ALGEBRAIC POINCARÉ DAULITY 2

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DEC, 2015

Abstract. This is the sequel of [6], [7]. In this paper we apply algebraic Poincaré duality to the maximal sub Hodge structures to show

(1) Generalized Hodge conjecture of level 1 is correct.

(2) The generalized Hodge conjecture of level 0, i.e. the usual Hodge conjecture, is correct.

1 Introduction

We work over the field $\mathbb{C}$ throughout. Let X be a smooth projective variety of dimension n. There is Poincaré duality: the intersection pairing $\mathcal{B}_X$ (or simply $\mathcal{B}$),

$$H^i(X;\mathbb{Q}) \times H^{2n-i}(X;\mathbb{Q}) \quad \rightarrow \quad \mathbb{Q}. \tag{1.1}$$

is non-degenerate.

We are interested in types of subspaces

$$G_2 \times G_1 \subset H^i(X;\mathbb{Q}) \times H^{2n-i}(X;\mathbb{Q}) \tag{1.2}$$

such that the restriction $\mathcal{B}|_{G_2\times G_1}$ is also non-degenerate. We call the non-degeneracy of $\mathcal{B}|_{G_2\times G_1}$, algebraic Poincaré duality in the title or APD for the abbreviation.

In [7], we proved two results:

Theorem 1.1. *(B. Wang) Let $G_2 \times G_1$ be the "algebraic part" of cohomology,*

$$H_a^{2p+1}(X;\mathbb{Q}) \times H_a^{2n-2p-1}(X;\mathbb{Q}) \subset H^{2p+1}(X;\mathbb{Q}) \times H^{2n-2p-1}(X;\mathbb{Q}) \tag{1.3}$$

where the "algebraic part" $H_a^{2p+1}(X;\mathbb{Q}) \otimes \mathbb{C}$ is also called "cohomology lying on a subvariety of codimension at least p", denoted by $Filt'^p H^{2p+1}(X;\mathbb{C})$. In this case, APD holds.

Theorem 1.2. *(B. Wang) Let $G_i, i = 1, 2$ be the subspaces A^p, A^{n-p} spanned by all rational algebraic cycles of corresponding codimensions p, q respectively. So*

$$G_2 \times G_1 = A^p(X) \times A^{n-p}(X) \subset H^{2p}(X;\mathbb{Q}) \times H^{2n-2p}(X;\mathbb{Q}). \tag{1.4}$$

In this case, APD holds.

In this paper, we are going to extend these two results in the following way.

Extension of theorem 1.1: Replace G_2 by a subspace called the linear span of families of algebro-singular cycles, denoted by

$$H_{sp}^{2p+1}(X;\mathbb{Q}). \tag{1.5}$$

$H_{sp}^{2p+1}(X;\mathbb{Q})$ will be proved to be the maximal sub Hodge structure of coniveau p (see claim 3.2).

THEOREM 1.3. *Then APD on*

$$H_{sp}^{2p+1}(X;\mathbb{Q}) \times H_a^{2n-2p-1}(X;\mathbb{Q}) \tag{1.6}$$

holds

Theorem 1.3 leads to an extension of theorem 1.2: Let

$$Hdg^{2i}(X) = H^{i,i}(X;\mathbb{C}) \cap H^{2i}(X;\mathbb{Q})$$

denote the subspace of $H^{2i}(X;\mathbb{Q})$, consisting of Hodge classes of degree $2i$.

THEOREM 1.4. *Then the APD on*

$$Hdg^{2p}(X) \times A^{n-p}(X) \tag{1.7}$$

holds.

COROLLARY 1.5. *The usual Hodge conjecture is correct, i.e.*

$$Hdg^{2i}(X) = A^i(X).$$

Remark It is clear that theorems 1.3, 1.4 imply theorems 1, 2. However the contents of theorem 1, 2 are more direct and more natural.

Notations:
(1) $(\bullet)^*$ denotes the dual of a vector space if $\bullet$ is a vector space or a vector.
(2) $(\bullet)^*$ also denotes a pullback from the cohomology or differential forms if $\bullet$ is a map.
(3) $(\bullet)_*$ denotes a pushforwad of the homology, or cycles, or currents if $\bullet$ is a map.
(4) $\overline{\bullet}$ denotes the complex conjugation on a complex vector space.
(5) $[a]$ denotes an equivalence class of the element a.
(6) CH denotes the Chow group, CH_{alg} denotes the subgroup of cycles algebraically equivalent to zero.
(7) J denotes the intermediate Jacobian or the Jacobian.
(8) $\mathcal{T}$ denotes the complex torus.

(9) All homology and cohomology are groups modulo their torsions.

The idea of the proof is the same as that for [7]. To implement it, we only need to replace the algebraic cycles in [7] by "algebro-singular cycles" defined in the following. Then verify all the steps in [7] are valid for "algebro-singular cycles". Let us be more specific on the idea. It suffices to show theorem 1.3, 1.4 for cohomology groups of complex coefficients. Let's prove theorem 1.3 first. This is an imitation of the proof of theorem 1.1 in [7], where the key ingredients are algebraic cycles and Abel-Jacobi maps. In this paper, we replace the algebraic cycles by topologically singular cycles of (p,p) type currents. We call them algebro-singular cycles. These are ordinary topological cycles, but locally as currents they are non-zero functional only on the $(n-p, n-p)$ type of differential forms. All such cycles homologous to zero form a subgroup in the group of simplicial cycles homologous to zero and there is a map from this subgroup to the intermediate Jacobian, which is an extension of the usual Abel-Jacobi map. We call it singular Abel-Jacobi map. We'll prove this is regular. Once this is all set-up. We'll repeat the proof of theorem 1.1. To be inclusive, we describe it in the following. For each fixed odd degree $2r-1$, those classes of degree $2r-1$ form a complex torus called "intermediate Jacobian $J^r(X)$". Subsequently the partially algebro-singular cycles form a subtorus

$$J^r_{sp}(X) \subset J^r(X)$$

which is Abelian inside of the intermediate Jacobian. Then the APD on the two subgroups of cohomology of complementary degrees

$$2p+1, 2n-2p-1$$

becomes a particular non-degeneracy property for the Poincaré line bundle— the induced map $\mathcal{P}$ from one Abelian variety to the dual of another Abelian variety

$$J^{p+1}_{sp}(X) \quad \stackrel{\mathcal{P}}{\to} \quad Pic^0(J^{n-p}_a(X))$$

is an isogeny. We call the isogeny the Saito's duality ([4]) between

$$J^{p+1}_{sp}(X) \quad and \quad Pic^0(J^{n-p}_a(X)).$$

This duality is not symmetric in the usual sense because $J^{p+1}_{sp}(X)$ is presumably larger than $J^{p+1}_a(X)$. Our proof explores the interpretation of the Abelian subvariety $J^{p+1}_{sp}(X)$ as follows. Let

$$C_2 \quad \stackrel{\rho}{\to} \quad J^{p+1}_{sp}(X) \tag{1.8}$$

be a regular map from a smooth projective curve to the Abelian variety. Then it is known that the Jacobian $J(C_2)$ of C_2 is mapped to $J^{p+1}_{sp}(X)$, whose image denoted by $\mathcal{T}_{C_2}$ is isomorphic to a quotient of the Jacobian $J(C_2)$. By the Jacobi inversion, if C_2 goes through 0, C_2 is contained in $\mathcal{T}_{C_2}$. Now such Abelian varieties $\mathcal{T}_{C_2}$ will cover the entire $J_{sp}^{p+1}(X)$ (also finitely many $\mathcal{T}_{C_2}$ will generate $J_{sp}^{p+1}(X)$). We call the set of $\mathcal{T}_{C_2}$ the curve-like sub Abelian structure of $J_{sp}^{p+1}(X)$. Then similarly the other part of the dual,

$$Pic^0(J_a^{n-p}(X)) \simeq J_a^{n-p}(X) \tag{1.9}$$

also has the curve-like sub Abelian structure, whose Abelian subvarieties are denoted by $\mathcal{T}_{C_2^1}$ for smooth projective curves C_2^1. [1] With the curve-like sub Abelian structures we obtain that the duality between

$$J_{sp}^{p+1}(X) \quad and \quad Pic^0(J_a^{n-p}(X))$$

is equivalent to the duality between

$$\mathcal{T}_{C_2} \quad and \quad \mathcal{T}_{C_2^1}$$

for a set of C_2 that covers $J_a^{p+1}(X)$ and all contain 0. Then it suffices to show the duality between one pair $\mathcal{T}_{C_2}$ and $\mathcal{T}_{C_2^1}$ with only one condition on C_2: it has to contain two points, an arbitrary given point and 0 (actually this condition with the help of Jacobi inversion is just to insure $\mathcal{T}_{C_2}$ covers $J_{sp}^{p+1}(X)$). Above steps were completed in the first paper [6] without presence of geometric X.

The last step, which is in this paper, is to show that the duality between two Abelian subvarieties

$$\mathcal{T}_{C_2} \quad and \quad \mathcal{T}_{C_2^1}$$

for selective curves C_2 holds. The ambient Abelian variety $J_{sp}^{p+1}(X)$ in general has another interpretation as the image of the Abel-Jacobi map on the parameter space of algebro-singular cycles of degree 1 less. Singular Abel-Jacobi map is compatible with the correspondence between two varieties, which means that the curve C_2 to the Abelian variety $J_{sp}^{p+1}(X)$ gives a rise to a correspondence Z of degree $-(n+1-p)$ from X to C_2, where the correspondence Z is some topological cycle (not an algebraic cycle as in algebraic geometry). Using this interpretation, when restricted to a curve C_2, above duality becomes the equality of two sub linear spaces of $H^{1,0}(C_2)$ of the curve C_2 (i.e. equality of two Abelian subvarieties inside of the Jacobian $J(C_2)$), obtained by pulling the cohomologies on X back to C_2 through Z. A proof of the equality is the key to reveal a deep relation on the topological cycles on X. The difficulty of it is apparent. However the difficulty turns into a quasi-Lefschetz-hyperplane-theorem (see footnote 4 after formula (3.44)) once the curve C_2, i.e. the correspondence Z is chosen in a particular way. Thus the flexibility in choosing the correspondence Z, which only requires C_2 goes through two

[1] Such a duality is so natural and spreads into the Abelian varieties. For instance, the dual curve C_2^1 is exactly $\mathcal{P}(C_2)$.

points, is the key. This is all done in section 3.2, step 1. This is the proof of theorem 1.3.

After theorem 1.3 we prove theorem 1.4 and corollary 1.5 in section 4. Through a product with an elliptic curve, the APD on partially algebro-singular cycle classes in cohomology of odd degrees descends to that on Hodge cycle classes in cohomology of even degrees.

2 Algebro-singular cycles

To apply the abstract formulation in [6], we must deal with cycles, not classes of cycles. So we use singular homology and cohomology for their singular cycles. [2] Thus the chains and cycles are all singular in the sense of singular homology from topology (not the "singular" in algebraic geometry). First we extend the concept of algebraic cycles to the category of singular cycles. The most part of this section consists of definitions. There are numerous definitions that bring in many new concepts. But if one notices that every definition below has a counterpart for algebraic cycles and all definitions are natural extensions of those for algebraic cycles, then the definitions become obvious. The key in this step is to pinpoint the properties of algebraic cycles that must be preserved during the extension.[3]

Throughout we always use whole numbers p, q satisfying $p + q = n - 1$. Let $\mathcal{Z}^{2p}(X)$ be the group of singular cycle group of real codimension $2p$ on X. There is a group injection

$$\begin{array}{ccc} \mathcal{Z}^{2p}(x) & \stackrel{\nu}{\to} & \mathcal{D}^{2p}(X;\mathbb{C}) \\ a & \to & \nu(a) \end{array} \tag{2.1}$$

where $\mathcal{D}^{2p}(X;\mathbb{C})$ is the space of complex valued C^∞ currents.

Definition 2.1. *(Algebro-singular cycles). Let*

$$\mathcal{Z}^{p,p}(X) \tag{2.2}$$

be the inverse image $\nu^{-1}(\mathcal{D}^{p,p}(X;\mathbb{C}))$, *where* $\mathcal{D}^{p,p}(X;\mathbb{C})$ *is the space of currents of* (p,p) *type. Let*

$$\mathcal{Z}^{p,p}_{hom}(X) \subset \mathcal{Z}^{p,p}(X) \tag{2.3}$$

[2] It is equivalent to use currents which are the functionals of differential forms.

[3] After modulo the boundaries of chains, those cycles form sub Hodge structures in cohomology. If the sub Hodge structure is the final goal as mentioned in abstract, why can't we use the already established theory of Hodge structure to avoid the tedious build-up in this paper? This is because the Hodge structure alone will not give an answer to the problem. One of the most important ingredient in the proof is the Abel-Jacobi map which allows some cycle classes in cohomology of odd degrees to carry two different structures: topological as integral classes in cohomology, and complex analytic as algebro-singular cycles of degree 1 less. The Abel-Jacobi map requires cycles, not classes of cycle.

be the subgroup consists of cycles homologous to zero. Elements in $\mathcal{Z}^{p,p}(X)$ are called algebro-singular cycles of codimension $2p$. Thus

$$\frac{\mathcal{Z}^{p,p}(X)\otimes\mathbb{Q}}{\mathcal{Z}^{p,p}_{hom}(X)\otimes\mathbb{Q}} = Hdg^{2p}(X).$$

Remark. It is well-known (for instance §3, section 1, 2 in [1]) that every Hodge class has a representative in

$$\mathcal{Z}^{p,p}(X)\otimes\mathbb{Q}.$$

Thus it suffices to consider the singular cycle group

$$\mathcal{Z}^{p,p}(X).$$

DEFINITION 2.2. *(Partially algebro-singular part of cohomology)*
Let p, q be non-negative integers satisfying $p+q=n-1$.
(a) (a 1 parameter family of algebro-singular cycles). Let S be a smooth projective curve. A singular cycle

$$I \in \mathcal{Z}^{p+1,p+1}(S\times X) \tag{2.4}$$

is called a 1 parameter family of algebro-singular cycles if
(1) All simplexes of I are C^∞, embedded, $2q+2$ dimensional and for each simplex $\delta : \Delta_{2q+2}\to S\times X$, the projection

$$\delta(\Delta_{2q+2})\to S$$

is proper, and the total projection

$$|I|\to S$$

is surjective.
(2) the fibre δ_γ of $\delta(\Delta_{2q+2})\to S$ over any singular chain γ of S is a singular chain, and each fibre $S_s=\sum\delta_s$ lies in

$$\mathcal{Z}^{p+1,p+1}(X).$$

For any chain γ, denote the projection of $\sum\delta_\gamma$ to X by $\nu_S(\gamma)$,
(b) Let $H^{2p+1}_S(X;\mathbb{Z})$ be the group of cohomology classes that are represented by $\nu_S(\gamma), \gamma\in\mathcal{Z}_1(S)$, where $\mathcal{Z}_1$ is the singular cycle group of real dimension 1. Let

$$H^{2p+1}_S(X;\mathbb{A}) = H^{2p+1}_S(X;\mathbb{Z})\otimes\mathbb{A} \tag{2.5}$$

for $\mathbb{A} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

(c) A class $c \in H^{2p+1}(X;\mathbb{C})$ *is called partially algebro-singular if*

$$c \in H_S^{2p+1}(X;\mathbb{C})$$

for some S. *All partially algebro-singular cycles span a subspace denoted by*

$$H_{as}^{2p+1}(X).$$

(d) In the definition (a), if

$$B = I_{s_1} - I_{s_0} \in \mathcal{Z}_{hom}^{p+1,p+1}(X) \tag{2.6}$$

we say B *is algebro-singularly equivalent to zero, and* S *goes through* B. *All such cycles that are algebro-singularly equivalent to zero form a subgroup, which is denoted by*

$$\mathcal{Z}_a^{p+1,p+1}(X). \tag{2.7}$$

It is clear that in even dimension, algebraic cycles are algebro-singular, and in odd dimension, partially algebraic cycles are partially algebro-singular, i.e.

$$H_a^{2p+1}(X) \subset H_{as}^{2p+1}(X).$$

The following lemma and theorem 2.7, part 1 have a fundamental importance in the algebraic Poincaré duality. It brings the complex structure into the topological intersection numbers.

Lemma 2.3. *There is a group homomorphism*

$$\mathcal{Z}_{hom}^{p,p}(X) \stackrel{\phi_s}{\to} J^p(X). \tag{2.8}$$

Proof. This is restricted to the Abel-Jacobi map on algebraic cycles. Let

$$B \in \mathcal{Z}_{hom}^{p,p}(X).$$

Then there exists a singular chain Γ_B such that

$$\partial \Gamma_B = B. \tag{2.9}$$

Define

$$\begin{array}{rcl} \phi_s : Z_{hom}^{p,p}(X) & \to & J^p(X). \\ B & \to & \int_{\Gamma_B}(\cdot), \end{array} \tag{2.10}$$

where $\int_{\Gamma_B}(\cdot)$ is defined to be the functional on the differential forms in

$$F^{q+2}H^{2q+3}(X).$$

Let's show ϕ_s is well-defined. It suffices to investigate two ambiguous factors:

(1) If Γ'_B is another chain such that

$$\partial\Gamma'_B = B, \tag{2.11}$$

and $\phi + \bar{\phi}$ is integral. Then

$$\int_{\Gamma_B - \Gamma'_B} \phi + \bar{\phi} \tag{2.12}$$

is an integer because

$$\Gamma_B - \Gamma'_B$$

is an integral cycle. Therefore

$$\int_{\Gamma_B} \phi + \bar{\phi} = \int_{\Gamma_{B'}} \phi + \bar{\phi},\ mod\ \mathbb{Z}. \tag{2.13}$$

(2) Suppose $\phi = d(\omega)$, and $\omega = \sum_{i+j=2q+2} \omega_{i,j}$ where $\omega_{i,j}$ are differential forms of (i,j) type. Because the type of ϕ,

$$\int_{\Gamma_B} \phi = \int_{\Gamma_B} \partial\omega_{q+1,q+1} + d(\sum_{i=0}^{q} \omega_{q+2+i,q-i}) \tag{2.14}$$

and $\omega_{q+1,q+1}$ is a holomorphic form. Then we may let

$$\omega_{q+1,q+1} = \bar{\partial}\theta_{q+1,q}$$

where $\theta_{q+1,q}$ is a $(q+1,q)$ form.

Now we have

$$\int_{\Gamma_B} \phi = \int_B \omega_{q+1,q+1} = \int_B d(\theta_{q+1,q}) = 0. \tag{2.15}$$

This completes the proof. □

Definition 2.4.

(1) The homomorphism ϕ_s will be called singular Abel-Jacobi map. We denote its image $\phi_s(\mathcal{Z}^{p,p}_{hom})$ by

$$J^p_{hom-as}(X)\ or\ J^{hom-as}_{n-p}(X). \tag{2.16}$$

We denote its subset, $\phi_s(\mathcal{Z}^{p,p}_a(X))$ by

$$J^p_{as}(X)\ or\ J^{as}_{n-p}(X). \tag{2.17}$$

We call $J^p_{as}(X)$ partially algebro-singular part of intermediate Jacobian, where as stands for algebro-singular.

(2) If S is a 1 parameter family of algebro-singular cycles of codimension $2p+2$ (the degree is changed from that in part (1)), the composition map

$$CH^1_a(S) \rightarrow \mathcal{Z}_a^{p+1,p+1}(X) \rightarrow J^{p+1}(X) \tag{2.18}$$

will also be denoted by ϕ_s (This should not cause any confusion in the context).

Definition 2.5.

(1) Let S, I be a one parameter family of algebo-singular cycles of codimension $2p+2$ as in definition 2.2, part (a). We'll repeat the definition in definition 2.2, then extend it to the singular chains. Let

$$H^S_{2q+1}(X;\mathbb{Z}) \tag{2.19}$$

be the subgroup of $H_{2q+1}(X;\mathbb{Z})$ defined to be the image of the map

$$\begin{array}{rcl} \nu_S : H_1(S;\mathbb{Z}) & \rightarrow & H_{2q+1}(X;\mathbb{Z}) \\ \gamma & \rightarrow & (Pr_X)_*((\gamma\times X)\cap \mathcal{S}). \end{array} \tag{2.20}$$

To avoid a mess in notations, the same map on the singular chains is also denoted by ν_S:

$$\begin{array}{rcl} \nu_S : C_1(S) & \rightarrow & C_{2q+1}(X) \\ \Gamma & \rightarrow & (Pr_X)_*((\Gamma\times X)\cap \mathcal{S}), \end{array} \tag{2.21}$$

where C_* is the group of chains. Note ν_S on chains is well-defined because of the definition of S.

Let

$$H^S_{2q+1}(X;\mathbb{A}) = H^S_{2q+1}(X;\mathbb{Z})\otimes \mathbb{A} \tag{2.22}$$

and

$$H^{2p+1}_S(X;\mathbb{A}) \tag{2.23}$$

be its Poincaré dual, for $\mathbb{A}=\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{C}$.

(2) Let Λ^{2p+1}_S be the projection of $H^{2p+1}_S(X;\mathbb{Z})$ into

$$\overline{F^{p+1}H^{2p+1}(X;\mathbb{C})}.$$

Denote

$$span_{\mathbb{C}}(\Lambda^{2p+1}_S)\subset \overline{F^{p+1}H^{2p+1}(X;\mathbb{C})}, \tag{2.24}$$

by

$$\overline{F^{p+1}H^{2p+1}_S(X;\mathbb{C})}. \tag{2.25}$$

Then we'll show in the immediate following, lemma 2.6 that Λ^{2p+1}_S is a lattice of

$$\overline{F^{p+1}H^{2p+1}_S(X;\mathbb{C})}. \tag{2.26}$$

Thus we let

$$J_S^{p+1}(X)=\frac{\overline{F^{p+1}H_S^{2p+1}(X;\mathbb{C})}}{\Lambda_S^{2p+1}}. \tag{2.27}$$

(3) Denote the kernel of

$$\phi_s:\mathcal{Z}_a^{p+1,p+1}(X)\quad\rightarrow\quad J^{p+1}(X) \tag{2.28}$$

by

$$\mathcal{Z}_{ab-jac}^{p+1,p+1}(X). \tag{2.29}$$

(4) Let $\mathcal{P}$ be the duality homomorphism

$$J^{p+1}(X)\quad\rightarrow\quad Pic^0(J^{q+1}(X)) \tag{2.30}$$

induced from the Poincaré line bundle. We denote the kernel of the composition $\mathcal{P}\circ\phi_s$

$$\mathcal{Z}_a^{p+1,p+1}(X)\ \stackrel{\phi_s}{\rightarrow}\ J^{p+1}(X)\ \stackrel{\mathcal{P}}{\rightarrow}\ Pic^0(J^{q+1}(X)) \tag{2.31}$$

by

$$\mathcal{Z}_{inc}^{p+1,p+1}(X). \tag{2.32}$$

A cycle $B\in\mathcal{Z}_{inc}^{p+1,p+1}(X)$ will be called "incidence equivalent to zero".

In the definition 2.5, part (2) is proved in the following lemma

Lemma 2.6. *Λ_S^{2p+1} is a lattice of*

$$\overline{F^{p+1}H_S^{2p+1}(X;\mathbb{C})}. \tag{2.33}$$

Proof. Let S, I be a one parameter family of algebo-singular cycles of codimension $2p+2$ as in definition 2.2, part (a). The Poincaré dual $H_S^{2p+1}(X;\mathbb{Z})$ of

$$\nu_S(H_1(S;\mathbb{Z})),$$

projects to Λ_S^{2p+1}.

Considering the diagram

$$\begin{array}{ccc} & S\times X & \\ \swarrow p_1 & & p_2\searrow \\ S & & X. \end{array} \tag{2.34}$$

we obtain that

$$H_S^{2p+1}(X;\mathbb{C})=(p_2)_*\bigg(p_1^*(H^1(S;\mathbb{C}))\cup P_I\bigg). \tag{2.35}$$

This implies that

$$\overline{F^{p+1}H_S^{2p+1}(X;\mathbb{C})} = (p_2)_*\Big(p_1^*(H^{0,1}(S;\mathbb{C}))\cup P_I\Big), \tag{2.36}$$

where P_I is the Poincaré dual to $I \subset S \times X$. Because

$$H^1(S;\mathbb{C}) = H^{0,1}(S;\mathbb{C}) + H^{1,0}(S;\mathbb{C}),$$

once applying the Künneth decomposition to P_I, we obtain that

$$dim_{\mathbb{C}}(\overline{F^{p+1}H_S^{2p+1}(X;\mathbb{C})}) = \frac{1}{2}rank(\Lambda_S^{2p+1}). \tag{2.37}$$

This completes the proof.
□

It is clear that $J_S^{p+1}(X)$ is a subtorus of the intermediate Jacobian $J^{p+1}(X)$.

THEOREM 2.7.
(1) ϕ_s is a regular map on a 1-parameter family S of algebro-singular cycles.
(2) Let S be a family of algebro-singular cycles of codimension $p+1$. Then

$$J_S^{p+1}(X) = \phi_s(CH^1_{alg}(S)). \tag{2.38}$$

and it is a sub-Abelian variety.

Proof. (1) Let (S, I) be a 1 parameter family of algebro-singular cycles , where S is smooth. For the simplicity we may assume I is a manifold of

$$S \times X.$$

Consider its correspondence

$$\begin{array}{ccc} & I & \\ \swarrow \pi_1 & & \pi_2 \searrow \\ S & & X. \end{array} \tag{2.39}$$

Let $\omega \in F^{q+1}H^{2q+1}(X;\mathbb{C})$. Let $s_0 \in S$ be a fixed point and $s \in S$ be general. We may assume ω is a smooth closed form that is

$$\omega = \sum_{i>j} \omega_{i,j}, \tag{2.40}$$

where $\omega_{i,j}$ is a form of (i, j) type. Choose s to be close to s_0 such that the homology of the analytic neighborhood is zero. So let Δ be a neighborhood of s_0 in S such that it is homotopic to s_0. Let

$$I_\Delta = I \cap (\Delta \times X) \tag{2.41}$$

(i.e. $\nu_S(\Delta)$) be the restriction of I to Δ. Then I_Δ is homotopic to I_{s_0}. This means that differential form

$$(\pi_2)^*(\omega) \tag{2.42}$$

is an exact form $d(\theta)$. Now we consider the well-defined function

$$F(s) = \int_{s_0}^{s} I^*(\omega) \tag{2.43}$$

in Δ.

$$dF(s) = (\pi_1)_* d(\theta) \tag{2.44}$$

where $(\pi_1)_*$ is the fibre integral of a differential form. Now $(\pi_1)_* d(\theta)$ must be a $(1,0)$ form because $d(\theta) = (\pi_2)^*(\omega)$ is extended to a form in

$$F^{q+1}H^{2q+1}(S \times X; \mathbb{C}).$$

Therefore

$$\bar{\partial} F(s) = 0. \tag{2.45}$$

So $F(s)$ is holomorphic.

(2) First we consider the commutative diagram

$$\begin{array}{ccc} H_1(S;\mathbb{C}) & \stackrel{\nu_S}{\longrightarrow} & H_{2q+1}(X;\mathbb{C}) \\ P \Big\downarrow \Big\uparrow P^{-1} & & P \Big\downarrow \Big\uparrow P^{-1} \\ H^1(S;\mathbb{C}) & \stackrel{I^*}{\longleftarrow} & H^{2p+1}(X;\mathbb{C}) \end{array} \tag{2.46}$$

where I^* is the induced from the pushforward of currents,

$$I \to S$$

of C^∞-forms on $S \times X$ restricted to I, and P is the Poincaré duality. By the Poincaré duality, the cokernel of ν_S is the kernel of I^* and vice versa. In the spaces of the diagram (2.46), we remove all cokernels and kernels. Then obtain the diagram

$$\begin{array}{ccc} W_1 & \stackrel{\nu_T}{\longrightarrow} & V_{2q+1} \\ P \Big\downarrow \Big\uparrow P^{-1} & & P \Big\downarrow \Big\uparrow P^{-1} \\ W^1 & \stackrel{I^*}{\longleftarrow} & V^{2p+1} \end{array} \tag{2.47}$$

where W_1 is the direct sum complement of the kernel of ν_S, V_{2q+1} is the image of ν_S, W^1 is the image of I^* and V^{2p+1} is a direct sum complement of the kernel of I^*. All homomorphism in (2.47) are isomorphisms.

Inside of these spaces there are spanning integral lattices denoted by $W_1^{\mathbb{Z}}$, $W^1_{\mathbb{Z}}$, $V^{\mathbb{Z}}_{2q+1}$ and $V^{2p+1}_{\mathbb{Z}}$. Applying the Hodge decomposition to the bottom row in (2.47), we obtain

$$(W^1)' \quad \stackrel{I^*}{\longleftarrow} \quad (V^{2p+1})' \tag{2.48}$$

where

$$(W^1)' = W^1 \cap H^{0,1}(X;\mathbb{C})$$
$$(V^{2p+1})' = V^{2p+1} \cap \overline{F^{p+1}H^{2p+1}(X;\mathbb{C})}.$$

Let

$$(W^1_{\mathbb{Z}})'$$
$$(V^{2p+1}_{\mathbb{Z}})'.$$

be their integral lattices. The existence of the integral lattices is guaranteed by the induced, sub-Hodge structure of $H_S^{2p+1}(X;\mathbb{C})$. Then by the isomorphism in (2.47), we obtain that

$$\frac{(W^1)'}{(W^1_{\mathbb{Z}})'} \simeq \frac{(V^{2q+1})'}{(V^{2q+1}_{\mathbb{Z}})'}. \tag{2.49}$$

By the definition, the right side of (2.49)

$$\frac{(V^{2p+1})'}{(V^{2p+1}_{\mathbb{Z}})'} \tag{2.50}$$

is

$$J_S^{p+1}(X).$$

To understand the left side of (2.49), we need to understand the Jacobian $J(S)$ of the curve S. There is projection map

$$J(S) \quad \rightarrow \quad \tfrac{(W^1)'}{(W^1_{\mathbb{Z}})'} \tag{2.51}$$

induced from the linear space projection ($(W^1)'$ is a subspace of $H^{0,1}(S;\mathbb{C})$),

$$H^{0,1}(S;\mathbb{C}) \rightarrow (W^1)'.$$

Let

$$\begin{array}{ccc} \phi' : CH^1_{alg}(S) & \rightarrow & \frac{(W^1)'}{(W^1_{\mathbb{Z}})'} \\ t & \rightarrow & \int_t \theta \end{array} \tag{2.52}$$

where θ is a $(1,0)$ closed form. Applying the diagram (2.46) to the maps ϕ_s and ϕ', we obtain that

$$\phi_s(CH^1_{alg}(S)) = \phi'(CH^1_{alg}(S)). \tag{2.53}$$

We conclude this with a commutative diagram

$$\begin{array}{ccc} CH^1_{alg}(S) & = & CH^1_{alg}(S) \\ \phi' \Big\downarrow & & \phi_s \Big\downarrow \\ \frac{(W^1)'}{(W^1_{\mathbb{Z}})'} & \xrightarrow{\simeq} & \frac{(V^{2p+1})'}{(V^{2p+1}_{\mathbb{Z}})'}. \end{array} \tag{2.54}$$

For the Abel-Jacobi map AJ, it is known that

$$AJ(CH^1_{alg}(S)) = J(S). \tag{2.55}$$

Hence ϕ' is onto, i.e.

$$\frac{(W^1)'}{(W^1_{\mathbb{Z}})'} = \phi'(CH^1_{alg}(S)). \tag{2.56}$$

So is ϕ_s. This completes the proof.

□

Definition 2.8. *Let $\overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{C})}$ be the linear span of all subspaces $\overline{F^{p+1}H^{2p+1}_{S}(X;\mathbb{C})}$ where S is a family of algebro-singular cycles. Let $\Lambda^{2p+1}_{sp}(\mathbb{Z})$ be its integral subgroup generated by all Λ^{2p+1}_S (all S). It is easy to see it spans $\overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{C})}$ over $\mathbb{C}$. We define*

$$J^{p+1}_{sp}(X) = \frac{\overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{C})}}{\Lambda^{2p+1}_{sp}(\mathbb{Z})}. \tag{2.57}$$

Here "sp" stands for span. This space is obtained by spanning of "curve-like" subspaces $\overline{F^{p+1}H^{2p+1}_{S}(X;\mathbb{C})}$.

We let

$$F^{p+1}H^{2p+1}_{sp}(X;\mathbb{C}) = \overline{(\overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{C})})}. \tag{2.58}$$

Define

$$H^{2p+1}_{sp}(X;\mathbb{C}) = \overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{C})} \oplus F^{p+1}H^{2p+1}_{sp}(X;\mathbb{C}). \tag{2.59}$$

Remark. We let $\overline{F^{p+1}_{as}H^{2p+1}(X;\mathbb{C})}$ be the tangent space of the torus $J^{p+1}_{as}(X)$. Let

$$H^{2p+1}_{as}(X;\mathbb{C}) = \overline{F^{p+1}_{as}H^{2p+1}(X;\mathbb{C})} + F^{p+1}_{as}H^{2p+1}(X;\mathbb{C}).$$

Then it is noticed that

$$H^{2p+1}_{sp}(X;\mathbb{C}) = H^{2p+1}_{as}(X;\mathbb{C}).$$

We are interested in the expression of the vector space $H^{2p+1}_{sp}(X;\mathbb{C})$ only.

3 APD on the cohomology of odd degrees

3.1 Review of the generalized APD for an application With the appropriate set-up of algebro-singular cycles, we can repeat the proof in [7]. It starts with the cohomology of odd degrees, then goes into the cohomology of even degrees. Let's recall theorem 4.5 in [6]. This theorem has a general setting, which does not require the presence of the projective variety X. Let V_1, V_2 be two isomorphic vector spaces over $\mathbb{C}$, equipped with a complex, non-degenerate, bilinear form $\mathcal{B}$ on $V_2 \times V_1$. Let Λ_1, Λ_2 be lattices of V_1, V_2, i.e. they are discrete subgroups of V_i such that

$$\Lambda_i \simeq \mathbb{Z}^{2dim(V_i)} \ and \ span_{\mathbb{R}}(\Lambda_i) = V_i.$$

Assume

$$\Lambda_i^a = \Lambda_i \cup V_i^a \tag{3.1}$$

are lattices of V_i^a. Let

$$\mathcal{B}_2 : V_2^a \xrightarrow{\mathcal{B}|_{V_2^a}} (V_1)^* \xrightarrow{restriction} (V_1^a)^*$$

$$\mathcal{B}_1 : V_1^a \xrightarrow{\mathcal{B}|_{V_1^a}} (V_2)^* \xrightarrow{restriction} (V_2^a)^*.$$

Also we assume the $\mathcal{B}$ satisfies

$$\Lambda_2 \times \Lambda_1 \to \mathbb{Z}.$$

We defined the lattice of the dual of the vector space to be the

$$hom(lattice, \mathbb{Z}). \tag{3.2}$$

We obtain the complex tori

$$\mathcal{T}(V_i^a) = \frac{V_i^a}{\Lambda_i^a}. \tag{3.3}$$

Let $C \stackrel{\rho}{\to} \mathcal{T}(V_i^a)$ be a regular map from a smooth complex curve C (not necessarily contains the origin).

Define Λ_C to be the image

$$H_1(C; \mathbb{Z}) \quad \to \quad H_1(\mathcal{T}(V_i^a)) = \Lambda_i^a. \tag{3.4}$$

Furthermore we denote $span_{\mathbb{C}}(\Lambda_C)$ by

$$H_C.$$

By [6], Λ_C is an integral lattice of H_C.

Define

$$\frac{H_C}{\Lambda_C} = \mathcal{T}_C, \tag{3.5}$$

to be the sub-torus of $\mathcal{T}(V_i^a)$. We call it a curve-like sub-torus.

Definition 3.1. *(curve-like sub-Abelian structure)*

We say the complex torus $\mathcal{T}(V_i^a)$ has a curve-like sub-Abelian structure if there are a projective variety M_i and a SURJECTIVE group homomorphism from the Chow group of 0-cycles of M_i, that are algebraically equivalent to zero to the complex torus,

$$\phi_i : CH_{alg}^{dim(M_i)}(M_i) \quad \to \quad \mathcal{T}(V_i^a) \tag{3.6}$$

for $i = 1, 2$, satisfying:

(1) it is regular, i.e. for any projective family T_i of algebraic 0-cycles of M_i, the composition map

$$\psi_{T_i} : T_i \quad \xrightarrow{(T_i)_t - m_i} \quad CH_{alg}^{dim(M_i)}(M_i) \quad \xrightarrow{\phi_i} \quad \mathcal{T}(V_i^a) \tag{3.7}$$

is regular, where m_i is a fixed point in the same irreducible component of M_i as T_i.

(2) If T_i is a smooth projective curve parametrizing 0-cycles on M_i, we'll denote the composition map

$$CH_{alg}^1(T_i) \quad \to \quad CH_{alg}^{dim(M_i)}(M_i) \quad \xrightarrow{\phi_i} \quad \mathcal{T}(V_i^a) \tag{3.8}$$

also by ϕ_i. Then for $\psi_{T_i}(T_i) \in \mathcal{T}(V_i^a)$,

$$\phi_i(CH_{alg}^1(T_i)) = \mathcal{T}_{\psi_{T_i}(T_i)} \tag{3.9}$$

and it is Abelian.

Let C_2 be a smooth curve on M_2, and $\psi_{C_2} : C_2 \to \mathcal{T}(V_2^a)$ be a morphism from definition 3.1. Denote the composition map

$$C_2 \quad \to \quad \mathcal{T}(V_2^a) \quad \stackrel{\tilde{\mathcal{B}}_R}{\to} \quad \mathcal{T}((V_1)^*) \tag{3.10}$$

by ρ_1, and

$$C_2 \quad \to \quad \mathcal{T}(V_2^a) \quad \stackrel{\tilde{\mathcal{B}}_2}{\to} \quad \mathcal{T}((V_1^a)^*) \tag{3.11}$$

by ρ_1^a. We define

Definition 3.2.

$$
\begin{aligned}
E_1 &= (\rho_1)^* \bigg(H^{1,0}(\mathcal{T}((V_1)^*))\bigg) \\
E_1^a &= (\rho_1^a)^* \bigg(H^{1,0}(\mathcal{T}((V_1^a)^*))\bigg),
\end{aligned} \tag{3.12}
$$

and

$$
\sigma(C_2) = dim(E_1) - dim(E_1^a). \tag{3.13}
$$

Similarly we define $\sigma(C_1)$ for $C_1 \subset M_1$.

Then the theorem 4.5 in [6] says

Theorem 3.3.
Let $\beta \in V_2^a$, $\beta' \in V_1^a$. Assume
(1) $\mathcal{T}(V_i^a)$ both are Abelian and both have curve-like sub Abelian structures,
(2) there exists a curve C_2 such that C_2 is through $\pi_2(\beta), 0$ with

$$
\sigma(C_2) = 0. \tag{3.14}
$$

and the same is true in opposite direction, i.e., there exists a curve C_1 such that C_1 is through $\pi_1(\beta'), 0$ for given $\beta' \in V_1^a$ with,

$$
\sigma(C_1) = 0. \tag{3.15}
$$

Then APD holds.

3.2 A geometric application

To apply theorem 3.3, we start with the geometric set-up. Let

$$
\begin{aligned}
V_2 &= \overline{F^{p+1}H^{2p+1}(X;\mathbb{C})} \\
V_1 &= F^{q+1}H^{2q+1}(X;\mathbb{C})
\end{aligned} \tag{3.16}
$$

Let

$$
\begin{aligned}
V_2^a &= \overline{F^{p+1}H_{sp}^{2p+1}(X;\mathbb{C})} \\
V_1^a &= F^{q+1}H_a^{2q+1}(X;\mathbb{C})
\end{aligned} \tag{3.17}
$$

Now due to the sub Hodge structures of algebraic parts, they both have lattices denoted by

$$
\begin{gathered}
\Lambda_{sp}^{2p+1}(\mathbb{Z}), \\
F^{q+1}H_a^{2q+1}(X;\mathbb{Z})
\end{gathered} \tag{3.18}
$$

respectively. Now it is known that the complex tori obtained from $V_i^a, i = 1, 2$ are just the images of Abel-Jacobi maps, and it is well-known that $J(V_1^a)$ is Abelian. Thus

$$\begin{aligned}\mathcal{T}(V_2^a) &\simeq J_{sp}^{p+1}(X)\\ \mathcal{T}(V_1^a) &\simeq J_a^{q+1}(X).\end{aligned} \tag{3.19}$$

Next we precisely define M_1, M_2. We start with more general set-up for Abel-Jacobi maps. Let Q be a smooth projective variety (irreducible) and

$$\mathcal{Q} \subset Q \times X \tag{3.20}$$

be a flat family of codimension r schemes in X. This defines a family of $n-r$ cycles by taking

$$\mathcal{Q}_q = (Pr_X)_*[\mathcal{Q} \cap (\{q\} \times X)], \tag{3.21}$$

where $q \in \mathcal{Q}$. Then there is a holomorphic map into the intermediate Jacobian, called Abel-Jacobi map ψ_Q:

$$\begin{array}{ccc} Q & \stackrel{\psi_Q}{\to} & J^r(X) = \frac{(F^{n-r+1}H^{2n-2r+1}(X))^*}{H_{2n-2r+1}(X;\mathbb{Z})} \\ q & \to & \frac{\int_{\Gamma_q}(\cdot)}{H_{2n-2r+1}(X;\mathbb{Z})}, \end{array} \tag{3.22}$$

where Γ_q is a singular chain on X satisfying $\partial\Gamma_q = \mathcal{Q}_q - \mathcal{Q}_{q_0}$ and $q_0 \in Q$ is fixed. If $\mathcal{Q}_q$ is rationally equivalent to $\mathcal{Q}_{q_0}$, then $\psi_Q(q) = 0$. This allows us to have a well-defined map

$$\begin{array}{ccc} CH_{alg}^{n-r}(X) & \stackrel{\phi}{\to} & J^r(X) \\ B & \to & \phi(B) \end{array} \tag{3.23}$$

where the map is denoted by ϕ, still called Abel-Jacobi map. By the proposition 1.2, [4], there exist an Abelian variety M and a cycle class

$$\mathcal{M} \subset CH^r(M \times X) \tag{3.24}$$

such that the composition map ψ_M

$$M \xrightarrow{\mathcal{M}_m - \mathcal{M}_{m_0}} CH_{alg}^r(X) \xrightarrow{\phi(\mathcal{M}_m - \mathcal{M}_{m_0})} J_a^r(X) \tag{3.25}$$

is an isogeny.

Coming back to our situation we let M_1 be the above Abelian variety for the Abelian variety $\mathcal{T}(V_1^a)$ and ϕ_1 be the Abel-Jacobi map induced from ψ_{M_1}. i.e.

$$CH_{alg}^{dim(M_1)}(M_1) \stackrel{\phi_1}{\to} J_a^{q+1}(X). \tag{3.26}$$

Next we consider $\mathcal{T}(V_2^a) \simeq J_{sp}^{p+1}(X)$, and would like to show it is also Abelian. Notice

$$span_{all\ S}(H_S^{2p+1}(X;\mathbb{C}))$$

is a finitely dimensional linear space over $\mathbb{C}$. Due to its linear span, there exist finitely many 1-parameter families of algebro-singular cycles $S_1, S_2, \cdots, S_m$ such that

$$\begin{array}{c}\sum_{i=1}^m H^{2p+1}_{S_i}(X;\mathbb{C}) = H^{2p+1}_{sp}(X;\mathbb{C}), \quad and \\ \sum_{i=1}^m H^{2p+1}_{S_i}(X;\mathbb{Q}) = H^{2p+1}_{sp}(X;\mathbb{Q}).\end{array} \tag{3.27}$$

Thus

$$\phi_s(CH^1_a(S_1)\oplus\cdots\oplus CH^1_a(S_m)) = J^{p+1}_{sp}(X) = \frac{\overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{C})}}{\Lambda^{2p+1}_{sp}(\mathbb{Z})}. \tag{3.28}$$

Now by the theorem 2.7 part (2), for fixed p_i on each S_i, the induced regular map Φ,

$$\begin{array}{ccc}(S_1)^{g_1}\times\cdots\times(S_m)^{g_m} & \stackrel{\Phi}{\rightarrow} & J^{p+1}_{sp}(X) \\ (t_1,\cdots,t_m) & \rightarrow & \phi_s(t_1-g_1p_1)+\cdots+\phi_s(t_m-g_mp_m)\end{array} \tag{3.29}$$

is surjective where g_i is the genus of S_i.

So we let

$$M_2 = (S_1)^{g_1}\times\cdots\times(S_m)^{g_m} \tag{3.30}$$

and $\phi_2 = \Phi$. Let's discuss a general situation for polarized Hodge structures. Let $K_{\mathbb{C}}, L_{\mathbb{C}}$ be two polarized rational Hodge structures. In general, a direct sum of polarized rational sub Hodge structures $K_{\mathbb{C}}, L_{\mathbb{C}}$ is still polarizable. If $K_{\mathbb{C}}, L_{\mathbb{C}}$ both are polarizable sub Hodge structures, then $K_{\mathbb{C}} + L_{\mathbb{C}}$ has a direct sum sub Hodge structure

$$K_{\mathbb{C}}\oplus(\Lambda_{\mathbb{Q}}\otimes\mathbb{C})$$

where $\Lambda_{\mathbb{Q}}$ is the direct summand in $L_{\mathbb{Q}}$ such that

$$L_{\mathbb{Q}} = \Lambda_{\mathbb{Q}}\oplus(L_{\mathbb{Q}}\cap K_{\mathbb{Q}}).$$

Hence the sum $K_{\mathbb{C}}+L_{\mathbb{C}}$ also has a polarizable Hodge structure. Now come back to our specific situation. Since $H^{2p+1}_{S_i}(X;\mathbb{C})$ has a polarized Hodge structure, by (3.27) the sum

$$H^{2p+1}_{sp}(X;\mathbb{C}) = \sum_{i=1}^m H^{2p+1}_{S_i}(X;\mathbb{C})$$

also has a polarizable Hodge structure. Hence the corresponding complex torus

$$\frac{\overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{C})}}{\Lambda^{2p+1}_{sp}(\mathbb{Z})}, \tag{3.31}$$

is also polarizable. Therefore $J^{p+1}_{sp}(X)$ is Abelian.

We prove theorem 1.3 in two steps.

Step 1: We assume $dim(H^1(X;\mathbb{Q})) \neq 0$. It suffices to verify assumption 2 of theorem 3.3

Let $u \in H^2(X;\mathbb{Z})$ be the class of a hyperplane section. Let

$$\overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{Z})} = \Lambda^{2p+1}_{sp}$$
$$\overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{Q})} = \Lambda^{2p+1}_{sp} \otimes \mathbb{Q}.$$

They are projections of

$$H^{2p+1}_{sp}(X;\mathbb{Z}), H^{2p+1}_{sp}(X;\mathbb{Q})$$

in the summand $\overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{C})}$.

Then for the vector space over $\mathbb{Q}$, we can decompose

$$\overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{Q})} = Z_1 \oplus Z_0, \tag{3.32}$$

where

$$Z_0 = \{\alpha \in \overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{Q})} : \alpha \cup u^q = 0\}. \tag{3.33}$$

(note $\alpha \cup u^q$ is a class of degree $2n-1$). Thus $\alpha \cup u^q \neq 0$ for all non-zero $\alpha \in Z_1$. By the assumption

$$H^{1,0}(X;\mathbb{Q}) \cup u^p \subset \overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{Q})} \tag{3.34}$$

is non zero. Hence Z_1 is non-zero whose dimension is at least

$$dim(H^{1,0}(X;\mathbb{Q})).$$

Upto to a finite index, we may assume the lattice also has a decomposition

$$\overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{Z})} = L_1 \oplus L_0, \tag{3.35}$$

where

$$L_0 = \{\alpha \in \overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{Z})} : \alpha \cup u^q = 0\}, \tag{3.36}$$

and for non-zero $\alpha \in L_1$, $\alpha \cup u^q \neq 0$. We denote

$$K_1 = span_{\mathbb{C}}(L_1), K_0 = span_{\mathbb{C}}(L_0).$$

By counting the dimensions, $L_i, i = 0, 1$ are lattices of $K_i, i = 0, 1$. Then

$$\overline{F^{p+1}H^{2p+1}_{sp}(X;\mathbb{C})} = K_1 \oplus K_0, \tag{3.37}$$

such that K_0, K_1 have lattices L_0, L_1 respectively. Note $\alpha \cup u^q \neq 0$ for all non-zero $\alpha \in K_1$. After shifting by the integral lattice and choosing an appropriate direct sum complement K_1, we may assume $\beta \in K_1$ (the new β is the sum of old β and an integral class). Then

$$J_{sp}^{p+1}(X) = \mathcal{T}(K_1) \oplus \mathcal{T}(K_0) \tag{3.38}$$

and $\pi_2(\beta) \in \mathcal{T}(K_1)$, i.e.

$$\pi_2(\beta) \in \mathcal{T}(K_1) \oplus \{0\}.$$

Since $J_{sp}^{p+1}(X)$ is Abelian, the complex torus $\mathcal{T}(K_1)$ is a non-empty Abelian subvariety. Then we can choose a curve R_2 on $\mathcal{T}(K_1) \oplus \{0\}$ through the points $\pi_2(\beta), 0$. Using the surjectivity of the singular Abel-Jacobi map ϕ_2, we obtain a smooth curve C_2 and a correspondence (an algebro-singular cycle)

$$Z \subset \mathcal{Z}^{p+1,p+1}(C_2 \times X) \tag{3.39}$$

such that $\psi_{C_2}(C_2) = R_2$.

Next we verify the key assumption $\sigma(C_2) = 0$.

Let $\omega_Z \in H^{2p+2}(C_2 \times X; \mathbb{C})$ be Poincaré dual to Z. Using Künneth decomposition

$$\omega_Z = \alpha_Z^1 + \alpha_Z^2 + \alpha_Z^0 \tag{3.40}$$

for

$$\alpha_Z^i \in H^i(C_2) \otimes H^{2p+2-i}(X), i = 0, 1, 2.$$

Let

$$\alpha_Z^1 = \sum_i l_i \otimes \omega_i \tag{3.41}$$

where

$$l_i \in H^1(C_2; \mathbb{C}), \omega_i \in H^{2p+1}(X; \mathcal{C}).$$

By the definitions, we immediately have

$$\begin{aligned} span(\bar{l}_i)_i &= (\rho_1)^*(H^{1,0}(\mathcal{T}((V_1)^*))) \\ span_i(\omega_i) &= H_Z^{2p+1}(X; \mathbb{C}). \end{aligned} \tag{3.42}$$

where $\bar{l}_i$ is the projection of l_i to $H^{1,0}(C_2; \mathbb{C})$ in the Hodge decomposition. Next we apply the consequence of the construction Z (This is the key step).

$$\overline{F^{p+1} H_Z^{2p+1}(X; \mathbb{Z})} \subset L_1. \tag{3.43}$$

Since $\alpha \cup u^q \neq 0$ in

$$H^{2p+1}(X; \mathbb{C})$$

for all non-zero $\alpha \in K_1$, the restriction $\alpha|_Y \neq 0$ in

$$H^{2p+1}(Y; \mathbb{C})$$

for all non-zero $\alpha \in K_1$, where Y is the complete q codimensional hyperplane intersection of X. Thus the restriction map

$$\begin{array}{ccc} \overline{F^{p+1}H_Z^{2p+1}(X;\mathbb{C})} & \to & \overline{F^{p+1}H^{2p+1}(Y;\mathbb{C})} \\ \alpha & \to & \alpha|_Y \end{array} \tag{3.44}$$

is injective.[4] Therefore the dual map

$$\begin{array}{ccc} (\overline{F^{p+1}H^{2p+1}(Y;\mathbb{C})})^* & \to & (\overline{F^{p+1}H_Z^{2p+1}(X;\mathbb{C})})^* \\ \theta & \to & \theta(\alpha|_Y) \end{array} \tag{3.45}$$

is surjective. By the Lefschetz hyperplane theorem, the restriction map

$$\begin{array}{ccc} H^{1,0}(X;\mathbb{C}) & \to & H^{1,0}(Y;\mathbb{C}) \\ l & \to & l|_Y \end{array} \tag{3.46}$$

is an isomorphism. Using the Poincaré duality of Y, the composition map

$$\begin{array}{ccccc} H^{1,0}(X;\mathbb{C}) & \to & H^{1,0}(Y;\mathbb{C}) \simeq (\overline{F^{p+1}H^{2p+1}(Y;\mathbb{C})})^* & \to & (\overline{F^{p+1}H_Z^{2p+1}(X;\mathbb{C})})^* \\ l & \to & l \cup u^q & \to & (l \cup u^q)^* \end{array} \tag{3.47}$$

is surjective. This implies

$$\begin{array}{ccccc} H^{1,0}(X;\mathbb{C}) & \to & (F^{p+1}H_Z^{2p+1}(X;\mathbb{C}))^* & \to & span_i(\bar{l}_i) \\ l & \to & (l \cup u^q, \sum l_i \otimes w_i) & \to & \sum_i l_i(l \cup u^q, w_i) \end{array} \tag{3.48}$$

is surjective, where the second map is the evaluation map

$$(F^{p+1}H_Z^{2p+1}(X;\mathbb{C}))^* \otimes H^1(C_2;\mathbb{C}) \otimes F^{p+1}H_Z^{2p+1}(X;\mathbb{C}) \quad \to \quad H^1(C_2;\mathbb{C}). \tag{3.49}$$

The final statement is equivalent to saying

$$\begin{array}{ccc} H^{1,0}(X;\mathbb{C}) & \to & span_i(\bar{l}_i) \\ l & \to & (\pi_1)_*(l \cup u^q \cup \omega_Z) \end{array} \tag{3.50}$$

is surjective where $\pi_1 : C_2 \times X \to C$ is the projection.

On the other hand, the image of the map

$$\begin{array}{ccc} F^{q+1}H_a^{2q+1}(X;\mathbb{C}) & \to & H^{1,0}(C_2;\mathbb{C}) \\ \omega_a & \to & (\pi_1)_*(\omega_a \cup \omega_Z) \end{array} \tag{3.51}$$

is just

$$(\rho_1^a)^*(H^{1,0}(\mathcal{T}((V_1^a)^*))).$$

[4]This injectivity statement is similar to the Lefschetz hyperplane theorem, but it is not because of the higher codimensions. We call it quasi-Lefschetz-hyperplane-theorem. It is incorrect in general. However it is true in our case because of the special construction of Z.

At last, $H^1 = H^1_a$ for any variety and cupping with hyperplane sections preserves the partial algebraicity. Hence $l \cup u^q \in V_1^a$. Then obtain that

$$\begin{array}{c} span(\bar{l}_i)_i \\ \bigcap \\ (\pi_1) * \left(H^{1,0}(X;\mathbb{C}) \cup u^q \cup \omega_Z \right) \\ \bigcap \\ (\pi_1) * (F^{q+1} H_a^{2q+1}(X;\mathbb{C}) \cup \omega_Z) \\ \bigcap \\ (\pi_1) * (F^{q+1} H^{2q+1}(X;\mathbb{C}) \cup \omega_Z) \\ \| \\ span(\bar{l}_i)_i. \end{array} \tag{3.52}$$

Finally $E_1 = E_1^a$. This completes the proof of vanishing $\sigma(C_2)$.

Step 2: Without the assumption $dim(H^1(X;\mathbb{Q})) \neq 0$.

APD can be easily handled in various situations once it is proved in a special case. Let's see this. Let E be an elliptic curve. Let $G = E \times X$. For any non-zero cycle $c \in H_{sp}^{2p+1}(X)$, $[E] \otimes c \in H_{sp}^{2p+1}(G)$ is non-zero. We can apply the step 1 to G. We obtain a $\beta \in H_a^{2q+3}(G)$ such that

$$\mathcal{B}_G([E] \otimes c, \beta) \neq 0. \tag{3.53}$$

Next we notice β lies in a subvariety of codimension q. Let's see it in the homology. There is a correspondence

$$\mathcal{I} \in CH^{q+1}(E \times X) \tag{3.54}$$

such that there is a singular cycle β_h Poincaré dual to β, contained in the support $|\mathcal{I}|$, and the projections from both $\mathcal{I}$ and β_h to X are finite-to-one. Let $\pi : G \to X$ be the projection. Then the cohomology class $\pi_*(\beta)$ lies in $\pi(\mathcal{I})$ which is a codimension q algebraic cycle. Hence

$$\pi_*(\beta) \in H_a^{2q+1}(X;\mathbb{C}).$$

On the other hand, using the projection formula, we obtain that

$$\mathcal{B}_G([E] \otimes c, \beta) = \mathcal{B}_X(c, \pi_*(\beta)) \neq 0, \tag{3.55}$$

This shows that the restricted intersection pairing is also non-degenerate on right. By the content of previous paper [7], the restricted intersection pairing is non-degenerate on left. This completes the proof of theorem 1.3

3.3 Generalized Hodge conjecture of level 1

In [2], Grothendieck made a slight correction of the Hodge's general conjecture to propose the topological description of the algebraic cycles lying on a subvariety of the least codimension. At the level 1, this is the following conjecture

Generalized Hodge Conjecture of level 1 3.1. *The maximal sub Hodge structure $\mathbb{M}^{2p+1}$ of weight $2p+1$ and level 1 is equal to*

$$H_a^{2p+1}(X;\mathbb{C}). \tag{3.56}$$

THEOREM 3.4. *Theorem 1.3 confirms the generalized Hodge conjecture of level 1.*

Proof. First we claim that

Claim 3.2. *$H_{sp}^{2p+1}(X;\mathbb{C})$ is the maximal sub Hodge structure $\mathbb{M}^{2p+1}$ of weight $2p+1$ and level 1.*

Proof of the claim:

$$H_{sp}^{2p+1}(X;\mathbb{C}) = \sum_{i=1}^{m} H_{S_i}^{2p+1}(X;\mathbb{C}), \tag{3.57}$$

where each $H_{S_i}^{2p+1}(X;\mathbb{C})$ is a sub Hodge structure of level 1. Hence

$$H_{sp}^{2p+1}(X;\mathbb{C}) \subset \mathbb{M}^{2p+1}. \tag{3.58}$$

Suppose

$$L \subset H^{2p+1}(X;\mathbb{Q}) \tag{3.59}$$

is a sub Hodge structure of level 1. By Voisin's argument in theorem 8, [5], there are a smooth projective curve C and a Hodge class

$$\mathcal{I} \in Hdg^{2p}(C \times X) \tag{3.60}$$

that send

$$H^1(C;\mathbb{Q}) \to L \subset H^{2p+1}(X;\mathbb{Q}). \tag{3.61}$$

Since any Hodge class can be represented by an algebro-singular cycle. Then by the definition

$$H_C^{2p+1}(X;\mathbb{Q}) = L. \tag{3.62}$$

Hence

$$\mathbb{M}^{2p+1} \subset H_{sp}^{2p+1}(X;\mathbb{C}). \tag{3.63}$$

This completes the proof of the claim.

Secondly we apply theorem 1.3 to obtain

$$\begin{aligned} dim(H^{2p+1}_{sp}(X)) = dim(H^{2q+1}_a(X)) \leq dim(H^{2q+1}_{sp}(X)) \\ = dim(H^{2p+1}_a(X)) \leq dim(H^{2p+1}_{sp}(X)). \end{aligned} \tag{3.64}$$

Hence

$$H^{2p+1}_{sp}(X) = H^{2p+1}_a(X). \tag{3.65}$$

Therefore

$$\mathbb{M}^{2p+1} = H^{2p+1}_a(X). \tag{3.66}$$

This confirms the generalized Hodge conjecture of level 1.

□

4 APD on the cohomology of even degrees

4.1 Proof of theorem 1.4

Proof. of theorem 1.4. This is the same as that in [7]. However we are going to repeat the proof to assure that it is valid in the category of algebro-singular cycles.

Let E be an elliptic curve. Let

$$G = E \times X. \tag{4.1}$$

Let $B \in \mathcal{Z}^{p,p}(X)$ be a algebro-singular cycle on X representing a non-zero cohomology class. Let c_1 be a real 1-dimensional singular cycle on E. Assume $[c_1] \in H^1(E)$ is non-zero. Take E as a 1 parameter family of the constant algebraic cycle B, i.e. take the correspondence

$$I = \{(e, e, x) \in E \times G : x \in B\} \tag{4.2}$$

Then E is the parameter space of the family of algebro-singular cycles $\{e\} \times B$. By the definition

$$[c_1 \otimes B] \in H^{2p+1}_{sp}(G; \mathbb{Z}) \tag{4.3}$$

is a non-zero cycle. By theorem 1.3, there exists a singular cycle Σ representing a class

$$[\Sigma] \in H^a_{2p+1}(G; \mathbb{Z}) \tag{4.4}$$

such that

$$\mathcal{B}_G([c_1] \otimes [B], [\Sigma]) \neq 0. \tag{4.5}$$

where the intersection between cohomology and homology is also denoted by $\mathcal{B}$. We can assume Σ is supported on algebraic varieties. We would like to dissect this cycle to extract the algebraic cycles from it.

Since Σ is partially algebraic, it is carried by irreducible, algebraic varieties

$$W_j \subset G, \tag{4.6}$$

of dimension $p+1$. Since $dim(E)=1$, there are only two kinds of W_j:

(1) those whose projection to E is a point, denoted by W_j^0

(2) those whose projection to E is surjective, denoted by W_j^1

For the second type of variety all fibres have the dimension p. It is easy to see that all components of Σ carried by W_j^0 will have zero intersection with $c_1 \times B$. Let

$$\Sigma_1$$

be the part of Σ carried by W_j^1. Assume Σ_1 is triangulated by the sum of simplexes

$$\begin{array}{rcl} \sigma_i^1 : \Delta_i^1 & \to & \Sigma_1 \\ \sigma_i^2 : \Delta_i^2 & \to & \Sigma_1, \end{array} \tag{4.7}$$

where $\sigma_i^k, k=1,2$ are such that the following compositions have k-dimensional (real) simplexes as their images,

$$\theta_i^k : \Delta_i^1 \quad \to \quad \Sigma_1 \quad \to \quad E \ .$$

Then we denote $\sum_i \sigma_i^k$ by Σ_k^1. Now we see that Σ_k^1 must be closed. It suffices to consider Σ_1^1 only. Notice that for each fixed j, fibres of W_1^j over E are all homopotic in X, thus represent the same cohomology class of X. Therefore by the Künneth decomposition

$$[\Sigma_1^1] = \sum_l c_l \otimes \beta_l \tag{4.8}$$

where $\sum_l c_l = \sum_i \theta_i^1(\Delta_i^1)$, and β_l are represented by fibres of W_j^1. Hence the classes β_l are all algebraic. We'll denote $\sum_l c_l = c$. Hence

$$\mathcal{B}_{E\times X}([c_1 \times B],[\Sigma]) = \mathcal{B}_{E\times X}([c_1 \times B],[\Sigma_1^1]) = \sum_h a_h \mathcal{B}_X([B],\beta_h) \neq 0. \tag{4.9}$$

where h represents an intersection point e_h of $c \cap c_1$ in E, a_h is some integer and

$$\beta_h = [\{e_h\} \times X) \cdot \Sigma_1^1]$$

are algebraic cycle classes of dimension p.

Then (4.9),

$$\mathcal{B}_X(B, \sum_h a_h \beta_h) \neq 0. \tag{4.10}$$

shows that the intersection pairing is non-degenerate on algebraic cycles on right. This completes the proof. □

4.2 Hodge conjecture

This is the proof of corollary 1.5. Hodge conjectured[5] in [3]

$$Hdg^{2p}(X) = A^p(X). \tag{4.11}$$

This conjecture follows from theorem 1.4 just as the generalized Hodge conjecture of level 1 follows from theorem 1.3 This is because

$$\begin{aligned} dim(Hdg^{2p}(X)) = dim(A^{q+1}) \leq dim(Hdg^{2(q+1)}(X)) \\ = dim(A^p) \leq dim(Hdg^{2p}(X)), \end{aligned} \tag{4.12}$$

then

$$dim(Hdg^{2p}(X)) = dim(A^p).$$

Since

$$A^p \subset Hdg^{2p}(X),$$

the conjecture is proved.

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[5] Hodge's original statement in [3], after a modification by Grothemdieck, is currently called"generalized Hodge conjecture". The "millennium" Hodge conjecture selected by Clay institute is referred to its special case which is the one in this section.