to the union $\mathcal{P}_1 \cup \mathcal{P}_2$ we found that it belongs only to $\mathcal{P}_2$, i.e. $\mathcal{P}_1 = \varnothing$.

And according to the fact that $\mathcal{P}_1 = \varnothing \Leftrightarrow \mathcal{P}_3 = \varnothing$ we have:

$$\mathcal{P}_3 = \mathcal{P}_1 = \varnothing, \quad \mathcal{P} = \mathcal{P}_2.$$

This proves the basic statement and the assumption which had been made by Bernhard Riemann about of the real parts of the nontrivial zeros of zeta function.

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