ALGEBRAIC POINCARÉ DUALITY 0 (APD)

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To human, it is the important one. To nature, it is just a fact like any others.

Abstract.
We discuss a structure that exists in many problems on smooth projective varieties over the field of complex numbers, and name it as "Algebraic Poincaré duality" or "APD" for abbreviation.
In particular, over the complex numbers with singular cohomology, it is a solution to
(1) Griffiths' conjecture on the incidence equivalence versus Abel-Jacobi equivalence,
(2) Generalized Hodge conjecture of level 1,
(3) Generalized Hodge conjecture of level 0, i.e. the usual Hodge conjecture,
(4) The standard conjectures,
(5) Grothendieck's "D" conjecture.

However it is not the goal of this paper to show APD implies these conjectures. In this paper we'll build the foundation for the structure by introducing the APD in its simplest form over the complex numbers.

1 Introduction

This is an introduction to the general algebraic Poincaré duality, not to this paper only.

Let X be a smooth projective variety of dimension n over $\mathbb{C}$. As a topological manifold of dimension $2n$, it has the Poincaré duality: a non-canonical isomorphism P,

$$H^i(X;\mathbb{Q}) \quad \rightarrow \quad H^{2n-i}(X;\mathbb{Q}). \tag{1.1}$$

Then for any subspace $G \subset H^i(X;\mathbb{Q})$, there is a NON-CANONICAL, but isomorphic subspace $P(G)$. Thus $P(G)$, which usually is not determined by G, is a symmetry of G. The pair $(G, P(G))$ forms a pair of Poincaré duality. It is non-canonical because even though the cup product gives a canonical isomorphism between

$$H^i(X;\mathbb{Q}) \quad \rightarrow \quad (H^{2n-i}(X;\mathbb{Q}))^*,$$

where $(H^{2n-i}(X;\mathbb{Q}))^* = hom(H^{2n-i}(X;\mathbb{Q}),\mathbb{Q})$, but there is no canonical isomorphism between

$$(H^{2n-i}(X;\mathbb{Q}))^* \quad \rightarrow \quad H^{2n-i}(X;\mathbb{Q}).$$

Thus the isomorphism P will not preserve additional structures. This means even if G carries an additional structure other than topological one, there is no way through P to find $P(G)$ carrying the similar structure. However there is a known canonical isomorphism P_{Lef} given by the cup product with the power of a hyperplane section, and Lefschetz hard theorem says that P_{Lef} preserves Hodge structures. It is unknown weather this canonical isomorphism preserves algebro-geometric structures. Our work focuses on this unknown territory. But we don't work with P_{Lef}. Precisely, we are interested in the converse of the question: for some priorly determined structural subspaces G, G', is G' the image of the G under some isomorphism P, i.e. do they form a pair of Poincaré duality? The algebraic Poincaré duality is the duality for general such structural pairs that arise from algebro-complex structures. Currently we only focus on the sub Hodge structures. So the subspaces, including those in [2], [3], are sub Hodge structures of X, each of which possesses both of the integral topological structure and the complex analytic structure. More specifically the following APDs for sub Hodge structures hold.

Table 1.1
Algebraic Poincaré duality over the complex numbers

	Subspaces of Algebraic cycles	**Subspaces of Hodge cycles**
Cohomology of odd degrees	$\left(H_a^{2p+1}(X;\mathbb{Q}), H_a^{2n-2p-1}(X;\mathbb{Q}) \right)$ Griffiths' conjecture on incidence equivalence versus Abel-Jacobi equivalence	$\left(H_{sp}^{2p+1}(X;\mathbb{Q}), H_a^{2n-2p-1}(X;\mathbb{Q}) \right)$ Generalized Hodge conjecture of level 1
Cohomology of even degrees	$\left(A^p(X), A^{n-p}(X) \right)$ Standard conjectures	$\left(Hdg^{2p}(X), A^{n-p}(X) \right)$ Usual Hodge conjecture

Notations:
(1) $H_{sp}^{\bullet}(X;\mathbb{Q})$ has the maximal subHodge structure of level 1.
(2) $Hdg^{\bullet}(X)$ is the subspace of Hodge classes.
(3) $A^{\bullet}(X)$ is the subspace of the rational cohomology, spanned by algebraic cycles.
(4) $H_a{}^{\bullet}(X;\mathbb{Q})$ is the algebraic part of cohomology (see example 3.1).

The table tells us the roles APD plays in geometry. However, as we said in the abstract, this paper only focuses on an abstract foundation in the algebra where the underlined variety X will be omitted. We work over the complex numbers only. We'll present the formal statements of APD in linear algebra and then in Lie groups (complex tori). Then reduce them to sufficient conditions on Jacobians of curves. All these are done without the presence of the projective variety X. The detailed geometric proof of these conditions will appear in the following papers [2], [3]. But to make the non-geometric statements more accessible, we provide multiple examples on projective varieties over $\mathbb{C}$.

Why is it necessary to have the algebraic notion of APD?

(1) Our study of APD led us to various similar situations. So a generalization to some extent will simplify and clarify the overall presentation.

(2) Our main technique is the full theory of intermediate Jacobians. There is no algebraically generalized version of such theory that will satisfy our need.† Thus selecting those needed axioms seems to be the only option.

This is our motivation to have a generalization of APD.

Notations:

(1) $(\bullet)^*$ denotes the dual of a vector space if $\bullet$ is a vector space or a vector.

(2) $(\bullet)^*$ also denotes a pullback from the cohomology or differential forms if $\bullet$ is a map.

(3) $(\bullet)_*$ denotes a pushforwad on the homology or cycles if $\bullet$ is a map.

(4) $\overline{\bullet}$ denotes the complex conjugation on a complex vector space.

(5) $[a]$ denotes an equivalence class of the element a.

(6) CH denotes the Chow group, CH_{alg} denotes the subgroup of cycles algebraically equivalent to zero.

(7) J denotes the intermediate Jacobian or the Jacobian.

(8) $\mathcal{T}$ denotes the complex torus.

(9) All homology and cohomology are groups modulo their torsions.

We organize the rest of the paper as follows. In section 2, we give an expression of APD in linear algebra. In section 3, we follow the expression of APD to the complex tori. The difference from section 1 is the beginning of the change of in the expression of APD. In section 4, we give sufficient conditions for the APD. This is the main theorem 4.5. In section 5, we give a proof of main theorem 4.5. In section 6, we use several examples to illustrate the connection of APD with the classical results.

† Theory of $l-adic$ intermediate Jacobians is a good candidate over a finite field.

2 The first expression in linear algebra

One of major differences between APD and the existing theory of Hodge structures is that APD is applied to a pair of sub Hodge structures. The only known invariant related to APD (but not the same in any aspects) is the polarization of Hodge structures. Also our current work only involves Hodge structures of weight 1. But this does not mean it will be limited to weight 1 forever. Since our work did not use much algebra from Hodge structures, to simplify the presentation, we'll avoid the notion of Hodge structures for the time being. Let's start with linear algebra.

Let V_1, V_2 be two isomorphic vector spaces over $\mathbb{C}$, equipped with a complex, non-degenerate, bilinear form $\mathcal{B}$ on $V_2 \times V_1$. The form $\mathcal{B}$ induces isomorphisms

$$\begin{array}{rcl} \mathcal{B}_R : V_2 & \to & (V_1)^* \\ \mathcal{B}_L : V_1 & \to & (V_2)^*. \end{array} \tag{2.1}$$

Now two subspaces

$$V_i^a \subset V_i, i = 1, 2 \tag{2.2}$$

satisfy the Algebraic Poincaré duality if:

Algebraic Poincaré duality 2.1. *The restriction $\mathcal{B}_a$ of $\mathcal{B}$ to*

$$V_2^a \times V_1^a \tag{2.3}$$

is non-degenerate

Given $V_1, V_2, \mathcal{B}$, it is trivial that there are infinitely many such pairs V_2^a, V_1^a. The goal of this paper is to introduce sufficient conditions for APD to hold.

APD deals with two composition maps:

$$\mathcal{B}_2 : V_2^a \xrightarrow{\mathcal{B}_R|_{V_2^a}} (V_1)^* \xrightarrow{restriction} (V_1^a)^* \tag{2.4}$$

$$\mathcal{B}_1 : V_1^a \xrightarrow{\mathcal{B}_L|_{V_1^a}} (V_2)^* \xrightarrow{restriction} (V_2^a)^*. \tag{2.5}$$

Since they are similar, we are going to concentrate on $\mathcal{B}_2$.

Assumption 2.2. *For each $\beta \in V_2^a$, there exists a β-dependent direct-sum decomposition*

$$V_1 = W_1(\beta) \oplus W_2(\beta) \tag{2.6}$$

such that $W_1(\beta) \subset V_1^a$, $\mathcal{B}(\beta, W_2(\beta)) = 0$.

Remark To be able to restrict to a single vector β is one of key steps in its application. It follows immediately that

PROPOSITION 2.1. *The assumption 2.2 for both directions is a sufficient condition for the algebraic Poincaré duality.*

In our application, V_i, V_i^a are subgroups of cohomology arising from the structures of projective varieties. Let's see the following examples.

Example 2.1. Let X be a compact Kähler manifold of dimension n. Let p, q be two whole numbers less than or equal to n.

Let

$$\begin{aligned} V_2 &= H^{p+q}(X;\mathbb{C}) \\ V_1 &= H^{2n-(p+q)}(X;\mathbb{C}) \\ V_2^a &= H^{p,q}(X;\mathbb{C}) \\ V_1^a &= H^{n-p,n-q}(X;\mathbb{C}). \end{aligned} \tag{2.7}$$

Let $\mathcal{B}$ be the topological intersection form. Then APD 2.1 holds. This follows from the Serre duality and Poincaré duality. This indicates APD 2.1 is a consequence of the geometry of X.

To have a more concrete understanding of the decomposition (2.6). Let's see it in this example. Let $\beta \in V_2^a = H^{p,q}(X;\mathbb{C})$ be a non-zero class. Let

$$W_2(\beta) = \{\alpha \in H^{2n-p-q}(X;\mathbb{C}) : \mathcal{B}(\beta,\alpha) = 0\}. \tag{2.8}$$

Let $\beta' \in H^{n-p,n-q}(X;\mathbb{C})$ such that $\mathcal{B}(\beta,\beta') = 1$. We let

$$W_1(\beta) = span(\beta') \tag{2.9}$$

be the line spanned by β'. Then $W_1(\beta), W_2(\beta)$ will satisfy the decomposition (2.6). This can be viewed as the theoretic basis for the decomposition (2.6) throughout. Based on this, $W_i(\beta)$ can be modified to different dimensions.

This example is the well-known situation of the hard Lefschetz theorem.

Example 2.2. Let X be a a compact Kähler manifold of dimension n. Let $i : Y \hookrightarrow X$ be a complex submanifold of dimension m. Let r, s be two whole numbers less than m. Let j be the homomorphism deduced from the Thom isomorphism

$$j : H^*(Y;\mathbb{C}) \quad \rightarrow \quad H^{*+2n-2m}(X;\mathbb{C}). \tag{2.10}$$

Let

$$\begin{aligned} V_2 &= i^*(H^{r+s}(X;\mathbb{C})) \\ V_1 &= j(H^{2m-(r+s)}(Y;\mathbb{C})) \\ V_2^a &= i^*(H^{r,s}(X;\mathbb{C})) \\ V_1^a &= j(H^{m-r,m-s}(Y;\mathbb{C})). \end{aligned} \tag{2.11}$$

Let $\mathcal{B}$ be the intersection form on X. Then APD holds on such V_i^a. Indeed, by the projection formula, this is just the APD of example 2.1 applied to the subvariety Y.

Proposition 2.1 is a trivial statement in linear algebra. But examples are not trivial because their truth comes from other structure, namely the Kählerian structure of the manifold. In any case, the assumption 2.2 can't be achieved within the linear algebra. This is the end of the first expression. Next we add discrete group actions to the vector spaces, then revisit the algebraic Poincaré duality. This leads to the second expression of the algebraic Poincaré duality.

3 The second expression on complex tori

The Algebraic Poincaré duality has another expression on the complex tori once the linear spaces are equipped with lattices. The goal of the section is to reduce the APD in linear algebra to homomorphisms between complex tori with finite kernel. In all applications these homomorphisms are isogenies between Abelian varieties.

Let Λ_1, Λ_2 be lattices of V_1, V_2, i.e. they are discrete subgroups of V_i such that

$$\Lambda_i \simeq \mathbb{Z}^{2dim(V_i)} \ and \ span_{\mathbb{R}}(\Lambda_i) = V_i.$$

We call them "integral lattices" to remind us the geometric background of them. Assume

$$\Lambda_i^a = \Lambda_i \cap V_i^a \tag{3.1}$$

are integral lattices of V_i^a. Also we assume the $\mathcal{B}$ satisfies

$$\Lambda_2 \times \Lambda_1 \to \mathbb{Z}.$$

We defined the integral lattice of the dual of the vector space to be the

$$hom(lattice, \mathbb{Z}). \tag{3.2}$$

We obtain the complex tori

$$\mathcal{T}(V_i^a) = \frac{V_i^a}{\Lambda_i^a}. \tag{3.3}$$

The commutative diagram follows

$$\begin{array}{ccc} V_2^a & \overset{\mathcal{B}_2}{\to} & (V_1^a)^* \\ {\scriptstyle \pi_2}\downarrow & & {\scriptstyle \pi_1}\downarrow \\ \mathcal{T}(V_2^a) & \overset{\tilde{\mathcal{B}}_2}{\to} & Pic^0(\mathcal{T}(V_1^a)), \end{array} \tag{3.4}$$

where $\tilde{\mathcal{B}}_2$ is induced from $\mathcal{B}_2$.

It follows immediately that

Theorem 3.1. *$\mathcal{B}_2$ is injective if and only if $\tilde{\mathcal{B}}_2$ has a finite kernel.*

Proof. of theorem 3.1.

"$\Rightarrow$": If $ker(\tilde{\mathcal{B}}_2)$ is not finite, then $dim(ker(\tilde{\mathcal{B}}_2)) \geq 1$. Then there are uncountable vectors $\alpha \in V_2^a$ such that $\mathcal{B}(\alpha, \cdot)|_{\Lambda_1^a} \in hom(\Lambda_1^a, \mathbb{Z})$. By the linearity, the group homomorphism $\mathcal{B}(\alpha, \cdot)|_{\Lambda_1^a}$ determines the linear homomorphism $\mathcal{B}(\alpha, \cdot)$. Because $\mathcal{B}_2$ is injective, there will be uncountable distinct $\mathcal{B}(\alpha, \cdot)|_{\Lambda_1^a}$ in $hom(\Lambda_1^a, \mathbb{Z})$. This is a contradiction because the set $hom(\Lambda_1^a, \mathbb{Z})$ is countable.

"$\Leftarrow$": If $\mathcal{B}_2$ is not injective, $dim(\ker(\mathcal{B}_2)) \geq 1$. There is a non-zero $\alpha \in V_2^a$ such that $\mathcal{B}(\alpha, \cdot)$ is a zero map. Then for any vectors α' on the line $span(\alpha)$, $\mathcal{B}(\alpha' + a, \cdot)$ will map each element a in the lattice to the lattice, i.e. $\pi_2(\alpha')$ lies in the kernel of $\tilde{\mathcal{B}}_2$. This shows the kernel of $\tilde{\mathcal{B}}_2$ is infinite. This proves the direction "$\Leftarrow$". □

Corollary 3.2.

$\mathcal{B}_2$ is injective if and only if the image of the lattice Λ_2^a has a finite index in the lattice (as a group) of $Pic^0(\mathcal{T}(V_1^a))$.

Proof. It is clear that the image of the lattice Λ_2^a $\mathbb{R}$-linearly span the vector space $(V_1^a)^*$ if and only if its index in the lattice group of $Pic^0(\mathcal{T}(V_1^a))$ is finite. □

Hence we have another expression of APD on the complex tori.

Algebraic Poincaré duality 3.1. *$\tilde{\mathcal{B}}_2$ and $\tilde{\mathcal{B}}_1$ have finite kernels.*

Corollary 3.3.

APD holds if and only if

$$\begin{array}{c} dim(V_2^a) = dim((V_1^a)^*), \ and \\ dim(V_1^a) = dim((V_2^a)^*) \end{array}.$$

Proof. This is due to the injectivity of $\mathcal{B}_2$ and $\mathcal{B}_1$. □

Even though the Poincaré duality 3.1 seems to be the same, but the assumption 2.2 is reduced significantly. This can be seen in the following proposition and corollary.

Proposition 3.4. *Let $\tilde{\mathcal{B}_R}$ be the homomorphism*

$$\mathcal{T}(V_2) \quad \rightarrow \quad Pic^0(\mathcal{T}(V_1)) \tag{3.5}$$

that is $\mathcal{B}_R$ modulo the lattice. Then the $\mathcal{B}_2$ is injective if for each $\tilde{\beta} \in \mathcal{T}(V_2^a)$, there is a decomposition

$$\nu : \mathcal{T}(V_1) \quad \xrightarrow{isogenous} \quad \mathcal{T}(W_1(\tilde{\beta})) \oplus \mathcal{T}(W_2(\tilde{\beta})), \tag{3.6}$$

where $\mathcal{T}(W_1(\tilde{\beta}))$, $\mathcal{T}(W_2(\tilde{\beta}))$ are complex tori and isogeny means a surjective homomorphism of Lie groups with a finite kernel, such that

(1) $\tilde{\mathcal{B}_R}(\tilde{\beta})$ is restricted to a trivial bundle on $\nu^{-1}(\mathcal{T}(W_2(\tilde{\beta})))$,

(2) and $\mathcal{T}(W_1(\tilde{\beta})) \subset \nu(\mathcal{T}(V_1^a))$.

Proof.

By theorem 3.1, it suffices to show $\tilde{\mathcal{B}_2}$ has a finite kernel. Let $\tilde{\beta} \in ker(\tilde{\mathcal{B}_2})$. Let

$$\mathcal{T}(W_1(\tilde{\beta})), \mathcal{T}(W_2(\tilde{\beta}))$$

be the tori for $\tilde{\beta}$ as in (3.6). Then $\tilde{\mathcal{B}_R}(\tilde{\beta})$ is a trivial bundle on

$$\nu^{-1}(\mathcal{T}(W_1(\tilde{\beta}))).$$

By the assumption it is also trivial on

$$\nu^{-1}(\mathcal{T}(W_2(\tilde{\beta})).$$

Hence $\tilde{\mathcal{B}_R}(\tilde{\beta})$ is trivial on the entire torus

$$\mathcal{T}(V_1).$$

Using the same version of theorem 3.1 for complex tori, we obtain that $\tilde{\beta}$ lies in the finite set $ker(\tilde{B}_R)$ because $\mathcal{B}$ is non-degenerate. Hence $\tilde{\mathcal{B}_2}$ has a finite kernel.

□

Corollary 3.5. *The linear map $\mathcal{B}_2$ is injective if for each $\beta \in V_2^a$, there is an $l_\beta \in \Lambda_2^a$ such that*

(1) there is a decomposition

$$V_1 = W_1(\beta + l_\beta) \oplus W_2(\beta + l_\beta). \tag{3.7}$$

satisfying assumption 2.2 for $\beta + l_\beta$.

(2) $W_1(\beta+l_\beta), W_2(\beta+l_\beta)$ have lattices that are all subsets of Λ_1 such that (3.6) holds.

Proof. If $\beta \in V_2^a$ satisfies the assumption in corollary 3.5, then $\pi_2(\beta)$ will satisfy the assumption in proposition 3.4. Therefore $\mathcal{B}_2$ is injective.

□

Example 3.1. This example is further than example 2.1, 2.2. It requires that the Kähler form in previous examples represents a rational class in the cohomology. To extend the examples 2.1 to a torus, $H^{p,q}(X; \mathbb{C})$ needs to have a natural lattice. To have lattices we use a different setting.

Let X be a smooth projective variety over $\mathbb{C}$ of dimension n. Let p, q be two whole numbers satisfying $p+q=n-1$. Let

$$\begin{array}{l} V_2 = \overline{F^{p+1}H^{2p+1}(X;\mathbb{C})}, \\ V_1 = F^{q+1}H^{2q+1}(X;\mathbb{C}). \end{array} \tag{3.8}$$

Let $\mathcal{B}$ be the usual topological intersection form on $V_2 \times V_1$. By the Poincaré duality, $\mathcal{B}$ is non-degenerate. Murre in [1] defined subgroups, called "algebraic part",

$$H_a^{2r+1}(X;\mathbb{C}).$$

To have a complete statement, we include the definition below.

We use the notation T to represent a 1-parameter family of r-cycles in the following set-up: there is an algebraic cycle

$$Z \in Z^{n-r}(T \times X), \tag{3.9}$$

such that,

(1) the support $|Z|$ of Z is projected onto the smooth T,

(2) the projection of $|Z| \to X$ is generically finite to one,

(3) Z intersects $\{t\} \times X$ properly.

We denote

$$Z(t) = (Pr_X)_*(Z \cdot (\{t\} \times X))$$

and Pr_X is the projection from $T \times X$ to X.

Definition 3.6.

(1) For any smooth 1-parameter space T of p-cycles in X, we let

$$H_{2p+1}^{T}(X;\mathbb{Z}) \tag{3.10}$$

be the subgroup of $H_{2p+1}(X;\mathbb{Z})$ defined to be the image of the topological homomorphism

$$\begin{array}{rcl} \nu_T : H_1(T;\mathbb{Z}) & \to & H_{2q+1}(X;\mathbb{Z}) \\ \gamma & \to & (Pr_X)_*((\gamma \times [X]) \cap [Z]). \end{array} \tag{3.11}$$

Let

$$H_{2p+1}^{T}(X;\mathbb{A}) = H_{2p+1}^{T}(X;\mathbb{Z}) \otimes \mathbb{A} \tag{3.12}$$

and

$$H_T^{2q+1}(X;\mathbb{A}) \tag{3.13}$$

be its Poincaé dual, for $\mathbb{A} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

(2) Let $H_{2p+1}^{a}(X;\mathbb{Z})$ be the subgroup of $H_{2p+1}(X;\mathbb{Z})$ generated by all

$$H_{2p+1}^{T}(X;\mathbb{Z}). \quad (all\ T)$$

Similarly let

$$H^a_{2p+1}(X;\mathbb{A}) = H^a_{2p+1}(X;\mathbb{Z}) \otimes \mathbb{A} \tag{3.14}$$

and

$$H_a^{2q+1}(X;\mathbb{A}) \tag{3.15}$$

be its Poincaré dual, for $\mathbb{A} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$*. The subspace*

$$H_a^{2q+1}(X;\mathbb{C}) \tag{3.16}$$

is called the "algebraic part" of cohomology. This coincides with the definition given by Murre in [1], and agrees with the Grothendieck's definition $Filt'^q H^{2q+1}(X)$*. The cycles in "algebraic part" of cohomology are called partially algebraic.*

Let $H_a^{i,j}(X;\mathbb{C})$ be its projection to the (i,j) type in the Hodge decomposition. Then we define the subgroups

$$\begin{array}{l} V_2^a = \overline{F^{p+1} H_a^{2p+1}(X;\mathbb{C})}, \\ V_1^a = F^{q+1} H_a^{2q+1}(X;\mathbb{C}). \end{array} \tag{3.17}$$

The algebraic parts are sub Hodge structures. Hence $V_i^a, i = 1, 2$ have lattices. Modulo the lattices, we thus obtain the Abelian varieties

$$\mathcal{T}(V_2^a) \simeq J_a^{p+1}(X), \mathcal{T}(V_1^a) \simeq J_a^{q+1}(X). \tag{3.18}$$

where $J_a^\bullet(X)$ is the image of the Abel-Jacobi map in the Griffiths' intermediate Jacobian, and the second isomorphism needs to go through the conjugation.

In [1], J. Murre interpreted the Griffiths' conjecture on incidence equivalence versus Abel-Jacobi equivalence as an isogeny between two Abelian varieties,

$$\frac{CH^r_{alg}(X)}{CH^r_{inc}(X)} \xrightarrow{isogenuous} J_a^r(X),$$

where CH stands for Chow groups and alg, inc stand for algebraically equivalent to zero and incidence equivalent to zero. It can be noticed that this Murre's isogeny (or Saito's isogeny) is identical to the isogeny of $\tilde{\mathcal{B}}_2$ above (it is non-trivial though). Hence the Griffiths' conjecture is equivalent to APD 3.1. Even though this example is still a conjecture but it at least gives an evidence that APD 3.1 is beyond the structure of linear algebra.

The APD 3.1 in this setting will be explored more in the following examples in section 6, and proved, in general, in another paper ([2]).

4 Algebraic formulation

In this section, we break the complex tori $\mathcal{T}(V_2^a), \mathcal{T}(V_1^a)$ to multiple Abelian subvarieties $\mathcal{T}_{C_2}$, $\mathcal{T}_{C_2^1}$ for projective curves C_2, C_2^1. Subsequently the APD between

$$\mathcal{T}(V_2^a) \quad and \quad \mathcal{T}(V_1^a) \simeq Pic^0(\mathcal{T}(V_1^a))$$

becomes the APD between $\mathcal{T}_{C_2}$ and $\mathcal{T}_{C_2^1}$ for selective, multiple curves C_2. This requires the existence of the full theory of intermediate Jacobians. Since there is no satisfactory algebraic version of intermediate Jacobians, we extract the needed conditions from the complex intermediate Jacobians and call it the curve-like sub Abelian structure. We'll start with an abstract definitions of Abelain subvarieties $\mathcal{T}_{C_2}$ and $\mathcal{T}_{C_2^1}$.

Curve-like sub Abelian structure

Algebraic Poincaré duality addresses a problem in linear algebra. The second expression of APD made the problem descend to its compact Lie groups, called complex tori once the linear spaces are equipped with a particular type of discrete group actions. This allows projective curves as correspondences to come in to play crucial roles. We are going to make a structural construction so that both linear spaces $W_1(\beta), W_2(\beta)$ arise from curves. There are many ways to see this arising. For instance, in the theory of Hodge structures, this is a Hodge structure of weight 1. Let's see the details.

Let $C \stackrel{\rho}{\to} \mathcal{T}(V_i^a)$ be a regular map from a smooth complex curve C (not necessarily contains the origin).

DEFINITION 4.1. *Define Λ_C to be the image*

$$H_1(C;\mathbb{Z}) \quad \to \quad H_1(\mathcal{T}(V_i^a);\mathbb{Z}) = \Lambda_i^a. \tag{4.1}$$

Furthermore we denote $span_{\mathbb{C}}(\Lambda_C)$ by

$$H_C. \tag{4.2}$$

Let $\mathcal{J}(V_i)$ be the collection of all such curves C to $\mathcal{T}(V_i^a)$ that Λ_C is an integral lattice of H_C.

LEMMA 4.2. *The complex analyticity of $C \stackrel{\rho}{\to} \mathcal{T}(V_i^a)$ implies*

$$C \in \mathcal{J}(V_i).$$

Proof.

There is a Gysin homomorphism ν

$$H^1(C;\mathbb{C}) \quad \to \quad H^{2m-1}(\mathcal{T}(V_i^a);\mathbb{C}),$$

where $m = dim(\mathcal{T}(V_i^a))$. Because C is complex analytic,

$$\nu = \nu' + \nu''$$

where

$$\begin{array}{ccc} H^{1,0}(C;\mathbb{C}) & \stackrel{\nu'}{\to} & H^{m,m-1}(\mathcal{T}(V_i^a;\mathbb{C})) \\ H^{0,1}(C;\mathbb{C}) & \stackrel{\nu''}{\to} & H^{m-1,m}(\mathcal{T}(V_i^a;\mathbb{C})). \end{array}$$

Then the kernel of ν' as a vector space over $\mathbb{R}$ is isomorphic to the kernel of μ

$$H_1(C;\mathbb{R}) \quad \to \quad H_1(\mathcal{T}(V_i^a);\mathbb{R}).$$

This is because the $ker(\nu)$ is projected isomorphically to $ker(\nu')$ as real vector spaces. Therefore their image will satisfy

$$dim_{\mathbb{C}}(H_C) = dim_{\mathbb{C}}(im(\nu')) = \frac{1}{2} dim_{\mathbb{R}}(im(\mu)).$$

□

For any $C \in \mathcal{J}(V_i)$, define

$$\frac{H_C}{\Lambda_C} = \mathcal{T}_C, \tag{4.3}$$

to be the sub-torus of $\mathcal{T}(V_i^a)$. We call it a curve-like sub-torus.

Definition 4.3. *(curve-like sub-Abelian structure)*

We say the complex torus $\mathcal{T}(V_i^a)$ has a curve-like sub-Abelian structure if there are a projective variety M_i and a SURJECTIVE group homomorphism from the Chow group of 0-cycles of M_i, that are algebraically equivalent to zero to the complex torus,

$$\phi_i : CH_{alg}^{dim(M_i)}(M_i) \quad \to \mathcal{T}(V_i^a) \tag{4.4}$$

for $i = 1, 2$, satisfying:

(1) it is regular, i.e. for any projective family T_i of algebraic 0-cycles of M_i, the composition map

$$\psi_{T_i} : T_i \quad \xrightarrow{(T_i)_t - m_i} \quad CH_{alg}^{dim(M_i)}(M_i) \quad \xrightarrow{\phi_i} \quad \mathcal{T}(V_i^a) \tag{4.5}$$

is regular, where m_i is a fixed point in the same irreducible component of M_i as T_i.

(2) If T_i is a smooth projective curve parametrizing 0*-cycles on M_i, we'll denote the composition map*

$$CH^1_{alg}(T_i) \ \rightarrow \ CH^{dim(M_i)}_{alg}(M_i) \ \xrightarrow{\phi_i} \ \mathcal{T}(V_i^a) \tag{4.6}$$

also by ϕ_i. Then for $\psi_{T_i}(T_i) \in \mathcal{T}(V_i)$,

$$\phi_i(CH^1_{alg}(T_i)) = \mathcal{T}_{\psi_{T_i}(T_i)} \tag{4.7}$$

and it is Abelian.

Remark We would like to emphasize the importance of such a definition. It is originated from the well-known facts in complex geometry. Part (1) provides Abel-Jacobi maps, part (2) provides Jacobi inversion for curves.

DEFINITION 4.4. *Let C_2 be a smooth curve on M_2, and $\psi_{C_2} : C_2 \to \mathcal{T}(V_2^a)$ be the restricted morphism from definition 4.3. Denote the composition map*

$$C_2 \ \rightarrow \ \mathcal{T}(V_2^a) \ \stackrel{\tilde{\mathcal{B}}_{\mathbb{R}}}{\rightarrow} \ \mathcal{T}((V_1)^*) \tag{4.8}$$

by ρ_1, and

$$C_2 \ \rightarrow \ \mathcal{T}(V_2^a) \ \stackrel{\tilde{\mathcal{B}}_2}{\rightarrow} \ \mathcal{T}((V_1^a)^*) \simeq \mathcal{T}(V_1^a) \tag{4.9}$$

by ρ_1^a. We define

$$\begin{aligned} E_1 &= (\rho_1)^*\bigg(H^{1,0}(\mathcal{T}((V_1)^*))\bigg) \\ E_1^a &= (\rho_1^a)^*\bigg(H^{1,0}(\mathcal{T}((V_1^a)^*))\bigg), \end{aligned} \tag{4.10}$$

and

$$\sigma(C_2) = dim(E_1) - dim(E_1^a). \tag{4.11}$$

Similarly we define $\sigma(C_1)$ for $C_1 \subset M_1$.

THEOREM 4.5. *(Main theorem) Let $\beta \in V_2^a, \beta' \in V_1^a$. Assume*

(1) $\mathcal{T}(V_i^a)$ are both Abelian and both have curve-like sub Abelian structures,

(2) there exists a curve $C_2 \subset M_2$ such that $\psi_{C_2}(C_2)$ is through $\pi_2(\beta), 0$ with

$$\sigma(C_2) = 0, \tag{4.12}$$

and the same is true in opposite direction, i.e., there exists a curve C_1 such that $\psi_{C_1}(C_1)$ is through $\pi_2(\beta'), 0$ for given $\beta' \in V_1^a$ with,

$$\sigma(C_1) = 0. \tag{4.13}$$

Then APD holds.

Remark Assumption 2 of the main theorem assumes that there is an APD between $\mathcal{J}_{C_2}$ and $\mathcal{J}_{C_2^1}$. They form curve-like sub Abelian structures for $\mathcal{T}(V_2^a), \mathcal{T}(V_1^a)$. In application, the assumption 2 is a condition originated from the topology of the underlined variety. Thus it does not seems to be possible to describe it algebraically without introducing the geometry.

5 The Proof

We'll provide a proof of theorem 4.5.

Proof. of theorem 4.5: It suffices to give a proof in one direction of the non-degeneracy of $\mathcal{B}_a$. So let $\beta \in V_2^a$ be a non-zero element. We first construct C_2: For each $\beta \in V_2^a$, choose a complex analytic curves C_2 as in the assumption 2, theorem 4.5. More specifically C_2 goes through $\pi_2(\beta), 0$ and $\sigma(C_2) = 0$. Let $C_2^1 = \rho_1^a(C_2) \subset \mathcal{T}(V_1^a)$ (but there is no canonical identification $\mathcal{T}(V_1^a) \simeq \mathcal{T}((V_1^a)^*)$). Then we define

$$\begin{aligned} W_1(\beta) &= H_{C_2^1} \subset V_1^a \\ W_2(\beta) &= \{\alpha \in V_1 : \mathcal{B}(H_{C_2}, \alpha) = 0\}. \end{aligned} \tag{5.1}$$

(The first equality is not canonical).

The subspaces $W_1(\beta), W_2(\beta)$ are derived from subspaces $H_{C_2^1}, H_{C_2}$. The construction simply says

$$\begin{aligned} W_1(\beta) &= H_{C_2^1} \\ W_2(\beta) &= (H_{C_2})^{\perp} \end{aligned} \tag{5.2}$$

where $\perp$ is the orthogonality with respect to $\mathcal{B}$. Then it suffices to show three things:

Claim 5.1. $W_1(\beta) \cap W_2(\beta) = \{0\}$
Claim 5.2. $dim(W_1(\beta)) + dim(W_2(\beta)) = dim(V_1)$.
Claim 5.3. $\pi_2(\beta) \in C_2 \subset \mathcal{T}_{C_2}$.

Let's see why they are sufficient. Assuming claims 5.1, 5.2, 5.3, the only condition in corollary 3.5 needs to be verified is

$$\mathcal{B}(\beta + l_\beta, W_2(\beta)) = 0. \tag{5.3}$$

By the claim 5.3, for each β, there exists an $l_\beta \in \Lambda_2$ such that

$$\mathcal{B}(\beta + l_\beta, W_2(\beta)) = 0. \tag{5.4}$$

Then by the corollary 3.5, appying the claim 5.1, 5.2, $\mathcal{B}_2$ is injective. Similarly $\mathcal{B}_1$ is also injective. Therefore APD holds.

Next we prove the claims 5.1, 5.2, 5.3. Proof of claims 5.1, 5.2: We'll see that all properties in the claims are originated from properties of curves C_2^1, C_2. The claim 5.1 and 5.2 all follow from the non-degeneracy of $\mathcal{B}_a$ restricted to the subspace

$$H_{C_2} \times H_{C_2^1}.$$

(This non-degeneracy is the duality between $\mathcal{T}_{C_2}, \mathcal{T}_{C_2^1}$). By corollary 3.3, this is equivalent to show that

$$dim(H_{C_2}) = dim(H_{C_2^1}). \tag{5.5}$$

Claim 5.4. *: Let V be a finitely dimensional complex vector space with a lattice, $\mathcal{T}(V) = V/lattice$ be the complex torus. Let*

$$C \stackrel{\rho}{\to} \mathcal{T}(V) \tag{5.6}$$

be a complex analytic map from a smooth projective curve C. Then

$$H_C \simeq \rho^*(H^{1,0}(\mathcal{T}(V))). \tag{5.7}$$

Proof of claim 5.4: By the Poincaré duality we have the diagram

$$\begin{array}{ccc} & & V \\ & & \| \\ H^{1,0}(C) & \stackrel{\rho^*}{\leftarrow} & H^{1,0}(\mathcal{T}(V)) \\ \| & & \| \\ H_{1,0}(C) & \stackrel{\rho_*}{\to} & H_{1,0}(\mathcal{T}(V)), \end{array} \tag{5.8}$$

where $H_{1,0}$ is the subgroup in the homology that is Poincaré dual to $H^{1,0}$ (with respect to a fixed Poincaré duality isomorphism). Since $\rho_* \circ \rho^* = identity$, we obtain that

$$\rho^*(H^{1,0}(\mathcal{T}(V))) \simeq \rho_*(H_{1,0}(C)). \tag{5.9}$$

Notice that

$$\rho_*(H_{1,0}(C)) = H_C. \tag{5.10}$$

This proves the claim 5.4.

It follows that

$$\begin{array}{c} dim(H_{C_2}) = dim(\rho_1)^*(H^{1,0}(\mathcal{T}((V_1)^*))) = dim(E_1) \\ dim(H_{C_2^1}) = dim(\rho_1^a)^*(H^{1,0}(\mathcal{T}((V_1^a)^*))) = dim(E_1^a). \end{array} \tag{5.11}$$

By the assumption 2 in theorem 4.5,

$$dim(H_{C_2}) = dim(H_{C_2^1}). \tag{5.12}$$

This proves claim 5.1, 5.2

Proof of claim 5.3: For any curve $C_i \in \mathcal{T}(V_i^a)$ through the 0, there is a projective curve T_i parametrizing zero cycles on M_i such that the composition, which is still denoted by ψ_{T_i},

$$T_i \stackrel{\{t\}-\{t_0\}}{\rightarrow} CH^1_{alg}(T_i) \stackrel{\phi_i}{\rightarrow} \mathcal{T}(V_i^a) \tag{5.13}$$

is onto C_i, where $t_0 \in T_i$ is fixed. Using this fact , for our choice of C_2, there is a curve T_2 on M_2 such that $C_2 = \psi_{T_2}(T_2)$. Then by part (2) of the definition, we have $\pi_2(\beta) \in C_2 \subset \mathcal{T}_{C_2}$. The claim is proved.
□

Remark After the proof of claims 5.1, 5.2 and 5.3, there is another way to see APD without the sections 2, 3. The claims 5.1, 5.2, 5.3 can be summarized as follows. The sufficient conditions for theorem 4.5 are (a) $\mathcal{B}_a$ restricted to the subspace

$$H_{C_2} \times H_{C_1}$$

is non-degenerate; (b) $\pi_2(\beta) \in C_2$. Once (a), (b) are proved, the non-degeneracy of $\mathcal{B}$ (on the entire $V_2 \times V_1$) and (b) will imply that $\beta \in H_{C_2}$. Then if $\beta \in ker(\mathcal{B}_2)$, it must be zero (otherwise it would contradict (a)).

6 Examples

Theorem 4.5 is abstract. So we give concrete examples to show its geometric interpretation. They are further studies of the example 3.1. Consider the set-up of example 3.1. Let $\beta \in V_2^a$ be any non-zero vector. Due to the existence of the Abel-Jacobi maps, the assumption 1 in theorem 4.5 is satisfied. Thus we consider assumption 2 in the theorem. Let C_2 be a smooth curve and

$$Z \in CH^{p+1}(C_2 \times X) \tag{6.1}$$

be any correspondence between points on C_2 and q cycles on X such that there are two points $c_0, c_1 \in C_2$ satisfying

$$\phi_2(Z_{c_0} - Z_{c_0}) = 0, \phi_2(Z_{c_1} - Z_{c_0}) = \pi_2(\beta). \tag{6.2}$$

(such curves C_2 always exist). Let $\pi_1 : C_2 \times X \to C_2$ be the projection.

Then $\sigma(C_2)$ in the assumption 2, theorem 4.5 can be expressed as the difference of the dimensions of two vector spaces

$$\begin{aligned} E_1 &= (\pi_1)_*(F^{q+1}H^{2q+1}(X;\mathbb{C}) \cup \omega_Z)) \\ E_1^a &= (\pi_1)_*(F^{q+1}H_a^{2q+1}(X;\mathbb{C}) \cup \omega_Z)), \end{aligned} \tag{6.3}$$

where ω_Z is the Poincaré dual class to Z in $C_2 \times X$. In the following we explore this example for specific p, q, n.

Example 6.1. The first example is for $p = 0, q = n-1$. Let $u \in H^2(X;\mathbb{Z})$ be the class of a hyperplane section. Using the hard Lefschetz theorem there is an isomorphism

$$\begin{array}{ccc} H_a^1(X;\mathbb{C}) = H^1(X;\mathbb{C}) & \to & H^{2n-1}(X;\mathbb{C}) \\ l & \to & l \cup u^{n-1} \end{array} \tag{6.4}$$

This implies that

$$H^{2q+1}(X;\mathbb{C}) = H^{2n-1}(X;\mathbb{C}) = H_a^{2q+1}(X;\mathbb{C}) \tag{6.5}$$

Thus

$$E_1 = E_1^a. \tag{6.6}$$

Hence $\sigma(C_2) = 0$. Similarly $\sigma(C_1) = 0$ for any curve C_1. By theorem 4.5, APD holds on such V_i^a. Indeed, in the case where $p = 0, q = n-1$, due to the hard Lefschetz theorem

$$V_1 = V_1^a, V_2 = V_2^a. \tag{6.7}$$

Then the APD is just a particular duality in example 2.1.

This case is trivial by the hard Lefschetz theorem. Next we consider a non-trivial case.

Example 6.2. Let X be a threefold, $p = q = 1$. Consider the same V_i, V_i^a in this case. Then, by the theorem 4.5, the APD boils town to the equality (6.6), i.e.

$$E_1 = E_1^a. \tag{6.8}$$

Consider the correspondence

$$Z \in CH^{p+1}(C_2 \times X)$$

above in this case. Using Künneth decomposition we can express

$$\omega_Z = \sum_i l_i \otimes \omega_i + \alpha_0 + \alpha_2 \tag{6.9}$$

where

$$\begin{array}{l} l_i \in H^1(C_2;\mathbb{C}), \omega_i \in H^{2p+1}(X;\mathbb{C}) \\ \alpha_i \in H^i(C_2;\mathbb{C}) \otimes H^{2p+2-i}(X;\mathbb{C}). \end{array} \tag{6.10}$$

It is clear that

$$E_1 = span(\bar{l}_i)_i, \tag{6.11}$$

where $\bar{l}_i$ is the projection of l_i to the direct summand $H^{1,0}(C_2;\mathbb{C})$ in the Hodge decomposition. Then it suffices to show that the Poincaré dual of

$$span(\bar{\omega}_i)_i \tag{6.12}$$

is contained in

$$F^{q+1}H_a^{2q+1}(X;\mathbb{C}),$$

where $\bar{\omega}_i$ is the projection of ω_i to the direct summand $H^{p,p+1}(X;\mathbb{C})$ in the Hodge decomposition.

We express it as

$$(span(\bar{\omega}_i)_i)^* \subset F^{q+1}H_a^{2q+1}(X;\mathbb{C}). \tag{6.13}$$

This is equivalent to say that for any non-zero

$$\xi \in span(\bar{\omega}_i)_i = \overline{F^{p+1}H_{C_2}^{2p+1}(X;\mathbb{C})},$$

there is an $\eta \in F^{q+1}H_a^{2q+1}(X;\mathbb{C})$ such that

$$\mathcal{B}(\xi, \eta) \neq 0. \tag{6.14}$$

By the definition,

$$span(\bar{\omega}_i)_i = \overline{F^{p+1}H_{C_2}^{2p+1}(X;\mathbb{C})}$$

is a subspace of $H_a^{2p+1}(X;\mathbb{C})$. Then ξ lies in $H_a^{2p+1}(X;\mathbb{C})$. †† To find η in (6.14), we use Murre's argument in [1]. Note $p = q = 1$. By the Lefschetz decomposition

$$\xi = u \cup \xi_1 + \xi_3, \tag{6.15}$$

where ξ_1, ξ_3 in $H^1(X;\mathbb{C})$ and $H^3(X;\mathbb{C})$ are primitive. Let

$$\xi_3 = \xi^{1,2} \in \overline{F^2H^3(X;\mathbb{C})}.$$

We may assume $\xi^{1,2} \neq 0$ (otherwise the proof is trivial). Then we let

$$\eta = \bar{\xi}^{1,2}. \tag{6.16}$$

Finally, by the primitive nature of ξ_1, ξ_3

$$\xi \cup \eta = \xi^{1,2} \cup \bar{\xi}^{1,2} \neq 0. \tag{6.17}$$

(the last inequality is the Hodge-Riemann relation). This shows $E_1 = E_1^a$.

Therefore, by theorem 4.5, APD holds in this case. Indeed APD in this case is confirmed in another source in the lemma 5.2, [1], whose proof is based on the argument after (6.15).

Example 6.3. Recall $u \in H^2(X;\mathbb{Z})$ is the class of a hyperplane section. For more general p, q, n, to repeat Murre's argument (that is after (6.15)), we need to ASSUME the correctness of the Lefschetz standard conjecture, which

†† Then the last statement goes back to the original APD.

says the inverse of the isomorphism in hard Lefschetz theorem is algebraic. Restricted to cohomology of odd degrees it says

Assumption 6.1. *For $p \leq q$, the injective homomorphism*

$$\begin{array}{ccc} H_a^{2p+1}(X;\mathbb{C}) & \to & H_a^{2q+1}(X;\mathbb{C}) \\ \alpha & \to & u^{q-p} \cup \alpha \end{array} \tag{6.18}$$

is an isomorphism.

As in the example above, it suffices to show

$$(span(\bar{\omega}_i)_i)^* \subset F^{q+1} H_a^{2q+1}(X;\mathbb{C}). \tag{6.19}$$

Since the correspondence is arbitrary, by the definition of "partially algebraic parts", (6.19) becomes

$$\overline{(F^{p+1} H_a^{2p+1}(X;\mathbb{C}))}^* \subset F^{q+1} H_a^{2q+1}(X;\mathbb{C}). \tag{6.20}$$

Next we imitate Murre's proof above. Let $\xi \in H_a^{2p+1}(X;\mathbb{C})$. Then consider Lefschetz decomposition

$$\xi = \sum_{i=0}^{p} u^i \cup \xi_{2p+1-2i} \tag{6.21}$$

where $\xi_{2p+1-2i} \in H^{2p+1-2i}(X;\mathbb{C})$ are primitive. Since ξ is partially algebraic. Then

$$u^{q-p+1} \cup (\xi - \xi_0) = u^{q-p+1} \cup \xi, \tag{6.22}$$

is partially algebraic (use the fact that ξ_0 is primitive). By the decomposition

$$\xi - \xi_0 = u \cup (\sum_{i=1}^{p} u^{i-1} \cup (\xi_i)) = u \cup \xi^1 \tag{6.23}$$

for some $\xi^1 \in H^{2p-1}(X;\mathbb{C})$. Apply assumption 6.1 to the following isomorphism,

$$u^{q-p+2} : H^{2p-1}(X;\mathbb{C}) \ \to \ H^{2q+3}(X;\mathbb{C}). \tag{6.24}$$

We then obtain that ξ^1 is partially algebraic because

$$u^{q-p+2} \cup \xi^1 = u^{q-p+1} \cup (u \cup \xi^1)$$

is partially algebraic. Hence $\xi - \xi_0 = u \cup \xi^1$ is also partially algebraic. This implies ξ_0 is partially algebraic. Using induction, we conclude all components ξ_i are partially algebraic.

If $\xi_0 = \xi^{p,p+1} \neq 0$ in $H^{p,p+1}(X;\mathbb{C})$ (use induction otherwise), we let

$$\eta = \bar{\xi}^{p,p+1} \cup u^{q-p}.$$

Easy to see

$$\mathcal{B}(\xi, \eta) \neq 0. \tag{6.25}$$

This shows (6.19) is correct. Hence $E_1 = E_1^a$. Therefore the assumption 2 in theorem 4.5 holds. Hence APD holds in this setting.$^{\dagger\,\dagger\,\dagger}$

Without theorem 4.5, examples have already directly proved the APD in this geometric setting (although for higher codimensions it requires the Lefschetz standard conjecture). So these examples are presented here not for the understanding of APD, rather for the understanding of theorem 4.5 which deal with abstract APD without current geometric setting. However the example reveals a fact that the Lefschetz standard conjecture plays a role in the APD. It also shows a possibility to avoid the Lefschetz standard conjecture by selecting a correspondence Z that meets certain criteria. This will be discussed in [2], [3]. Overall these examples hinted at close relations among the APD, Griffiths' conjecture and the Lefschetz standard conjecture.

REFERENCES

[1] J. Murre, *Abel-Jacobi equivalence versus incidence equivalence for algebraic cycles of codimension two*, Topology **24** (1985), pp 361-367.

[2] B. Wang , *Algebraic Poincaré Duality 1*, Preprint 2015.

[3] ——— , *Algebraic Poincaré Duality 2*, Preprint 2015.

$^{\dagger\,\dagger\,\dagger}$ The only difference between this example and example 6.2 is that this example requires the Lefschetz standard conjecture. The example 6.2 has no need for the conjecture simply because of its low degree on the cohomology. Examples 6.2, 6.3 clearly illustrate the reason why J. Murre in [1] could prove the Griffiths' conjecture for codimension 2 cycles but failed in higher codimensions.