The real parts of the nontrivial Riemann zeta function zeros

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ABSTRACT

This theorem is based on the study of holomorphy functions and on the fact that near the singularity point of the imaginary part of some rational function can accept an arbitrary preassigned value.

Registration contains colored markers:
- ● - a fact which is not proven at present or an assumption;
- ● - the statement which requires additional attention;
- ● - statement which is proved earlier or clearly undestandable.

THEOREM

● The real parts of all the nontrivial Riemann zeta function zeros ρ lie on the line $\Re(\rho) = \frac{1}{2}$.

PROOF:

● According to the functional equality [10, p. 22], [5, p. 8-11]:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta(1-s), \qquad \Re(s) > 0 \quad (1)$$

$\zeta(s)$ - the Riemann zeta function, $\Gamma(s)$ - the gamma function.

- From [5, p. 8-11] $\zeta(\bar{s}) = \overline{\zeta(s)}$, it means that $\forall \rho = \sigma + it$: $\zeta(\rho) = 0$ and $0 \leqslant \sigma \leqslant 1$ we have:

$$\zeta(\bar{\rho}) = \zeta(1-\rho) = \zeta(1-\bar{\rho}) \tag{2}$$

- From [11], [9, p. 128], [10, p. 45] we know that $\zeta(s)$ it has no nontrivial zeros on the line $\sigma = 1$ and consequently on the line $\sigma = 0$ also, in accordance with (2) they are not present.

- We denote the set of nontrivial zeros $\zeta(s)$ through $\mathcal{P}$:

$$\mathcal{P} \equiv \{\rho: \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ 0 < \sigma < 1\},$$

and

$$\begin{aligned} \mathcal{P}_1 &\equiv \left\{\rho: \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ 0 < \sigma < \frac{1}{2}\right\} \\ \mathcal{P}_2 &\equiv \left\{\rho: \ \zeta(\rho) = 0, \ \rho = \frac{1}{2} + it\right\} \\ \mathcal{P}_3 &\equiv \left\{\rho: \ \zeta(\rho) = 0, \ \rho = \sigma + it, \ \frac{1}{2} < \sigma < 1\right\} \end{aligned} \tag{3}$$

Then:

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \text{ and } \mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P}_2 \cap \mathcal{P}_3 = \mathcal{P}_1 \cap \mathcal{P}_3 = \varnothing, \ \|\mathcal{P}_1\| = \|\mathcal{P}_3\|$$

- Hadamard's theorem (Weierstrass preparation theorem) the decomposition of function through the roots gives us the following result [10, p. 30], [5, p. 31], [12]:

$$\zeta(s) = \frac{\pi^{\frac{s}{2}} e^{as}}{s(s-1)} \prod_{\rho \in \mathcal{P}} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \qquad \Re(s) > 0 \tag{4}$$

$$a = \ln 2\sqrt{\pi} - \frac{\gamma}{2} - 1, \ \gamma - \text{Euler's constant and}$$

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\ln\pi + a - \frac{1}{s} + \frac{1}{1-s} - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \sum_{\rho \in \mathcal{P}} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \tag{5}$$

- According to the fact that $\dfrac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$ - Digamma function of [10, p. 31], [5, p. 23] should be:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + \sum_{\rho\in\mathcal{P}}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n}-\frac{1}{2n}\right) + C, \qquad (6)$$

$$C = const$$

.

- From [4, p. 160], [8, p. 272], [3, p. 81]:

$$\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = 1 + \frac{\gamma}{2} - \ln 2\sqrt{\pi} = 0,0230957\ldots \qquad (7)$$

- Indeed, from (2):

$$\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = \frac{1}{2}\sum_{\rho\in\mathcal{P}}\left(\frac{1}{1-\rho}+\frac{1}{\rho}\right)$$

- From (5):

$$2\sum_{\rho\in\mathcal{P}}\frac{1}{\rho} = \lim_{s\to 1}\left(\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{1-s} + \frac{1}{s} - a - \frac{1}{2}\ln\pi + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right).$$

- Exactly the same, for example, [10, p. 49], [3, p. 98] the number of nontrivial zeros of $\rho = \sigma + it$ in strip $0 < \sigma < 1$, the imaginary parts of which t is less than some number $T > 0$ is limited, i.e.

$$\|\{\rho:\ \rho\in\mathcal{P},\ \rho = \sigma + it,\ |t| < T\}\| < \infty.$$

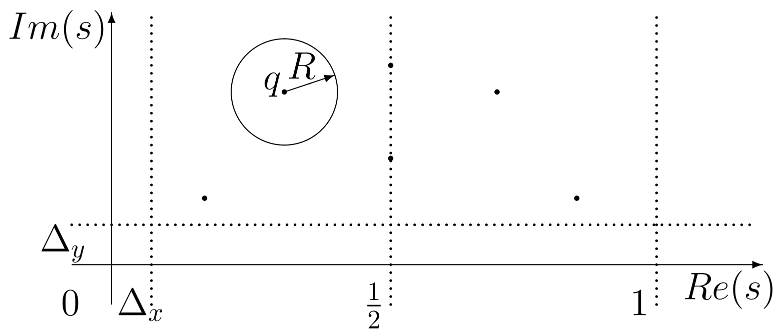
- Indeed, it can be presented as the sum of $\sum_{\rho\in\mathcal{P}}\dfrac{1}{\rho}$ which would be unlimited.

- Thus $\forall T > 0 \; \exists \; \Delta_x > 0, \; \Delta_y > 0$ such that

$$\text{in area } 0 < t \leqslant \Delta_y, 0 < \sigma \leqslant \Delta_x \text{ no zeros } \rho = \sigma + it \in \mathcal{P}. \tag{8}$$

Let's consider that any root $q \in \mathcal{P}_1 \cup \mathcal{P}_2$
Let $k(q)$ be the multiplicity of the root of q.
Let's consider area $Q\,(R) \stackrel{\text{def}}{=} \{s : \|s - q\| \leqslant R, R > 0\}$.

- From the fact of finiteness of set of nontrivial zeros $\zeta(s)$ in the limited area is as follows $\exists \; R > 0$, such that $Q(R)$ does not contain any roots from $\mathcal{P}$ except q.



- From [1], [10, p. 31], [5, p. 23] we know that the Digamma function $\dfrac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$ in the field $Q(R)$ has no poles, i.e. $\forall s \in Q(R)$

$$\left| \frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \right| < \infty.$$

Let us denote by:

$$I_{\mathcal{P}}(s) \stackrel{\text{def}}{=} -\frac{1}{s} + \frac{1}{1-s} + \sum_{\rho \in \mathcal{P}} \frac{1}{s - \rho}$$

and

$$I_{\mathcal{P} \setminus \{q\}}(s) = -\frac{1}{s} + \frac{1}{1-s} + \sum_{\rho \in \mathcal{P} \setminus \{q\}} \frac{1}{s - \rho}. \tag{9}$$

Hereinafter $\mathcal{P} \setminus \{q\} \stackrel{\text{def}}{=} \mathcal{P} \setminus \{(q, k(q))\}$.

- Note that $I_{\mathcal{P}\setminus\{q\}}(s)$ complex differentiable function $s \in Q(R)$.

Then from (5) we have:

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\ln\pi + a - \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + \sum_{\rho\in\mathcal{P}} \frac{1}{\rho} + I_{\mathcal{P}}(s). \tag{10}$$

And in view of (7):

$$Im\frac{\zeta'(s)}{\zeta(s)} = Im\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + I_{\mathcal{P}}(s)\right). \tag{11}$$

Note that the equality of

$$\sum_{\rho\in\mathcal{P}} \frac{1}{1-s-\rho} = -\sum_{(1-\rho)\in\mathcal{P}} \frac{1}{s-(1-\rho)} = -\sum_{\rho\in\mathcal{P}} \frac{1}{s-\rho} \tag{12}$$

follows that:

$$I_{\mathcal{P}}(1-s) = -I_{\mathcal{P}}(s),\ I_{\mathcal{P}\setminus\{q\}}(1-s) = -I_{\mathcal{P}\setminus\{q\}}(s),\ \Re(s) > 0.$$

- Except for that

$$I_{\mathcal{P}\setminus\{q\}}(s) = I_{\mathcal{P}}(s) - \frac{k(q)}{s-q}$$

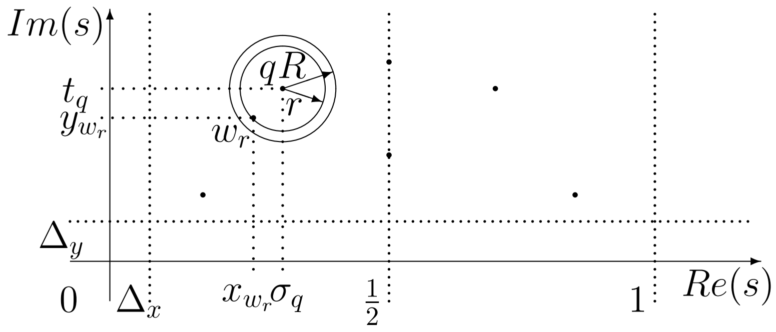
and $I_{\mathcal{P}\setminus\{q\}}(s)$ it is limited in the field of $s \in Q(R)$ in connection with absence of poles at it in this area and differentiability in each point of this area.

- From (1), (5) follows:

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta'(1-s)}{\zeta(1-s)} = -\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} + C, \tag{13}$$

$$C = const,\ C \in \mathbb{R}$$

- Let's consider a circle with the center in a point q and radius $r \leqslant R$, laying in the field of $Q(R)$:



- For $s = x + iy,\ q = \sigma_q + it_q$

$$Im\frac{k(q)}{s-q} = Im\frac{k(q)}{x+iy-\sigma_q-it_q} = \frac{k(q)(t_q-y)}{(x-\sigma_q)^2+(y-t_q)^2} = k(q)\frac{t_q-y}{r^2}.$$

- On each of the semicircles: left - $\{s : \|s-q\| = r,\ \sigma_q - r \leqslant x \leqslant \sigma_q\}$ and the right - $\{s : \|s-q\| = r,\ \sigma_q \leqslant x \leqslant \sigma_q + r\}$ function $Im\frac{k(q)}{s-q}$ is continuous and takes values from $-\frac{k(q)}{r}$ to $\frac{k(q)}{r}$, $r > 0$ and $k(q) \geqslant 1$ as the multiplicity of the root.

- From (11) function

$$Im\frac{\zeta'(s)}{\zeta(s)} - Im\frac{k(q)}{s-q} = Im\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + I_{\mathcal{P}\setminus\{q\}}(s)\right)$$

in the field of $Q(R)$ limited so $\exists H_1(R) > 0,\ H_1(R) \in \mathbb{R}$:

$$\left|Im\frac{\zeta'(s)}{\zeta(s)} - Im\frac{k(q)}{s-q}\right| < H_1(R), \qquad \forall s \in Q(R). \tag{14}$$

- From the properties of continuous functions on the interval taking all intermediate values between its extremes, it follows that $\exists R_1 > 0 :$

$$R_1 < R, \ \frac{k(q)}{R_1} > H_1(R)$$

and $\forall r > 0, \ r < R_1$ there is for example on the left point of the semicircle $w_r \stackrel{\text{def}}{=} x_{w_r} + iy_{w_r}$ such that:

$$Im\frac{\zeta'(w_r)}{\zeta(w_r)} - Im\frac{k(q)}{w_r - q} = -Im\frac{k(q)}{w_r - q},$$

i.e.

$$Im\frac{\zeta'(w_r)}{\zeta(w_r)} = 0, \quad \forall r > 0, r < R_1. \tag{15}$$

- From (11), similar to the (14) function

$$Im\frac{\zeta'(1-s)}{\zeta(1-s)} + Im\frac{k(q)}{s-q} = Im\left(-\frac{1}{2}\frac{\Gamma'\left(\dfrac{1-s}{2}\right)}{\Gamma\left(\dfrac{1-s}{2}\right)} - I_{\mathcal{P}\setminus\{q\}}(s)\right)$$

is limited $\forall s \in Q(R)$ i.e. $\exists H_2(R) > 0, \ H_2(R) \in \mathbb{R}$:

$$\left|Im\frac{\zeta'(1-s)}{\zeta(1-s)} + Im\frac{k(q)}{s-q}\right| < H_2(R), \qquad \forall s \in Q(R).$$

which means: $\exists R_2 > 0 :$

$$R_2 < R, \ \frac{k(q)}{R_2} > H_2(R)$$

and $\forall r > 0, \ r < R_2$ there is for example on the left point of the semicircle $z_r = x_{z_r} + iy_{z_r}$ such as:

$$Im\frac{\zeta'(1-z_r)}{\zeta(1-z_r)} = 0, \quad \forall r > 0, \ r < R_2. \tag{16}$$

- From (11), (15), (16) follows:

$$\begin{cases} k(q)\dfrac{t_q - y_{w_r}}{r^2} = -Im\left(-\dfrac{1}{2}\dfrac{\Gamma'\left(\dfrac{w_r}{2}\right)}{\Gamma\left(\dfrac{w_r}{2}\right)} + I_{\mathcal{P}\setminus\{q\}}(w_r)\right) \\ k(q)\dfrac{t_q - y_{z_r}}{r^2} = Im\left(-\dfrac{1}{2}\dfrac{\Gamma'\left(\dfrac{1-z_r}{2}\right)}{\Gamma\left(\dfrac{1-z_r}{2}\right)} - I_{\mathcal{P}\setminus\{q\}}(z_r)\right) \end{cases} \tag{17}$$

for $\forall r > 0,\ r < min\,(R_1, R_2)\,.$

I.e. $t_q - y_{w_r} = O(r^2)$ and $t_q - y_{z_r} = O(r^2)$ when $r \to 0$.

For definiteness we will consider that w_r and z_r we find on the left semicircle $\|s - q\| = r,\ \Re(s) \leqslant \Re(q) = \sigma_q$ because on the right semicircle there are points with the same - (15), (16) properties.

- Let's choose arbitrary of $r_1,\ r_2$ will be right to inequalities:

$$r_1 > 0,\ r_2 > 0,\ r_2 < r_1,\ r_1 < \min(R_1, R_2).$$

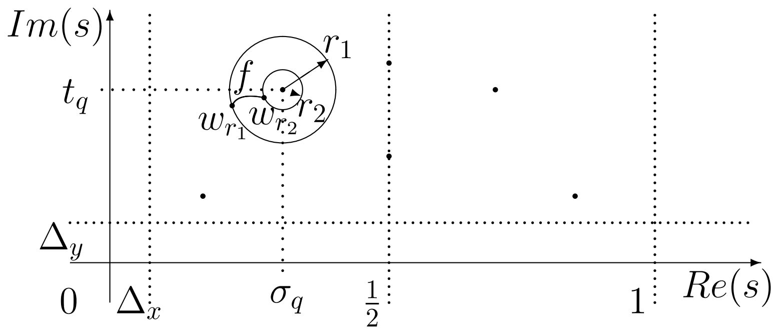
For them:

$$Im\frac{\zeta'\left(w_{r_1}\right)}{\zeta\left(w_{r_1}\right)} = Im\frac{\zeta'\left(w_{r_2}\right)}{\zeta\left(w_{r_2}\right)} = 0$$

and

$$Im\frac{\zeta'\left(1 - z_{r_1}\right)}{\zeta\left(1 - z_{r_1}\right)} = Im\frac{\zeta'\left(1 - z_{r_2}\right)}{\zeta\left(1 - z_{r_2}\right)} = 0.$$

- Let's connect the dots w_{r_1} and w_{r_2} by the curve line given by the function $f_{r_1,r_2}(\tau) \overset{\text{def}}{=} f_x(\tau) + if_y(\tau)$ which has a derivative at every point of the interval:



In accordance with the construction let's denote τ_{r_1} and τ_{r_2} as follows:

$$f_{r_1,r_2}(\tau_{r_1}) = w_{r_1},\ f_{r_1,r_2}(\tau_{r_2}) = w_{r_2}.$$

- Function $\dfrac{\zeta'(s)}{\zeta(s)}$ is meromorphic as a quotient of two holomorphic functions, consequently in a ring $r_2 \leqslant \|s-q\| \leqslant r_1$ function $Im\dfrac{\zeta'(s)}{\zeta(s)}$ is differentiable and therefore continuous and differentiable on τ function $Im\dfrac{\zeta'(f_{r_1,r_2}(\tau))}{\zeta(f_{r_1,r_2}(\tau))}$ for any differentiable $f_{r_1,r_2}(\tau)$.

  This means the real function which is continuous and differentiable on the segment takes on the ends of the same value:

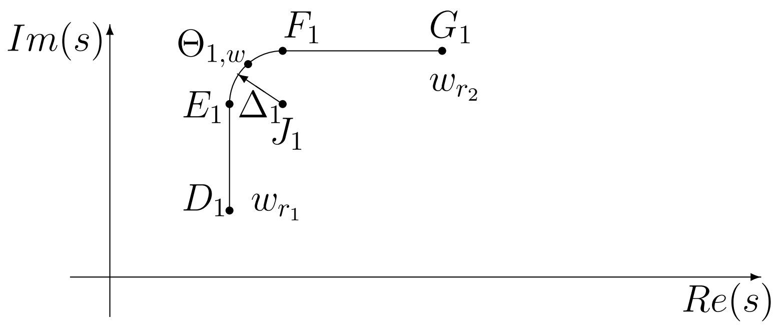
$$Im\frac{\zeta'(f_{r_1,r_2}(\tau_{r_1}))}{\zeta(f_{r_1,r_2}(\tau_{r_1}))} = Im\frac{\zeta'(f_{r_1,r_2}(\tau_{r_2}))}{\zeta(f_{r_1,r_2}(\tau_{r_2}))} = 0.$$

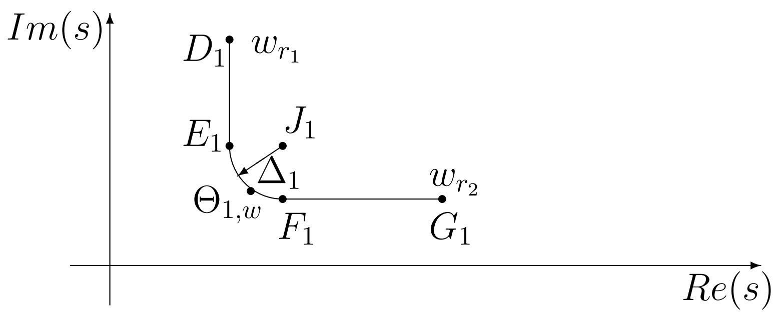
- According to the theorem on the extremum of a differentiable function on the interval we have:

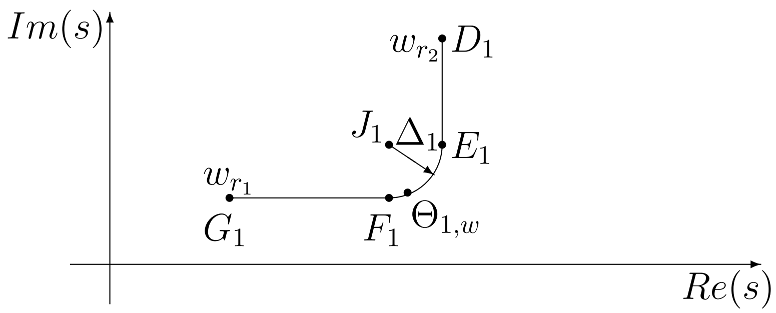
$$\exists\tau_1 \in (\tau_{r_1}, \tau_{r_2}):\quad \left(Im\frac{\zeta'(f_{r_1,r_2}(\tau_1))}{\zeta(f_{r_1,r_2}(\tau_1))}\right)' = 0. \tag{18}$$

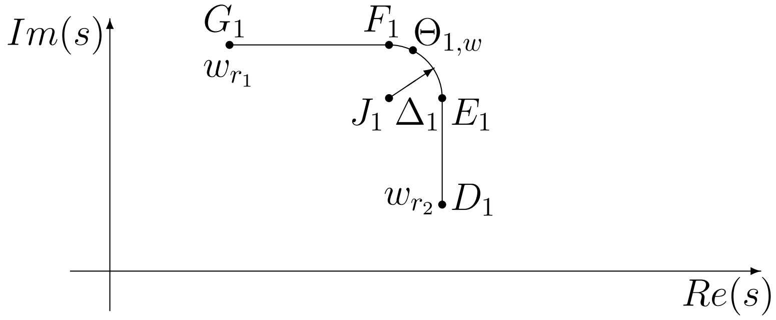
  I.e. on a curve line described by function $f_{r_1,r_2}(\tau),\ \ \tau \in (\tau_{r_1}, \tau_{r_2})$ there is a point of $\Theta_{1,w} \stackrel{\text{def}}{=} f_{r_1,r_2}(\tau_1)$ for which is true (18).

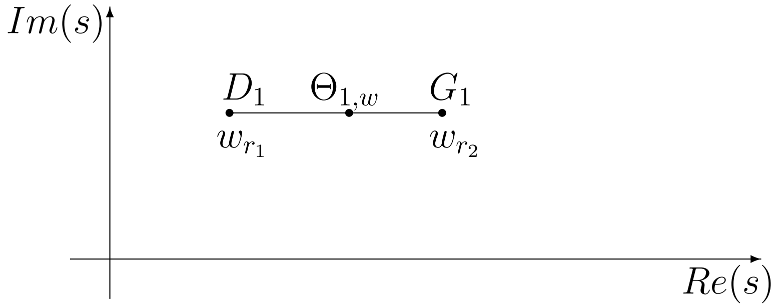
- Let's consider the following five versions as a curve line which is connecting points w_{r_1} and w_{r_2}:











Taking into account the fact that we have chosen the left semicircle and $r_2 < r_1$ these five variants of connection of the points w_{r_1} and w_{r_2} using the rounded corner are exhaustive.

The curve line is connecting w_{r_1} and w_{r_2}, it is consisted of three sections $D_1E_1,\ E_1F_1,\ F_1G_1$ or one - D_1G_1. Segments $D_1E_1,\ F_1G_1$ or like in the fifth case D_1G_1 is a parallel to the relevant axes, in the case of existence of E_1F_1, we will construct it like a quarter of a circle with radius $\Delta_1 > 0$ and centered at $J_1 \stackrel{\text{def}}{=} J_1(x_{\Delta_1}, y_{\Delta_1})$.
It's clear that $\Delta_1 < |Im(w_{r_1} - w_{r_2})|\,,\quad \Delta_1 < |\Re(w_{r_1} - w_{r_2})|$.

- Way $f_{r_1,r_2}(\tau)$ strictly speaking $f_{r_1,r_2}(\tau,\Delta_1) = f_x(\tau,\Delta_1) + i f_y(\tau,\Delta_1)$ in these five cases it will be described by the following equations systems.

For the segment D_1E_1:

$$\begin{cases} f_x(\tau,\Delta_1) = x_{w_{r_1}} \\ f_y(\tau,\Delta_1) = \tau, \end{cases} \tag{19}$$

where $y_{w_{r_1}} \leqslant \tau \leqslant (y_{w_{r_2}} - \Delta_1)$ for the first case and $(y_{w_{r_2}} + \Delta_1) \leqslant \tau \leqslant y_{w_{r_1}}$ for the second.
And

$$\begin{cases} f_x(\tau,\Delta_1) = x_{w_{r_2}} \\ f_y(\tau,\Delta_1) = \tau, \end{cases} \tag{20}$$

where $(y_{w_{r_1}} + \Delta_1) \leqslant \tau \leqslant y_{w_{r_2}}$ for the third case and $y_{w_{r_2}} \leqslant \tau \leqslant (y_{w_{r_1}} - \Delta_1)$ for the fourth.

For the segment E_1F_1:

$$\begin{cases} f_x(\tau,\Delta_1) = \tau \\ f_y(\tau,\Delta_1) = y_{w_{r_2}} - \Delta_1 + \sqrt{\Delta_1^2 - (\tau - x_{w_{r_1}} - \Delta_1)^2}, \end{cases} \tag{21}$$

where $x_{w_{r_1}} \leqslant \tau \leqslant (x_{w_{r_1}} + \Delta_1)$ for the first variant.
And for the second variant:

$$\begin{cases} f_x(\tau,\Delta_1) = \tau \\ f_y(\tau,\Delta_1) = y_{w_{r_2}} + \Delta_1 - \sqrt{\Delta_1^2 - (\tau - x_{w_{r_1}} - \Delta_1)^2}, \end{cases} \tag{22}$$

where $x_{w_{r_1}} \leqslant \tau \leqslant (x_{w_{r_1}} + \Delta_1)$.
For the third variant:

$$\begin{cases} f_x(\tau,\Delta_1) = \tau \\ f_y(\tau,\Delta_1) = y_{w_{r_1}} + \Delta_1 - \sqrt{\Delta_1^2 - (\tau - x_{w_{r_2}} + \Delta_1)^2}, \end{cases} \tag{23}$$

where $(x_{w_{r_2}} - \Delta_1) \leqslant \tau \leqslant x_{w_{r_2}}$.
For the fourth variant:

$$\begin{cases} f_x(\tau,\Delta_1) = \tau \\ f_y(\tau,\Delta_1) = y_{w_{r_1}} - \Delta_1 + \sqrt{\Delta_1^2 - (\tau - x_{w_{r_2}} + \Delta_1)^2}, \end{cases} \tag{24}$$

where $(x_{w_{r_2}} - \Delta_1) \leqslant \tau \leqslant x_{w_{r_2}}$.
For the segment F_1G_1:

$$\begin{cases} f_x(\tau,\Delta_1) = \tau \\ f_y(\tau,\Delta_1) = y_{w_{r_2}}, \end{cases} \tag{25}$$

where $(x_{w_{r_1}} + \Delta_1) \leqslant \tau \leqslant x_{w_{r_2}}$ for the first and second cases.
And

$$\begin{cases} f_x(\tau,\Delta_1) = \tau \\ f_y(\tau,\Delta_1) = y_{w_{r_1}}, \end{cases} \tag{26}$$

where $x_{w_{r_1}} \leqslant \tau \leqslant (x_{w_{r_2}} - \Delta_1)$ for the third and fourth cases.
For the segment D_1G_1 in the fifth case:

$$\begin{cases} f_x(\tau,\Delta_1) = \tau \\ f_y(\tau,\Delta_1) = y_{w_{r_2}}, \end{cases} \tag{27}$$

where $x_{w_{r_1}} \leqslant \tau \leqslant x_{w_{r_2}}$.

- Let's assume that the point $\Theta_{1,w}$ lies on the segment D_1E_1:
Then (18) which is given (19) can be written to $s = x + iy$, $\quad y_{\tau_1} \stackrel{\text{def}}{=} f_y(\tau_1)$, $\quad x_{\tau_1} \stackrel{\text{def}}{=} f_x(\tau_1)$ follows:

$$\left.\frac{d\left(Im\dfrac{\zeta'\left(x_{w_{r_1}}+i\tau\right)}{\zeta\left(x_{w_{r_1}}+i\tau\right)}\right)}{d\tau}\right|_{\tau=\tau_1} = \left.\frac{\partial\left(Im\dfrac{\zeta'\left(s\right)}{\zeta\left(s\right)}\right)}{\partial y}\right|_{s=f_{r_1,r_2}(\tau_1)} = 0 \qquad (28)$$

- Or given (11):

$$\left(-\frac{1}{2}Im\frac{\Gamma'\left(\dfrac{f_{r_1,r_2}(\tau_1)}{2}\right)}{\Gamma\left(\dfrac{f_{r_1,r_2}(\tau_1)}{2}\right)}+ImI_{\mathcal{P}\setminus\{q\}}(f_{r_1,r_2}(\tau_1))\right)' =$$

$$= -\left.\frac{\partial\dfrac{k(q)(t_q-y)}{(x_{w_{r_1}}-\sigma_q)^2+(y-t_q)^2}}{\partial y}\right|_{y=y_{\tau_1}} = -\frac{k(q)((y_{\tau_1}-t_q)^2-(x_{w_{r_1}}-\sigma_q)^2)}{((x_{w_{r_1}}-\sigma_q)^2+((y_{\tau_1}-t_q)^2)^2}$$

- The left part of last equality is limited $\forall r_1,\ r_2 : r_1 > 0,\ r_2 > 0,\ r_2 < r_1,\ r_1 < \min(R_1, R_2)$ and $f_{r_1,r_2}(\tau)$ from (19):

$$\left(-\frac{1}{2}Im\frac{\Gamma'\left(\dfrac{x_{r_1}+iy}{2}\right)}{\Gamma\left(\dfrac{x_{r_1}+iy}{2}\right)}+ImI_{\mathcal{P}\setminus\{q\}}(x_{r_1}+iy)\right)'_{y=y_{r_1}},$$

because in brackets is the imaginary part of a holomorphic $\forall s \in Q(R)$ functions

$$-\frac{1}{2}\frac{\Gamma'\left(\dfrac{s}{2}\right)}{\Gamma\left(\dfrac{s}{2}\right)}+I_{\mathcal{P}\setminus\{q\}}(s). \qquad (29)$$

The right part is equal to:

$$-\frac{k(q)(((y_{\tau_1}-t_q)^2-(x_{w_{r_1}}-\sigma_q)^2)}{((x_{w_{r_1}}-\sigma_q)^2+((y_{\tau_1}-t_q)^2)^2}=-\frac{k(q)(((y_{\tau_1}-t_q)^2-(x_{w_{r_1}}-\sigma_q)^2)}{r_{\theta_w}^4},$$

where r_{θ_w} - radius from point q to $\Theta_{1,w}$.

Letting $r_2 \to r_1$ and $r_1 \to 0$.
While considering (17):

$$r_{\theta_w}=O(r_1),\ \ (y_{\tau_1}-t_q)^2=O(r_1^4),\ \ (x_{w_{r_1}}-\sigma_q)^2=O(r_1^2). \tag{30}$$

- Therefore the expression

$$-\frac{k(q)(((y_{\tau_1}-t_q)^2-(x_{w_{r_1}}-\sigma_q)^2)}{r_{\theta_w}^4}=O(r_1^{-2}),\ \ r_1\to 0,$$

i.e. unlimited with a decrease r_1, which leads us to contradiction and means that the vertical stretch of the D_1E_1 consideration of the curve line doesn't have the point $\Theta_{1,w}$.

- On the basis of similar reasonings it is also true for the second, third and fourth version of a curve line. I.e. point $\Theta_{1,w}$ should be on a curve line between points E_1 and G_1 in first four versions of a curve line.

- Suppose that $\Theta_{1,w}$ lies in the quarter circle between points E_1 and F_1 as shown in the first four versions of a curve.

Considering that $J_1=J_1(x_{\Delta_1},y_{\Delta_1})$ and $\Theta_{1,w}=\Theta_{1,w}(x_{\tau_1},y_{\tau_1})$ we have:

$$(x_{\tau_1}-x_{\Delta_1})^2+(y_{\tau_1}-y_{\Delta_1})^2=\Delta_1^2.$$

Then the equations (21), (22), (23) and (24) can be written as a function of $y=y(x)$:

$$y=y_{\Delta_1}+\sqrt{\Delta_1^2-(x-x_{\Delta_1})^2},\ \ y\geqslant y_{\Delta_1}$$

and

$$y=y_{\Delta_1}-\sqrt{\Delta_1^2-(x-x_{\Delta_1})^2},\ \ y\leqslant y_{\Delta_1}.$$

- For definiteness we will consider that $y \geqslant y_{\Delta_1}$, then:

$$y(x)' = -\frac{x - x_{\Delta_1}}{\sqrt{\Delta_1^2 - (x - x_{\Delta_1})^2}} = -\frac{x - x_{\Delta_1}}{y - y_{\Delta_1}} \tag{31}$$

.

- And the equation (18) can be written as follows:

$$\left(-\frac{1}{2} Im \frac{\Gamma'\left(\dfrac{x+iy(x)}{2}\right)}{\Gamma\left(\dfrac{x+iy(x)}{2}\right)} + Im I_{\mathcal{P}\setminus\{q\}}(x+iy(x)) \right)'_{x=x_{\tau_1}} =$$

$$= -\frac{d\dfrac{k(q)(t_q - y(x))}{(x-\sigma_q)^2 + (y(x)-t_q)^2}}{dx}\Bigg|_{x=x_{\tau_1}}$$

The expression on the left side of the last equality in the partial derivatives:

$$\frac{\partial}{\partial x}\left(-\frac{1}{2} Im \frac{\Gamma'\left(\dfrac{x+iy_{\tau_1}}{2}\right)}{\Gamma\left(\dfrac{x+iy_{\tau_1}}{2}\right)} + Im I_{\mathcal{P}\setminus\{q\}}(x+iy_{\tau_1}) \right)\Bigg|_{x=x_{\tau_1}} +$$

$$+y(x_{\tau_1})' \frac{\partial}{\partial y}\left(-\frac{1}{2} Im \frac{\Gamma'\left(\dfrac{x_{\tau_1}+iy}{2}\right)}{\Gamma\left(\dfrac{x_{\tau_1}+iy}{2}\right)} + Im I_{\mathcal{P}\setminus\{q\}}(x_{\tau_1}+iy) \right)\Bigg|_{y=y_{\tau_1}} =$$

$$= O_{r_1\to 0}(1) + y(x_{\tau_1})' O_{r_1\to 0}(1)$$

- Considering continuity of sums in brackets at $\forall s = x + iy(x) \in Q(R)$ we have:

$$-\frac{d\dfrac{k(q)(t_q - y(x))}{(x-\sigma_q)^2 + (y(x)-t_q)^2}}{dx}\Bigg|_{x=x_{\tau_1}} = O(1) + y(x_{\tau_1})' O(1), \quad r_1 \to 0. \tag{32}$$

And reveal the derivative:

$$y(x_{\tau_1})'\frac{(y_{\tau_1}-t_q)^2-(x_{\tau_1}-\sigma_q)^2}{((x_{\tau_1}-\sigma_q)^2+(y_{\tau_1}-t_q)^2)^2}-\frac{2(\sigma_q-x_{\tau_1})(t_q-y_{\tau_1})}{((x_{\tau_1}-\sigma_q)^2+(y_{\tau_1}-t_q)^2)^2}=$$
$$=O_{r_1\to 0}(1)+y(x_{\tau_1})'O_{r_1\to 0}(1).$$

At small Δ_1, for example, $\Delta_1 = O_{r_1\to 0}(r_1^2)$ in accordance with (17) it follows that:

$$t_q-y_{\tau_1}=O_{r_1\to 0}(r_1^2),\quad \sigma_q-x_{\tau_1}=O_{r_1\to 0}(r_1)$$

and (32) can be written as:

$$y(x_{\tau_1})'O(r_1^{-2})-O(r_1^{-1})=O(1)+y(x_{\tau_1})'O(1),\quad r_1\to 0,$$

i.e.

$$y(x_{\tau_1})'=O_{r_1\to 0}(r_1). \tag{33}$$

- Let's note, that a sign in the case of $y \leqslant y_{\Delta_1}$ does not affect on the result (33) i.e. the ratio $\dfrac{x-x_{\Delta_1}}{y-y_{\Delta_1}}$ tends to 0 at $r_1 \to 0$ for $\forall\Delta_1 > 0$, $\Delta_1 = O_{r_1\to 0}(r_1^2)$.

And besides:

$$|y(x)'|=\left|\frac{x-x_{\Delta_1}}{y-y_{\Delta_1}}\right|\leqslant\left|\frac{x_{\tau_1}-x_{\Delta_1}}{y_{\tau_1}-y_{\Delta_1}}\right|=|y(x_{\tau_1})'|\,,$$

$\forall x : x_{\tau_1} < x < x_{\Delta_1}$ in the first and second version of the curve line or $\forall x : x_{\Delta_1} < x < x_{\tau_1}$ in the third and fourth.

I.e.

$$y(x)'=O_{r_1\to 0}(r_1) \tag{34}$$

for all the real parts of the x points $s = x+iy$ lying on the curve line under consideration between F_1 and $\Theta_{1,w}$ for $\forall\Delta_1 > 0$, $\Delta_1 = O_{r_1\to 0}(r_1^2)$.
Let's consider the third circle with the center in the point q and radius $r_3 > 0$, $r_3 < r_2$.
For couples $(r_2,\ r_3)$ apply the same reasoning as for couples $(r_1,\ r_2)$.

- This means that on the left semicircle $s : \|s-q\| = r_3, \quad \Re(s) \leqslant \Re(q)$ where there is a point w_{r_3}, in accordance with the (15):

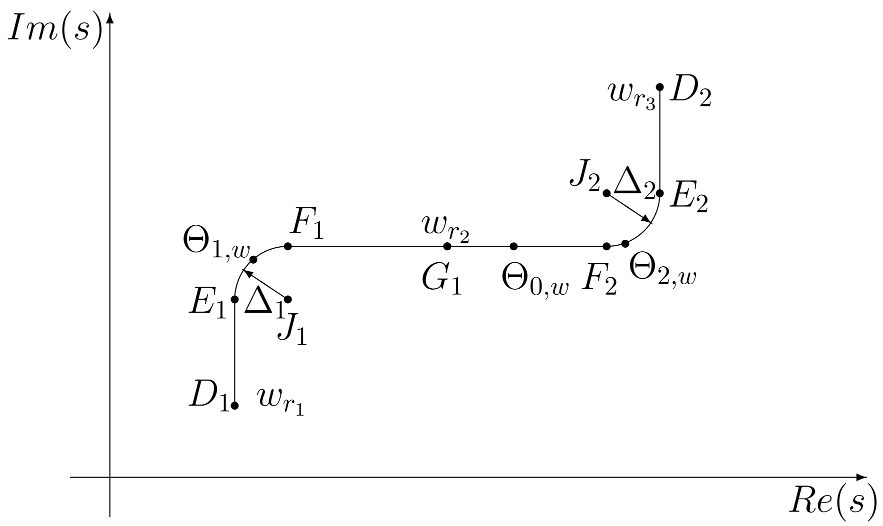
$$Im\frac{\zeta'(w_{r_3})}{\zeta(w_{r_3})} = 0.$$

Let's connect points w_{r_2} and w_{r_3} by a curved line in a same way as how we have connected w_{r_1} and w_{r_2}.

At the same time we will choose those options of the curve lines at which point w_{r_2} lies on the segment parallel to the real axis, so that at this point there were equal right and left derivative of the function describing the line connection points w_{r_1} and w_{r_3}.

Depending on the location of the points w_{r_1} and w_{r_2} let's choose line connection like in the first, second or fifth variant that was taken into consideration earlier.
These three variants exhaustive, noting that $r_2 < r_1$.
To connect w_{r_2} and w_{r_3} we choose respectively third, fourth or fifth a variant considered earlier connection.

- Without loss of generality, let's assume that $Im(w_{r_1}) < Im(w_{r_2}) < Im(w_{r_3})$ then the searched line will look as follows:



- On a line that passes through the points G_1, F_2, E_2 there is a point $\Theta_{2,w} \stackrel{\text{def}}{=} x_{\tau_2} + iy_{\tau_2}$ such as:

$$\left(Im\frac{\zeta'\left(x+iy(x)\right)}{\zeta\left(x+iy(x)\right)}\right)'_{x=x_{\tau_2}} = 0,$$

  where $y(x) = Imf_{r_2,r_3}(x)$.

- So that on a line which passes through points E_1, F_1, G_1, F_2, E_2 and consists of a horizontal site F_1F_2 and two quarters of circles with appropriating radiuses $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_1 = O_{r_1\to 0}(r_1^2)$, $\Delta_2 = O_{r_1\to 0}(r_1^2)$ by virtue of holomorphy $\dfrac{\zeta'(s)}{\zeta(s)}$ at all points $s \in Q(R) \setminus \{q\}$ function

$$\left(Im\frac{\zeta'\left(x+iy(x)\right)}{\zeta\left(x+iy(x)\right)}\right)'_{x}$$

  is differentiable everywhere along the stated line and takes in the points x_{r_1} and x_{r_2} the same value that is equal to 0.

- Let's note that $x_{r_1} < x_{r_2}$. This follows from the theorem on the extreme of it the point where the derivative becomes 0 does not coincide with the end point of the segment, i.e., it lies strictly inside it. So $\Theta_{1,w} \neq G_1$ and $\Theta_{2,w} \neq G_1$.

- Applying again the theorem on the extreme of the function, differentiable on the segment and takes the same values at the ends, we find that between $\Theta_{1,w}$ and $\Theta_{2,w}$ on the line $E_1F_1G_1F_2E_2$ must be a point $\Theta_{0,w} \stackrel{\text{def}}{=} x_{\tau_0} + iy_{\tau_0}$, $\tau_0 \in (\tau_1, \tau_2)$:

$$\left(Im\frac{\zeta'\left(x+iy(x)\right)}{\zeta\left(x+iy(x)\right)}\right)''_{x=x_{\tau_0}} = 0. \qquad (35)$$

  This can be rewritten as follows:

$$\left(-\frac{1}{2}Im\frac{\Gamma'\left(\dfrac{x+iy(x)}{2}\right)}{\Gamma\left(\dfrac{x+iy(x)}{2}\right)}+ImI_{\mathcal{P}\setminus\{q\}}(x+iy(x))\right)''_{x=x_{\tau_0}}=$$

$$=\left(-\frac{k(q)(t_q-y(x))}{(\sigma_q-x)^2+(t_q-y(x))^2}\right)''_{x=x_{\tau_0}}. \qquad (36)$$

The left part of last equality in private derivatives looks as follows:

$$\frac{d}{dx}\left(\frac{\partial}{\partial x}\left(-\frac{1}{2}Im\frac{\Gamma'\left(\dfrac{x+iy(x)}{2}\right)}{\Gamma\left(\dfrac{x+iy(x)}{2}\right)}+ImI_{\mathcal{P}\setminus\{q\}}(x+iy(x))\right)\right)\Bigg|_{x=x_{\tau_0}}+$$

$$+\frac{d}{dx}\left(y(x)'\,\frac{\partial}{\partial y}\left(-\frac{1}{2}Im\frac{\Gamma'\left(\dfrac{x+iy(x)}{2}\right)}{\Gamma\left(\dfrac{x+iy(x)}{2}\right)}+ImI_{\mathcal{P}\setminus\{q\}}(x+iy(x))\right)\right)\Bigg|_{\substack{x=x_{\tau_0}\\ y=y_{\tau_0}}}=$$

$$=\frac{\partial^2}{\partial x^2}\left(-\frac{1}{2}Im\frac{\Gamma'\left(\dfrac{x+iy_{\tau_0}}{2}\right)}{\Gamma\left(\dfrac{x+iy_{\tau_0}}{2}\right)}+ImI_{\mathcal{P}\setminus\{q\}}(x+iy_{\tau_0})\right)\Bigg|_{x=x_{\tau_0}}+$$

$$+2y(x_{\tau_0})'\,\frac{\partial^2}{\partial x\partial y}\left(-\frac{1}{2}Im\frac{\Gamma'\left(\dfrac{x+iy}{2}\right)}{\Gamma\left(\dfrac{x+iy}{2}\right)}+ImI_{\mathcal{P}\setminus\{q\}}(x+iy)\right)\Bigg|_{\substack{x=x_{\tau_0}\\ y=y_{\tau_0}}}+$$

$$+\left(y(x_{\tau_0})'\right)^2\frac{\partial^2}{\partial y^2}\left(-\frac{1}{2}Im\frac{\Gamma'\left(\dfrac{x_{\tau_0}+iy}{2}\right)}{\Gamma\left(\dfrac{x_{\tau_0}+iy}{2}\right)}+ImI_{\mathcal{P}\setminus\{q\}}(x_{\tau_0}+iy)\right)\Bigg|_{y=y_{\tau_0}}+$$

$$+y(x_{\tau_0})''\frac{\partial}{\partial y}\left(-\frac{1}{2}Im\frac{\Gamma'\left(\dfrac{x_{\tau_0}+iy}{2}\right)}{\Gamma\left(\dfrac{x_{\tau_0}+iy}{2}\right)}+ImI_{\mathcal{P}\setminus\{q\}}(x_{\tau_0}+iy)\right)\Bigg|_{y=y_{\tau_0}} \overset{\text{def}}{=}$$

$$\overset{\text{def}}{=} A_0(x_{\tau_0},y_{\tau_0})+A_1(x_{\tau_0},y_{\tau_0})y(x_{\tau_0})'+$$
$$+A_2(x_{\tau_0},y_{\tau_0})(y(x_{\tau_0})')^2+A_3(x_{\tau_0},y_{\tau_0})y(x_{\tau_0})''.$$

This is similar to (29) all $A_i(x,y),\ i\in[0;3]$ limited to $\forall s=x+iy\in Q(R)$.

The right side of (36) in turn:

$$\left(-\frac{k(q)(t_q-y(x))}{(\sigma_q-x)^2+(t_q-y(x))^2}\right)''_{x=x_{\tau_0}}=$$
$$=k(q)\left(\frac{y(x)'((t_q-y(x))^2-(\sigma_q-x)^2)}{((\sigma_q-x)^2+(t_q-y(x))^2)^2}\right)'_{x=x_{\tau_0}}-$$
$$-k(q)\left(\frac{2(t_q-y(x))(\sigma_q-x)}{((\sigma_q-x)^2+(t_q-y(x))^2)^2}\right)'_{x=x_{\tau_0}}=$$
$$=k(q)y(x_{\tau_0})''\frac{(t_q-y_{\tau_0})^2-(\sigma_q-x_{\tau_0})^2}{((\sigma_q-x_{\tau_0})^2+(t_q-y_{\tau_0})^2)^2}+$$
$$+k(q)y(x_{\tau_0})'\frac{\partial}{\partial x}\left(\frac{(t_q-y_{\tau_0})^2-(\sigma_q-x)^2}{((\sigma_q-x)^2+(t_q-y_{\tau_0})^2)^2}\right)\Bigg|_{x=x_{\tau_0}}+$$
$$+k(q)(y(x_{\tau_0})')^2\frac{\partial}{\partial y}\left(\frac{(t_q-y)^2-(\sigma_q-x_{\tau_0})^2}{((\sigma_q-x_{\tau_0})^2+(t_q-y)^2)^2}\right)\Bigg|_{y=y_{\tau_0}}-$$
$$-k(q)\frac{\partial}{\partial x}\left(\frac{2(t_q-y_{\tau_0})(\sigma_q-x)}{((\sigma_q-x)^2+(t_q-y_{\tau_0})^2)^2}\right)'\Bigg|_{x=x_{\tau_0}}-$$
$$-k(q)y(x_{\tau_0})'\frac{\partial}{\partial y}\left(\frac{2(t_q-y)(\sigma_q-x_{\tau_0})}{((\sigma_q-x_{\tau_0})^2+(t_q-y)^2)^2}\right)'\Bigg|_{y=y_{\tau_0}}\overset{\text{def}}{=}$$
$$\overset{\text{def}}{=} B_0(x_{\tau_0},y_{\tau_0})+B_1(x_{\tau_0},y_{\tau_0})y(x_{\tau_0})'+$$
$$+B_2(x_{\tau_0},y_{\tau_0})(y(x_{\tau_0})')^2+B_3(x_{\tau_0},y_{\tau_0})y(x_{\tau_0})''.$$

- And (36) can be written as follows:

$$\begin{aligned} A_0(x_{\tau_0}, y_{\tau_0}) + A_1(x_{\tau_0}, y_{\tau_0}) y(x_{\tau_0})' + \\ + A_2(x_{\tau_0}, y_{\tau_0})(y(x_{\tau_0})')^2 + A_3(x_{\tau_0}, y_{\tau_0}) y(x_{\tau_0})'' = \\ = B_0(x_{\tau_0}, y_{\tau_0}) + B_1(x_{\tau_0}, y_{\tau_0}) y(x_{\tau_0})' + \\ + B_2(x_{\tau_0}, y_{\tau_0})(y(x_{\tau_0})')^2 + B_3(x_{\tau_0}, y_{\tau_0}) y(x_{\tau_0})''. \end{aligned}$$

Or:

$$\begin{aligned} (B_3(x_{\tau_0}, y_{\tau_0}) - A_3(x_{\tau_0}, y_{\tau_0})) y(x_{\tau_0})'' = \\ A_0(x_{\tau_0}, y_{\tau_0}) - B_0(x_{\tau_0}, y_{\tau_0}) + (A_1(x_{\tau_0}, y_{\tau_0}) - B_1(x_{\tau_0}, y_{\tau_0})) y(x_{\tau_0})' + \\ + (A_2(x_{\tau_0}, y_{\tau_0}) - B_2(x_{\tau_0}, y_{\tau_0}))(y(x_{\tau_0})')^2. \end{aligned} \tag{37}$$

- Still to be specific we will assume that $y \geqslant y_{\Delta_1}$, then from the (31):

$$\begin{aligned} y(x)'' = \left(-\frac{x - x_{\Delta_1}}{\sqrt{\Delta_1^2 - (x - x_{\Delta_1})^2}} \right)' = \\ = -\frac{\sqrt{\Delta_1^2 - (x - x_{\Delta_1})^2} - (x - x_{\Delta_1})(-\frac{x - x_{\Delta_1}}{\sqrt{\Delta_1^2 - (x - x_{\Delta_1})^2}})}{\Delta_1^2 - (x - x_{\Delta_1})^2} = \\ = -\frac{\Delta_1^2}{(\Delta_1^2 - (x - x_{\Delta_1})^2)^{\frac{3}{2}}}. \end{aligned}$$

- Let's suppose that $\Theta_{0,w}$ lies in one of the quarters of circles centered at the points J_1, J_2 and radius Δ_1, Δ_2, between the points $\Theta_{1,w}$ and F_1 or between F_2 and $\Theta_{2,w}$. To be specific we will assume that it is - the first quarter of the circle.

- Then using the (34) we have:

$$\frac{x_{\tau_0} - x_{\Delta_1}}{y_{\tau_0} - y_{\Delta_1}} = O_{r_1 \to 0}(r_1), \quad (x_{\tau_0} - x_{\Delta_1})^2 + (y_{\tau_0} - y_{\Delta_1})^2 = \Delta_1^2.$$

- Consequently:

$$(x_{\tau_0} - x_{\Delta_1})^2 = \frac{\Delta_1^2}{1 + O_{r_1 \to 0}(r_1^2)}$$

and

$$y(x_{\tau_0})'' = -\frac{\Delta_1^2}{(\Delta_1^2 - (x_{\tau_0} - x_{\Delta_1})^2)^{\frac{3}{2}}} = -\frac{1}{\Delta_1(1 - (\frac{1}{1 + O_{r_1 \to 0}(r_1^2)}))^{\frac{3}{2}}}.$$

- It means that $\exists\ R_3 > 0,\quad R_3 < \min(R_1, R_2)$:for any fixed $r_1 < R_3$ and $\forall\forall\ r_3 < r_2 < r_1,\ \ r_2 > 0,\ r_3 > 0$, the second derivative $y(x)''$ at the point x_{τ_0} is unlimited at $\Delta_1 \to 0$.

Then function

$$\begin{aligned} B_3(x_{\tau_0}, y_{\tau_0}) = &\quad k(q)\frac{(t_q - y_{\tau_0})^2 - (\sigma_q - x_{\tau_0})^2}{((\sigma_q - x_{\tau_0})^2 + (t_q - y_{\tau_0})^2)^2} = \\ = &\quad k(q)\frac{(t_q - y_{\tau_0})^2 - (\sigma_q - x_{\tau_0})^2}{r_{\tau_0}^4}, \end{aligned}$$

where r_{τ_0} the length of the radius of a circle with center at the point q passing through $\Theta_{0,w}$ such $\forall M_1 > 0\ \exists\ R_4 > 0,\ R_4 < R_3$: $\forall\ r_1 < R_4,\ \ \exists\ \delta > 0,\ \forall r_2,\ r_3 : |r_1 - r_3| < \delta$

$$|B_3(x_{\tau_0}, y_{\tau_0})| > M_1.$$

- This follows from the (30).

As a M_1 constant we will take:

$$M_1 = \max_{x+iy \in Q(R)}(|A_3(x, y)|),$$

then $\forall\ r_1 < R_4,\ \ \exists\ \delta > 0,\ \forall r_2,\ r_3 : |r_1 - r_3| < \delta :$

$$|B_3(x_{\tau_0}, y_{\tau_0}) - A_3(x_{\tau_0}, y_{\tau_0})| > 0. \tag{38}$$

Then for $\forall M_2 > 0 \ \exists \ R_5 > 0, \ R_5 < R_4$: $\forall \ r_1 < R_5, \quad \exists \ \delta > 0, \ \forall r_2, \ r_3 :$ $|r_1 - r_3| < \delta, \exists \ \Delta_1 > 0 :$

$$|(B_3(x_{\tau_0}, y_{\tau_0}) - A_3(x_{\tau_0}, y_{\tau_0}))\, y(x_{\tau_0})''| > M_2.$$

- I.e. in the left part of equality (37) is infinitely increasing function with $\Delta_1 \to 0$ for any fixed three $r_1, \ r_2, \ r_3$ that satisfies the condition (38).

In the right part (37), in view of (34) there is a sum of the limited functions for the same fixed three $r_1, \ r_2, \ r_3$ and $\Delta_1 \to 0$.
Indeed, from the (34) it follows that $\forall \epsilon > 0, \ \exists R_6 > 0 : \ \forall r_1 < R_6$ fulfilled $|y(x_{\tau_0})'| < \epsilon$.

- And taking M_2 for fixed $r_1, \ r_2, \ r_3$ that satisfy the condition (38) and $r_1 < R_6$:

$$M_2 = \max_{s = x+iy : r_3 \leqslant \|s-q\| \leqslant r_1} \left(|A_0(x,y) - B_0(x,y)| + |A_1(x,y) - B_1(x,y)|\, \epsilon + \right.$$
$$\left. + \, |A_2(x,y) - B_2(x,y)|\, \epsilon^2 \right),$$

we come to the contradiction in (37).

- It means that the assumption that the point $\Theta_{0,w}$ lays on the open arch of the circle between points $\Theta_{1,w}$ and F_1 since instant place is incorrectly. Similarly it is false from a certain moment and the assumption that the point $\Theta_{0,w}$ lays on the opened arch of the second circle between points F_2 and $\Theta_{2,w}$.

- I.e. for any r_1 since some threshold exist $r_2 > 0, \ r_3 > 0, \ \Delta_1 > 0$ so that the point $\Theta_{0,w}$ lies on the segment $F_1 F_2$.

And (36) can be written as:

$$\left(-\frac{1}{2}Im\frac{\Gamma'\left(\dfrac{x+iy_{\tau_0}}{2}\right)}{\Gamma\left(\dfrac{x+iy_{\tau_0}}{2}\right)}+ImI_{\mathcal{P}\setminus\{q\}}(x+iy_{\tau_0})\right)''_{x=x_{\tau_0}}=$$

$$=\left(-\frac{k(q)(t_q-y_{\tau_0})}{(\sigma_q-x)^2+(t_q-y_{\tau_0})^2}\right)''_{x=x_{\tau_0}}. \tag{39}$$

Or

$$\left(-\frac{k(q)(t_q-y_{\tau_0})}{(\sigma_q-x)^2+(t_q-y_{\tau_0})^2}\right)''_{x=x_{\tau_0}}=O_{r_1\to 0}(1). \tag{40}$$

Let's open the brackets of the last equality:

$$\left(-\frac{k(q)(t_q-y_{\tau_0})}{(\sigma_q-x)^2+(t_q-y_{\tau_0})^2}\right)''_{x=x_{\tau_0}}=$$
$$-k(q)\left(\frac{-2(\sigma_q-x)(t_q-y_{\tau_0})}{((\sigma_q-x)^2+(t_q-y_{\tau_0})^2)^2}\right)'_{x=x_{\tau_0}}=$$
$$=-2k(q)(t_q-y_{\tau_0})\frac{(t_q-y_{\tau_0})^2-3(\sigma_q-x_{\tau_0})^2}{((\sigma_q-x_{\tau_0})^2+(t_q-y_{\tau_0})^2)^3}$$

- From (17) with condition $r_3\to r_2\to r_1\to 0$:

$$(t_q-y_{\tau_0})^2=O(r_1^4),\ \ (\sigma_q-x_{\tau_0})^2=O(r_1^2)$$

and (40) can be written as follows:

$$(t_q-y_{\tau_0})O(r_1^{-4})=O(1),\ \ r_3\to r_2\to r_1\to 0.$$

- According to the fact that $y_{\tau_0}=y_{\tau_2}$ because the point $\Theta_{0,w}$ lies on the segment F_1F_2 we get:

$$t_q-y_{\tau_2}=O(r_2^4),\ \ r_2\to 0,$$

or

$$t_q - y_{w_r} = O(r^4), \quad r \to 0. \tag{41}$$

If our assumption that for any $R_7 > 0,\ R_7 < R:\ \exists\, r_1 < R_7$: point $\Theta_{1,w}$ lays on an arch of a circle of one of versions about connection of points w_{r_1} and w_{r_2} is incorrect that $\exists R_7 > 0,\ R_7 < R:\ \forall\, r_1 < R_7$: point $\Theta_{1,w}$ lies on the segment F_1G_1 so we will not constract a third circle with a radius r_3 and search for the point $\Theta_{0,w}$.

- That's enough to rewrite (18) as:

$$\left(-\frac{1}{2}Im\frac{\Gamma'\left(\dfrac{x+iy_{\tau_1}}{2}\right)}{\Gamma\left(\dfrac{x+iy_{\tau_1}}{2}\right)} + ImI_{\mathcal{P}\setminus\{q\}}(x+iy_{\tau_1})\right)'_{x=x_{\tau_1}} =$$

$$= \left(-\frac{k(q)(t_q - y_{\tau_1})}{(\sigma_q - x)^2 + (t_q - y_{\tau_1})^2}\right)'_{x=x_{\tau_1}}.$$

Or

$$2k(q)\frac{(\sigma_q - x_{\tau_1})(t_q - y_{\tau_1})}{((\sigma_q - x_{\tau_1})^2 + (t_q - y_{\tau_1})^2)^2} = O(1), \quad r_2 \to r_1 \to 0.$$

Then on the basis of (17), under condition of $r_2 \to r_1 \to 0$:

$$(t_q - y_{\tau_1})O(r_1^{-3}) = O(1), \quad r_2 \to r_1 \to 0.$$

And according to the fact that $y_{\tau_1} = y_{w_{r_2}}$ as the point $\Theta_{1,w}$ lies on the segment F_1G_1 we receive:

$$t_q - y_{w_{r_2}} = O(r_2^3), \quad r_2 \to 0,$$

i.e.

$$t_q - y_{w_r} = O(r^3), \quad r \to 0. \tag{42}$$

Let's combine all of the cases from (41) and (42):

$$t_q - y_{w_r} = O(r^3), \quad r \to 0. \tag{43}$$

- Similarly acting with the point z_r we come to conclusion that:

$$t_q - y_{z_r} = O(r^3), \quad r \to 0. \tag{44}$$

Let's consider the limit:

$$\lim_{r\to 0}\left(Im\frac{\zeta'(w_r)}{\zeta(w_r)} + Im\frac{\zeta'(1-z_r)}{\zeta(1-z_r)}\right).$$

- By the construction of (15) and (16):

$$\lim_{r\to 0}\left(Im\frac{\zeta'(w_r)}{\zeta(w_r)} + Im\frac{\zeta'(1-z_r)}{\zeta(1-z_r)}\right) = 0 \tag{45}$$

And if we expand:

$$\lim_{r\to 0} Im\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{w_r}{2}\right)}{\Gamma\left(\frac{w_r}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-z_r}{2}\right)}{\Gamma\left(\frac{1-z_r}{2}\right)} + I_{\mathcal{P}\setminus\{q\}}(w_r) - I_{\mathcal{P}\setminus\{q\}}(z_r) + \right.$$
$$\left. + k(q)\frac{t_q - y_{w_r}}{r^2} - k(q)\frac{t_q - y_{z_r}}{r^2}\right) \stackrel{\text{def}}{=} I_1 + I_2 + I_3,$$

where:

$$I_1 = \lim_{r\to 0} Im\left(-\frac{1}{2}\frac{\Gamma'\left(\frac{w_r}{2}\right)}{\Gamma\left(\frac{w_r}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-z_r}{2}\right)}{\Gamma\left(\frac{1-z_r}{2}\right)}\right),$$

$$I_2 = \lim_{r \to 0} Im \left(I_{\mathcal{P}\setminus\{q\}}(w_r) - I_{\mathcal{P}\setminus\{q\}}(z_r) \right),$$

$$I_3 = \lim_{r \to 0} Im \left(k(q)\frac{t_q - y_{w_r}}{r^2} - k(q)\frac{t_q - y_{z_r}}{r^2} \right).$$

By the continuity of the digamma function and the function $ImI_{\mathcal{P}\setminus\{q\}}(s), \quad \forall s \in Q(R)$ we have:

$$I_1 = Im \left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)} \right),$$

$$I_2 = 0,$$

In accordance with (43) and (44)

$$I_3 = 0.$$

- Thus, from the (45) we have:

$$Im \left(-\frac{1}{2}\frac{\Gamma'\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} - \frac{1}{2}\frac{\Gamma'\left(\frac{1-q}{2}\right)}{\Gamma\left(\frac{1-q}{2}\right)} \right) = 0, \quad \forall q \in \mathcal{P}_1 \cup \mathcal{P}_2. \tag{46}$$

- From (6) equality (46) can be written as follows:

$$\sum_{n=1}^{\infty}\left(\frac{t_q}{(2n+\sigma_q)^2+t_q^2}-\frac{t_q}{(2n+1-\sigma_q)^2+t_q^2}\right)=0.$$

I.e.

$$\sum_{n=1}^{\infty}\frac{t_q((2n+1-\sigma_q)^2-(2n+\sigma_q)^2)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)}=$$
$$=\sum_{n=1}^{\infty}\frac{t_q(1-2\sigma_q)(4n+1)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)}=$$
$$=(1-2\sigma_q)\sum_{n=1}^{\infty}\frac{t_q(4n+1)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)}=0.$$

Sum

$$\sum_{n=1}^{\infty}\frac{t_q(4n+1)}{((2n+\sigma_q)^2+t_q^2)((2n+1-\sigma_q)^2+t_q^2)}$$

exists and is not equal to 0 when $t_q\neq 0$ so the equality (46) is performed exclusively at

$$\sigma_q=\frac{1}{2}.$$

So, assuming that an arbitrary nontrivial root of zeta functions q belongs to the union $\mathcal{P}_1\cup\mathcal{P}_2$ we found that it belongs only to $\mathcal{P}_2$, i.e. $\mathcal{P}_1=\varnothing$.

And according to the fact that $\|\mathcal{P}_3\|=\|\mathcal{P}_1\|=0$ we have:

$$\mathcal{P}_3=\mathcal{P}_1=\varnothing,\ \ \mathcal{P}=\mathcal{P}_2,$$

Which is given above proves the basic statement and the assumption which had been made by Bernhard Riemann about the location of the real parts of the nontrivial zeros of zeta function.

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