Two Theorems on Solutions in Eulerian Description

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Abstract – We present two proofs of theorems needed to the major work we are doing on existence and breakdown solutions of the Navier-Stokes equations for incompressible case without external force in n = 3 spatial dimensions.
Keywords – Navier-Stokes equations, velocity, pressure, Eulerian description, Lagrangian description, formulation, equivalent equations, exact solutions, existence, inexistence.

Let $u^0(x,y,z)$ and $p^0(x,y,z)$ be respectively the initial velocity and initial pressure of the three-dimensional incompressible $(\nabla\cdot u = \nabla\cdot u^0 = 0)$ Navier-Stokes equations without external force

(1) $$\frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t)\frac{\partial u_i(X,t)}{\partial x_j} = \nu\nabla^2 u_i(X,t),$$

$1 \le i \le 3,\ X = (x_1,x_2,x_3) \in \mathbb{R}^3,\ x_1 \equiv x,\ x_2 \equiv y,\ x_3 \equiv z,\ x_i, t \in \mathbb{R},\ t \ge 0.$

Then in $t = 0$ is valid, for each integer i belongs to $1 \le i \le 3$,

(2) $$\frac{\partial p^0(X)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} + \sum_{j=1}^{3} u_j^0(X)\frac{\partial u_i^0(X)}{\partial x_j} = \nu\nabla^2 u_i^0(X).$$

Supposing that $u(x,y,z,t) = u^0(x+t,y+t,z+t)$ and $p(x,y,z,t) = p^0(x+t,y+t,z+t)$ is a solution (u,p) for (1), we have

(3) $$\frac{\partial p^0(\xi)}{\partial x_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j} = \nu\nabla^2 u_i^0(\xi),$$

where $\xi = (\xi_1,\xi_2,\xi_3)$ and $\xi_i = \xi_i(X,t) = x_i + t,\ 1 \le i \le 3$.

For $t = 0$ the equations (2) and (3) are equals, because in $t = 0$ we have $\xi_i = x_i$ and therefore $\xi = (\xi_1,\xi_2,\xi_3) = (x_1,x_2,x_3) = X$.

For $t > 0$, if (2) is valid for any $X = (x,y,z) \in \mathbb{R}^3$ then (3) is valid for any $\xi \in \mathbb{R}^3$ substituting $x \mapsto \xi_1 = x+t,\ y \mapsto \xi_2 = y+t,\ z \mapsto \xi_3 = z+t,\ x,y,z \in \mathbb{R}, t \ge 0$, so $u(x,y,z,t) = u^0(x+t,y+t,z+t)$ and $p(x,y,z,t) = p^0(x+t,y+t,z+t)$, i.e., $u(X,t) = u^0(\xi)$ and $p(X,t) = p^0(\xi)$, solve equation (3) and therefore the Navier-Stokes equation (1).

The initial motivation to prove it is as follows. Let $A(x), B(x), C(x)$ and $D(x)$ functions such that is always valid, for any $x \in \mathbb{R}$, the relation

(4) $$A(x) + B(x) + C(x) = D(x).$$

Then, as $(x + t) \in \mathbb{R}, x, t \in \mathbb{R}, t \geq 0$, need be valid too the relation

(5) $$A(x + t) + B(x + t) + C(x + t) = D(x + t).$$

The same argument can be used for functions of two and three spatial dimensions (or better, for n spatial dimensions), for example, $\forall x, y, z, t \in \mathbb{R}, t \geq 0$,

(6) $$\begin{aligned} &A_i(x,y,z) + B_i(x,y,z) + C_i(x,y,z) = D_i(x,y,z) \\ &\Longrightarrow A_i(x+t,\ y+t,\ z+t) + B_i(x+t,\ y+t,\ z+t) + \\ &\quad + C_i(x+t,\ y+t,\ z+t) = D_i(x+t,\ y+t,\ z+t). \end{aligned}$$

Applying the previous relation (6) to the Navier-Stokes equations (2) for $t = 0$, if

(7.1) $$A_i(x,y,z) = \frac{\partial p^0(X)}{\partial x_i},$$

(7.2) $$B_i(x,y,z) = \frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0},$$

(7.3) $$C_i(x,y,z) = \sum_{j=1}^{3} u_j^0(X)\frac{\partial u_i^0(X)}{\partial x_j},$$

(7.4) $$D_i(x,y,z) = \nu\nabla^2 u_i^0(X),$$

(7.5) $$A_i(x,y,z) + B_i(x,y,z) + C_i(x,y,z) = D_i(x,y,z),$$

$X = (x,y,z)$, then, using $\xi = \xi(X,t) = (x+t, y+t, z+t)$, need be valid too the equalities

(8.1) $$A_i(x+t,\ y+t,\ z+t) = \frac{\partial p^0(\xi)}{\partial x_i},$$

(8.2) $$B_i(x+t,\ y+t,\ z+t) = \left(\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0}\right)(\xi),$$

(8.3) $$C_i(x+t,\ y+t,\ z+t) = \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j},$$

(8.4) $$D_i(x+t,\ y+t,\ z+t) = \nu\nabla^2 u_i^0(\xi),$$

(8.5) $$\begin{aligned} &A_i(x+t,\ y+t,\ z+t) + B_i(x+t,\ y+t,\ z+t) + \\ &+ C_i(x+t,\ y+t,\ z+t) = D_i(x+t,\ y+t,\ z+t). \end{aligned}$$

The expression $\left(\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0}\right)(\xi)$ in (8.2) means that first is calculated the value of $\frac{\partial u_i(X,t)}{\partial t}$, next we assign the value $t = 0$ in this result and then we substitute $x \mapsto \xi_1 = x + t,\ y \mapsto \xi_2 = y + t,\ z \mapsto \xi_3 = z + t$, i.e., $X \mapsto \xi$.

Note that the right side of the relations (8.1) to (8.4) corresponds to each parcel of the Navier-Stokes equations (8.5) with the solution (u, p) such that

(9.1) $\qquad u(X,t) = u^0(\xi),$

(9.2) $\qquad p(X,t) = p^0(\xi),$

$X = (x, y, z),\ \xi = \xi(X,t) = (x + t,\ y + t,\ z + t)$, then (9) is a solution for (1) if $u^0(X)$ and $p^0(X)$ are initial conditions.

We will now prove that if the variables (9.1) and (9.2) solve (1) for $t \geq 0$ then $u^0(x, y, z)$ and $p^0(x, y, z)$ solve (1) for $t = 0$, i.e., then both $u^0(x, y, z)$ and $p^0(x, y, z)$ solve (2). This is an important result of this section. We'll use the chain rule[1].

Proof: Starting from (1), the three-dimensional incompressible Navier-Stokes equations, where $\nabla \cdot u = \nabla \cdot u^0 = 0$,

(10) $$\frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t)\frac{\partial u_i(X,t)}{\partial x_j} = \nu\nabla^2 u_i(X,t),$$

$1 \leq i \leq 3,\ X = (x, y, z)$, if a solution (u, p) for them is (9), i.e.,

(11.1) $\qquad u(X,t) = u^0(\xi),$

(11.2) $\qquad p(X,t) = p^0(\xi),$

$\xi = \xi(X,t) = (x + t,\ y + t,\ z + t)$, then we have, according (3),

(12) $$\frac{\partial p^0(\xi)}{\partial x_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j} = \nu\nabla^2 u_i^0(\xi).$$

How $\xi_i = x_i + t$ then $\frac{\partial \xi_i}{\partial x_i} = \frac{\partial \xi_i}{\partial t} = 1$ and $\frac{\partial \xi_i}{\partial x_j} = 0$ if $i \neq j$, so using the chain rule[1] we have, for each parcel in (10) and (12),

(13.1) $$\frac{\partial p(X,t)}{\partial x_i} = \frac{\partial p^0(\xi)}{\partial x_i} = \sum_{j=1}^{3}\frac{\partial p^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_i} = \frac{\partial p^0(\xi)}{\partial \xi_i}$$

(13.2) $$\frac{\partial u_i(X,t)}{\partial t} = \frac{\partial u_i^0(\xi)}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3}\frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

(13.3) $$u_j(X,t)\frac{\partial u_i(X,t)}{\partial x_j} = u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j} = u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j} =$$

$$= u_j^0(\xi)\,\frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

(13.4) $$\nabla^2 u_i(X,t) = \nabla^2 u_i^0(\xi) = \left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_3}\right) u_i^0(\xi) =$$
$$= \sum_{j=1}^3 \left(\frac{\partial}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j}\frac{\partial}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j}\right) u_i^0(\xi) = \sum_{j=1}^3 \left(\frac{\partial}{\partial \xi_j}\frac{\partial}{\partial \xi_j}\right) u_i^0(\xi) =$$
$$= \nabla_\xi^2 u_i^0(\xi)$$

Adding the parcels (13), with (13.3) for each integer $j = 1, 2, 3$ and the multiplication of (13.4) by viscosity coefficient ν, we come to

(14) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^3 \frac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^3 u_j^0(\xi)\,\frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

which is equivalent to previous Navier-Stokes equations (10) and (12) with the solution (11), although it is not a conventional Navier-Stokes equation because the time derivative disappears, i.e.,

(15) $$\frac{\partial u_i(X,t)}{\partial t} \mapsto \sum_{j=1}^3 \frac{\partial u_i^0(\xi)}{\partial \xi_j}.$$

In $t = 0$, when $\xi_i = x_i$, the equation (14) became

(16) $$\frac{\partial p^0(X)}{\partial x_i} + \sum_{j=1}^3 \frac{\partial u_i^0(X)}{\partial x_j} + \sum_{j=1}^3 u_j^0(X)\,\frac{\partial u_i^0(X)}{\partial x_j} = \nu \nabla^2 u_i^0(X).$$

If this equation is equivalent to (2) then

(17) $$\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^3 \frac{\partial u_i^0(X)}{\partial x_j},$$

which is thereby a good manner of define or choose the temporal derivative of velocity at $t = 0$ when the solution for velocity is $u(X,t) = u^0(\xi)$.

Similarly, for $t > 0$ we have

(18) $$\frac{\partial u_i(X,t)}{\partial t} = \sum_{j=1}^3 \frac{\partial u_i^0(\xi)}{\partial \xi_j},$$

$X = (x, y, z),\ \xi = (\xi_1, \xi_2, \xi_3),\ \xi_i = \xi_i(X,t) = x_i + t,\ 1 \le i \le 3.$

Concluding, assuming that (9), identical to (11), is a solution for (1), identical to (10), we come to (16) for $t = 0$, which is equivalent to (2) with the additional initial condition (17) and it has a solution $(u^0(X), p^0(X))$. This is what we wanted to prove. □

Next, we will prove the opposite way of the previous demonstration: if $u^0(x, y, z)$ and $p^0(x, y, z)$ solve (1) for $t = 0$, i.e., if both $u^0(x, y, z)$ and $p^0(x, y, z)$ solve (2), then the variables (u, p) given in (9.1) and (9.2) solve (1) for $t \geq 0$. This is the fundamental result of this section. The proof basically follows what we write from beginning of this section until the equations (9), increasing the transformations (13) and the conditions (17) and (18). We'll use the chain rule[1] again.

Proof: If $u^0(x, y, z)$ and $p^0(x, y, z)$ solve the three-dimensional incompressible $(\nabla \cdot u = \nabla \cdot u^0 = 0)$ Navier-Stokes equations

(19) $$\frac{\partial p(X,t)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t} + \sum_{j=1}^{3} u_j(X,t) \frac{\partial u_i(X,t)}{\partial x_j} = \nu \nabla^2 u_i(X,t)$$

for $t = 0$, with $1 \leq i \leq 3$, $X = (x_1, x_2, x_3) \in \mathbb{R}^3$, $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, $x_i, t \in \mathbb{R}$, $t \geq 0$, then in $t = 0$ is valid, for each integer i belongs to $1 \leq i \leq 3$,

(20) $$\frac{\partial p^0(X)}{\partial x_i} + \frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} + \sum_{j=1}^{3} u_j^0(X) \frac{\partial u_i^0(X)}{\partial x_j} = \nu \nabla^2 u_i^0(X).$$

Supposing that $u(x, y, z, t) = u^0(x + t, y + t, z + t)$ and $p(x, y, z, t) = p^0(x + t, y + t, z + t)$ is a solution (u, p) for (19), we have

(21) $$\frac{\partial p^0(\xi)}{\partial x_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial x_j} = \nu \nabla^2 u_i^0(\xi),$$

using $\xi = (\xi_1, \xi_2, \xi_3)$ and $\xi_i = \xi_i(X, t) = x_i + t$, $1 \leq i \leq 3$.

For $t = 0$ the equations (20) and (21) are equals, because in $t = 0$ we have $\xi_i = x_i$ and therefore $\xi = (\xi_1, \xi_2, \xi_3) = (x_1, x_2, x_3) = X$.

For $t > 0$, if (20) is valid for any $X = (x, y, z) \in \mathbb{R}^3$ then (21) is valid for any $\xi \in \mathbb{R}^3$ substituting $x \mapsto \xi_1 = x + t$, $y \mapsto \xi_2 = y + t$, $z \mapsto \xi_3 = z + t$, $x, y, z \in \mathbb{R}, t \geq 0$, according transformations (22) below, so $u(x, y, z, t) = u^0(x + t, y + t, z + t)$ and $p(x, y, z, t) = p^0(x + t, y + t, z + t)$, i.e., $u(X, t) = u^0(\xi)$ and $p(X, t) = p^0(\xi)$, solve equation (21) and therefore the Navier-Stokes equation (19).

How $\xi_i = x_i + t$ then $\frac{\partial \xi_i}{\partial x_i} = \frac{\partial \xi_i}{\partial t} = 1$ and $\frac{\partial \xi_i}{\partial x_j} = 0$ if $i \neq j$, so using the chain rule[1] we have, for each parcel in (21),

(22.1) $$\frac{\partial p^0(\xi)}{\partial x_i} = \frac{\partial p(\xi(X,t))}{\partial x_i} = \sum_{j=1}^{3} \frac{\partial p^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_i} = \frac{\partial p^0(\xi)}{\partial \xi_i}$$

(22.2) $$\frac{\partial u_i^0(\xi)}{\partial t} = \frac{\partial u_i(\xi(X,t))}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

(22.3) $$u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial x_j} = u_j(\xi(X,t))\frac{\partial u_i(\xi(X,t))}{\partial x_j} = u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j} = = u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j}$$

(22.4) $$\nabla^2 u_i^0(\xi) = \nabla^2 u_i(\xi(X,t)) = \sum_{j=1}^3 \left(\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}\right) u_i^0(\xi(X,t)) = = \sum_{j=1}^3 \left(\frac{\partial}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j}\frac{\partial}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_j}\right) u_i^0(\xi) = \sum_{j=1}^3 \left(\frac{\partial}{\partial \xi_j}\frac{\partial}{\partial \xi_j}\right) u_i^0(\xi) = = \nabla_\xi^2 u_i^0(\xi)$$

The equation (21) transformed through by (22) gives

(23) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^3 \frac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^3 u_j^0(\xi)\frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

that is, we transform $X \longmapsto \xi$ and from $\xi_i = x_i + t$ we have $\frac{\partial \xi_i}{\partial x_i} = 1$ and therefore $\partial x_i = \partial \xi_i$.

The unexpected transformation is

(24) $$\frac{\partial u_i^0(\xi)}{\partial t} = \frac{\partial u_i(\xi(X,t))}{\partial t} = \sum_{j=1}^3 \frac{\partial u_i^0(\xi)}{\partial \xi_j},$$

making (23) not be in the form of a standard Navier-Stokes equation. In $t = 0$ the transformation (24) becomes

(25) $$\frac{\partial u_i^0(\xi)}{\partial t}\Big|_{t=0} = \frac{\partial u_i(\xi(X,t))}{\partial t}\Big|_{t=0} = \frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^3 \frac{\partial u_i^0(X)}{\partial x_j},$$

$\xi_j = x_j,\ \xi = X$, for $t = 0$, thus we need to assume the additional initial condition

(26) $$\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^3 \frac{\partial u_i^0(X)}{\partial x_j}$$

when the solution for Navier-Stokes equation (1), identical to (19), is given by (9), i.e.,

(27.1) $u(X,t) = u^0(\xi)$,
(27.2) $p(X,t) = p^0(\xi)$,

$X = (x,y,z),\ \xi = \xi(X,t) = (x+t,\ y+t,\ z+t)$.

Concluding, if $(u^0(X), p^0(X))$ solve (2), identical to (20), substituting in (20) the transformation $X \mapsto \xi,\ X = (x,y,z),\ \xi = (\xi_1,\xi_2,\xi_3),\ \xi_i = x_i + t$, we come to (23),

(28) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

assuming the additional initial condition (26)

(29) $$\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(X)}{\partial x_j}$$

due to transformation (24),

(30) $$\frac{\partial u_i^0(\xi)}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j}.$$

Using (30) in (28) comes finally to

(31) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \frac{\partial u_i^0(\xi)}{\partial t} + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} = \nu \nabla_\xi^2 u_i^0(\xi),$$

the Navier-Stokes equations with the solution $(u^0(\xi), p^0(\xi))$, i.e., $(u(X,t), p(X,t))$, according (27), identical to (9). This is what we wanted to prove. □

What we see in the two previous proofs can be applied, with the obvious adaptations, to solutions of the form

(32.1) $u(X,t) = u^0(\xi)$,
(32.2) $p(X,t) = p^0(\xi)$,

$X = (x,y,z),\ \xi = (\xi_1, \xi_2, \xi_3),\ \xi_i = x_i + T_i(t),\ 1 \le i \le 3,$

with the conditions

(33) $$\frac{\partial u_i(X,t)}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} \frac{\partial \xi_j}{\partial t} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(t),$$

and

(34) $$\frac{\partial u_i(X,t)}{\partial t}\Big|_{t=0} = \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(0),$$

being the functions $T_i(t)$ differentiable of class $C^1([0,\infty))$. In this case the equations (23) and (28) are

(35) $$\frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(t) + \sum_{j=1}^{3} u_j^0(\xi) \frac{\partial u_i^0(\xi)}{\partial \xi_j} =$$

$$= \frac{\partial p^0(\xi)}{\partial \xi_i} + \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} \left[T_j'(t) + u_j^0(\xi)\right] = \nu \nabla_\xi^2 u_i^0(\xi).$$

Note that the equation (33) implies

(36) $$u_i(X,t) = u_i^0(X) + \int_0^t \sum_{j=1}^{3} \frac{\partial u_i^0(\xi)}{\partial \xi_j} T_j'(t)\, dt =$$
$$= u_i^0(\xi_1, \xi_2, \xi_3) = u_i^0\big(x_1 + T_1(t),\ x_2 + T_2(t),\ x_3 + T_3(t)\big),$$

that must be true for all differentiable function $u_i^0(\xi)$ with $\xi_i = x_i + T_i(t)$, $T_i(t)$ differentiable, $1 \le i \le 3$.

It is clear that in the Eulerian description[2] the computational and analytical challenges will be, more than solving the Navier-Stokes equations for $t > 0$, solve these equations for $t = 0$, the initial instant. Unfortunately, it is not for all pair of values (u^0, p^0) that exists solution to the equation (28) and related equations, so or u^0 is a function of p^0, or p^0 is a function of u^0, or both u^0 and p^0 are functions of another functions. Nevertheless, must have at least one solution to these equations, what is easy to resolve transforming (28) and similar equations in the Lagrangian formulation[2] for velocity, assuming that the correspondent derivatives and integrations are possible.

References

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