Is “Dark Energy” Just an Effect of Gravitational Time Dilation?

Steven Kenneth Kauffmann*

Abstract

During an epoch when a uniform-density expanding dust ball’s radius doesn’t sufficiently exceed the Schwarzschild value, the dust ball’s expansion rate will actually be increasing because the dominant gravitational time dilation effect diminishes as the dust ball expands. However this “acceleration of expansion” is absent in the “comoving” FLRW dust-ball model because the “comoving” fixing of the 00 component of the metric tensor to unity suppresses gravitational time dilation, an obvious fact in the static gravitational field limit and manifested in the “comoving” FLRW model by the precisely Newtonian form of its Friedmann equation of motion. Therefore we extend to all dust-ball initial conditions the singular Oppenheimer-Snyder transformation from “comoving” to “standard” coordinates which they carried out for a particular initial condition. In “standard” coordinates we find that gravitational and speed time dilation is wired into the equation of motion of the dust ball’s dynamical radius. All sufficiently small dust balls undergo acceleration rather than deceleration of expansion, but dust balls with sufficiently highly “hyperbolic” initial conditions actually undergo acceleration of expansion at any size. Attempts to account for the observed acceleration of the expansion of the universe by means of an ad hoc nonzero cosmological constant thus seem to be quite unnecessary.

Introduction

The Friedmann equation for the spherically-symmetric, uniform-density FLRW dust-ball model in “comoving coordinates” is mathematically indistinguishable from the *strictly Newtonian* equation of motion for a test mass moving purely radially under the gravitational influence of a point mass [1, 2]. That fact certainly appears to defy physical credibility in the context of GR, wherein the details of the motion of a test mass moving radially under the influence of a spherically-symmetric gravitational source ought to reflect *purely relativistic* phenomena, such as the effects of gravitational and speed time dilation, not *merely* Newton’s *completely nonrelativistic* laws of gravitation and motion.

The *essential characteristic* of “comoving coordinates”, however, is the *fixing* of the metric component g_{00} to *unity* [3], which, in view of the fact that $(g_{00})^{-\frac{1}{2}}$ is the gravitational time dilation factor in the static gravitational field limit [4], implies that “comoving coordinates” in fact *extinguish gravitational time dilation* in the limit of a static gravitational field. This makes it easier to make sense of the fact that the Friedmann equation for the FLRW dust-ball model in “comoving coordinates” comes out to be mathematically Newtonian in form, and therefore *as well* extinguishes *all trace* of gravitational time dilation. (As a matter of fact, it of course *even* extinguishes relativistic *speed* time dilation.)

A major clue to the *nature* of “comoving coordinates” is that in order to *accomplish* the fixing of the metric component g_{00} to unity, “comoving time” is defined by the clock readings of *an infinite number of different observers* [5], a “coordinate” definition *that is completely incompatible with Einstein’s observer/coordinate-system paradigm.*

This GR-*unphysical* nature of “comoving coordinates” is further *underlined* by the fact that the metric tensor for the FLRW dust-ball model in “comoving coordinates” has a *singularity* at the particular “comoving time” when its Newtonian-analog radially-moving “test mass” *coincides in location* with its Newtonian-analog “point mass”.

That *metric singularities* are indeed *GR-unphysical* follows via Einstein’s equivalence principle from the fact that *coordinate-transformation Jacobian-matrix singularities* are incompatible with the tensor contraction theorem—the tensor contraction theorem is of course *indispensable* to the general covariance of the Einstein equation because the Einstein tensor is constructed from *contractions* of the Riemann tensor.

The incompatibility of coordinate-transformation Jacobian-matrix singularities with the tensor contraction theorem stems from the fact that the proof of the tensor contraction theorem requires the Jacobian matrix of any candidate coordinate transformation $\bar{x}^\alpha(x^\mu)$ (and of its inverse transformation $x^\nu(\bar{x}^\alpha)$) to satisfy the Jacobian-matrix relation [6],

$$(\partial\bar{x}^\alpha/\partial x^\mu)(\partial x^\nu/\partial\bar{x}^\alpha) = \delta^\nu_\mu, \tag{1}$$

which, if each component of the Jacobian matrix $\partial\bar{x}^\alpha/\partial x^\mu$ is well-defined in terms of the finite real numbers at a given space-time point x^μ, and *also* each component of its inverse matrix is thus well-defined in terms

* Retired, American Physical Society Senior Life Member, E-mail: SKKauffmann@gmail.com

of the finite real numbers, follows at that space-time point from the chain rule of the calculus. However, because the right-hand side δ^ν_μ of Eq. (1) is *always* well-defined in terms of the finite real numbers, Eq. (1) *becomes self-inconsistent* at any space-time point x^μ where any component of the Jacobian matrix $\partial\bar{x}^\alpha/\partial x^\mu$ or any component of its inverse matrix *fails to be a well-defined finite real number*. Thus at a singularity of a coordinate transformation's Jacobian matrix or at a singularity of the inverse of that matrix the underpinning of the proof of the GR-indispensable tensor contraction theorem is destroyed.

Einstein's equivalence principle implies that a metric tensor is at each space-time point the congruence transformation of the Minkowski metric tensor with the Jacobian matrix of some coordinate transformation [7]. Therefore given the foregoing discussion of GR-physical coordinate transformations, a metric tensor is GR-physical *only* at those space-time points where both it and its inverse consist solely of components which are well-defined finite real numbers and its signature is equal to the $\{+,-,-,-\}$ signature of the Minkowski metric tensor [8]. Thus the metric tensor of the expanding FLRW dust-ball model in "comoving coordinates" is clearly GR-unphysical at its particular singularity in "comoving time", namely at the "comoving time" when its Newtonian-analog radially-moving "test mass" coincides in location with its Newtonian-analog "point mass".

In a *GR-physical* coordinate system where g_{00} *isn't* fixed to unity, the FLRW dust-ball model *will* of course be affected by gravitational time dilation. Oppenheimer and Snyder transformed an FLRW dust-ball model with a very particular initial condition (namely that the radial velocity of the "test mass" initially vanishes) from GR-unphysical "comoving coordinates"—where its metric tensor is governed by the GR-unphysical Newtonian-analogous Friedmann equation—to "standard" coordinates [9], and found that in this *GR-physical* coordinate system gravitational time dilation *completely blocks* the Newtonian-analog radially-moving "test mass" from ever coming as near to the Newtonian-analog "point mass" as the Schwarzschild radius of that "point mass" [10, 11, 2], thus *preventing* the "comoving coordinate" metric *singularity* from *ever occurring* in *GR-physical* "standard" coordinates.

The time-dilation *introduced* by the Oppenheimer-Snyder transformation from "comoving" to "standard" coordinates is entirely naturally *infinite* at the Schwarzschild radius of the Newtonian-analog "point mass". That fact, however, makes the Oppenheimer-Snyder coordinate transformation a GR-unphysical *singular* one. Of course it is mathematically *obvious* that a GR-unphysical *singular* metric can *only* be transformed into a GR-physical *nonsingular* metric by a GR-unphysical *singular* coordinate transformation [11], such as the one of Oppenheimer and Snyder. If the relevant Einstein equations were all analytically solvable, the GR practitioner would have *no call* to *ever* become thus embroiled in GR-unphysical singular metrics or coordinate transformations. For example, the GR sensible approach to the FLRW dust ball model would be to straightaway solve its Einstein equation in a GR-physical coordinate system such as the "standard", "isotropic" or "harmonic" one, *shunning* the patently *GR-unphysical* $g_{00} = 1$ "comoving system" like the proverbial plague. Unfortunately, of course, the brutal fact of the matter is that it isn't known how to analytically solve the Einstein equation for the dust ball in *other* than GR-unphysical "comoving coordinates". But the GR practitioner certainly mustn't proceed under the delusion that the use of GR-unphysical $g_{00} = 1$ "coordinates" produces an Einstein-equation solution which is GR-physical; indeed the *resulting* very precisely Newtonian Friedmann equation for a "test mass" moving radially in the gravitational field of a "point mass", which is bereft of *any trace* of purely relativistic phenomena such as gravitational or speed time dilation, but which *does* deliver a wholly GR-unphysical *metric singularity*, absolutely confirms *the opposite*.

Therefore, if we want GR-physically *legitimate* analytic information about the FLRW dust ball model, there apparently is no viable option *other* than to follow the Oppenheimer-Snyder lead of (singularly) *transforming* the patently GR-unphysical singular "comoving" dust-ball metric to a GR-physical coordinate system such as a "standard", "isotropic" or "harmonic" one in which we *would* have analytically solved the dust-ball Einstein equation *in the first place*, had we but been *able* to do so.

The *experience* of Oppenheimer and Snyder clearly shows that this coordinate-transformation course *indeed fills in* GR-physically *fundamental phenomena* such as gravitational time dilation *that are entirely absent from the GR-unphysical "comoving metric" results*. Proper understanding of gravitational time dilation in the dust-ball model is of considerable importance, for example, to working out the behavior of an expanding dust ball in the distant past [11]. Gravitational time dilation can, just on its face, be expected to *reverse* the intuitive Newtonian "deceleration of expansion" of any dust ball which isn't sufficiently larger than its Schwarzschild radius. Such implications of gravitational time dilation for dust balls are of special interest in light of observations that the universe is undergoing "acceleration of expansion" [12, 13] (which is most commonly modeled by ad hoc selection of a nonzero value of the "cosmological constant", thereby producing a space-time permeating "expansive pressure" [14], the ether-reminiscent "dark energy").

Therefore this article will *extend* the Oppenheimer-Snyder transformation from the specialized case they treated of initially stationary dust-ball uniform energy density [15] in “comoving coordinates” to *any* initial rate of change of uniform dust-ball energy density in those “coordinates”. The specialized Oppenheimer-Snyder initial condition is guaranteed to be immediately followed by an epoch of *increasing* uniform energy density of the dust ball in “comoving coordinates”, which implies a *contracting* dust-ball radius in *GR-physical* coordinates such as “standard” coordinates. Oppenheimer and Snyder were of course intent on zeroing in on “gravitational collapse”, for which their choice of initial condition is certainly convenient, but here we are *more interested* in expanding dust balls, making Oppenheimer and Snyder's very particular *specialization* of the initial rate of change of dust energy density a counterproductive one for our purposes.

That extension to arbitrary dust energy-density initial rates of change impels a generalization of the form of the particular Friedmann equation which was relevant for Oppenheimer and Snyder. Therefore, *before* we launch into the calculation of the extended Oppenheimer-Snyder transformation *itself*, we shall in the next section detail important properties of the *solution* of that extended Friedmann equation, which solution is, after all, *a key constituent* of the GR-unphysical singular “comoving metric tensor” which is to be (singularly) transformed to GR-physical nonsingular “standard” form.

Friedmann-equation solutions for “comoving” dust balls

In GR-unphysical “comoving coordinates” all individual dust particles always have zero three-velocity [16], so a uniform energy-density dust ball of radius a never changes that radius in “comoving coordinates. However, the value of the uniform energy density within the dust ball *can* change in GR-unphysical “comoving time”; the Friedmann equation is a *consequence* of the Einstein equation in “comoving coordinates” which governs the evolution of the dust ball's uniform energy density and the accompanying “comoving metric tensor” within the dust ball. The dimensionless function which the first-order Friedmann differential equation in GR-unphysical “comoving time” describes can be conceptualized in different ways, one convenient one being as the cube root of the reciprocal of the ratio of the dust ball's uniform energy density to its initial uniform energy density [17],

$$R(t) = (\rho(t_0)/\rho(t))^{\frac{1}{3}}, \tag{2a}$$

so that $R(t_0) = 1$. But *in addition* to the Eq. (2a) relationship that is satisfied by $R(t)$, the *square* of $R(t)$ *also occurs as the unique “comoving time-dependent” factor* of *both* nontrivial components of the spherically-symmetric “comoving metric tensor” [18].

The Friedmann equation for $R(t)$ which follows from the Einstein equation for the uniform energy-density dust ball in “comoving coordinates” can be conveniently written as [19],

$$(\dot{R}(t))^2 = \omega^2((1/R(t)) + \gamma), \tag{2b}$$

for which, from Eq. (2a),

$$R(t_0) = 1. \tag{2c}$$

The convenient abbreviation ω^2 is defined by,

$$\omega^2 \stackrel{\text{def}}{=} (8\pi/3)G\rho(t_0)/c^2, \tag{2d}$$

and the dimensionless constant γ is readily seen to inherently relate to the $t = t_0$ *initial value of* $(\dot{R}(t))^2$, namely,

$$\gamma = (\dot{R}(t_0)/\omega)^2 - 1, \tag{2e}$$

which implies in particular that,

$$\gamma \geq -1. \tag{2f}$$

Oppenheimer and Snyder of course specifically restricted their work to the particular case that $\gamma = -1$.

The *wholly Newtonian analog* of the Friedmann equation emerges upon taking the radial coordinate $r(t)$ of the purely radially moving “test mass” to be,

$$r(t) \stackrel{\text{def}}{=} aR(t), \tag{3a}$$

and the mass M of the “point mass” to be the initial effective mass of the dust ball, i.e.,

$$M = (4\pi/3)\rho(t_0)a^3/c^2 = \omega^2 a^3/(2G), \tag{3b}$$

which implies that,

$$\omega^2 = 2GM/a^3. \tag{3c}$$

Inserting the Eq. (3a) and (3c) substitutions for $R(t)$ and ω^2 into the Friedmann Eq. (2b) and also into Eq. (2e) for γ, and furthermore taking account of the Eq. (2c) initial condition yields,

$$\tfrac{1}{2}(\dot{r}(t))^2 - GM/r(t) = \tfrac{1}{2}(\dot{r}(t_0))^2 - GM/r(t_0), \tag{4a}$$

which when multiplied through by the arbitrary value m of the "test mass" yields the very familiar *conservation* of the *strictly Newtonian* kinetic plus gravitational potential energy of the "test mass" in the gravitational field of the "point mass" M.

Of course when the location of the "test mass" is coincident with that of the "point mass", namely when $r(t) = 0$ (which is when $R(t) = 0$ in the language of the Friedmann equation), then $(\dot{r}(t))^2$ is *infinite* (and the *same* is true of $(\dot{R}(t))^2$ in Friedmann equation language). Furthermore, since $(R(t))^2$ is the "comoving" time-dependent *factor* of both nontrivial *components* of the "comoving metric" which applies within the dust ball, that metric is *singular* when $R(t) = 0$.

An alternative way to write the "test mass" Eq. (4a) is readily verified to be,

$$(dr/dt)^2 = 2GM\big((1/r) + (\gamma/a)\big), \tag{4b}$$

where, of course, $a = r(t_0)$. In "standard" coordinates it turns out that the dynamically changing radius of the dust ball obeys Eq. (4b) in the nonrelativistic limit $c \to \infty$, but for a *finite* values of c, Eq. (4b) is *modified* by a relativistic gravitational cum initial-speed reciprocal squared *time dilation factor* on its right-hand side that makes $|dr/dt|$ not only less than c but also *linearly diminishing toward zero* as r approaches the Schwarzschild radius value $r_S = 2GM/c^2$ of the dust ball. Those results show that the GR-physical issue with "comoving coordinates" is that they *extinguish relativistic time dilation*, which is, of course, exactly what the "comoving coordinate" *fixing* of g_{00} to unity *patently* does in *the static gravitational field limit*.

The solution of the Friedmann equation can be *directly* expressed in terms of elementary functions *only* for "parabolic" initial conditions wherein $\gamma = 0$. In that case the Friedmann equation simplifies to,

$$(\dot{R}(t))^2 = \omega^2/R(t) \ \text{ or } \ \dot{R}(t) = \pm\omega/(R(t))^{\frac{1}{2}}, \tag{5a}$$

which with the initial condition $R(t_0) = 1$ yields the solution,

$$R(t) = (1 \pm \tfrac{3}{2}\omega(t - t_0))^{\frac{2}{3}}, \tag{5b}$$

where $\pm$ is the *sign* of $\dot{R}(t_0)$.

Even in those cases where $\gamma \neq 0$, however, the Friedmann equation *and* its initial condition $R(t_0) = 1$ can be cast into the integral form,

$$\int_1^{R(t)} R^{\frac{1}{2}} dR/(1 + \gamma R)^{\frac{1}{2}} = \pm\omega(t - t_0). \tag{6}$$

where $\pm$ is again the *sign* of $\dot{R}(t_0)$ when $\dot{R}(t_0) \neq 0$, and equals -1 in the case that $\dot{R}(t_0)$ vanishes (which is precisely the $\gamma = -1$ case that was treated by Oppenheimer and Snyder).

In the "parabolic" $\gamma = 0$ initial-condition case, Eq. (6) quickly leads to the solution for $R(t)$ that is given in Eq. (5b).

In the "hyperbolic" $\gamma > 0$ initial-condition case, the change of variable $R = [\sinh(u)]^2/\gamma$, i.e., $u = \sinh^{-1}((\gamma R)^{\frac{1}{2}})$, permits evaluation of the integral on the left side of Eq. (6) in terms of elementary functions. But the *consequence* of that evaluation is *only* an *implicit* algebraic expression for $R(t)$, namely,

$$(R(t))^{\frac{1}{2}}(1 + \gamma R(t))^{\frac{1}{2}} - (1 + \gamma)^{\frac{1}{2}} + \gamma^{-\frac{1}{2}}\sinh^{-1}(\gamma^{\frac{1}{2}}) - \gamma^{-\frac{1}{2}}\sinh^{-1}((\gamma R(t))^{\frac{1}{2}}) = \pm\gamma\omega(t - t_0). \tag{7a}$$

Since $\sinh^{-1}(x) = \ln\big((1 + x^2)^{\frac{1}{2}} + x\big)$, we can also express Eq. (7a) in the form,

$$(R(t))^{\frac{1}{2}}(1 + \gamma R(t))^{\frac{1}{2}} - (1 + \gamma)^{\frac{1}{2}} + \gamma^{-\frac{1}{2}}\ln\left(\frac{(1+\gamma)^{\frac{1}{2}} + \gamma^{\frac{1}{2}}}{(1+\gamma R(t))^{\frac{1}{2}} + (\gamma R(t))^{\frac{1}{2}}}\right) = \pm\gamma\omega(t - t_0). \tag{7b}$$

In the "periodic" $0 > \gamma \geq -1$ initial-condition case, the change of variable $R = -[\sin(u)]^2/\gamma$, i.e., $u = \arcsin((-\gamma R)^{\frac{1}{2}})$, likewise permits evaluation of the integral on the left side of Eq. (6) in terms of elementary functions. The consequence of that evaluation is the following *implicit* algebraic expression for $R(t)$,

$$(R(t))^{\frac{1}{2}}(1+\gamma R(t))^{\frac{1}{2}} - (1+\gamma)^{\frac{1}{2}} + (-\gamma)^{-\frac{1}{2}} \arcsin((-\gamma)^{\frac{1}{2}}) - (-\gamma)^{-\frac{1}{2}} \arcsin((-\gamma R(t))^{\frac{1}{2}}) = \pm\gamma\omega(t-t_0). \quad (8)$$

We have pointed out that at the "comoving time" t when $R(t) = 0$ it is the case that $(\dot{R}(t))^2$ is *infinite* and the "comoving" metric is *singular*. With Eqs. (5b), (7b) and (8) in hand, we can now *explicitly* write down the *value* of the "comoving time" t_s when $R(t_s) = 0$, namely the *value* of the "comoving time" *when this singularity occurs*.

In the "parabolic" initial-condition case that $\gamma = 0$ we see from Eq. (5b) that if $R(t_s) = 0$,

$$t_s = t_0 \mp \tfrac{2}{3}\omega^{-1}. \quad (9a)$$

In the "hyperbolic" initial-condition case that $\gamma > 0$ we see from Eq. (7b) that if $R(t_s) = 0$,

$$t_s = t_0 \mp (\gamma\omega)^{-1}\left[(1+\gamma)^{\frac{1}{2}} - \gamma^{-\frac{1}{2}} \ln\left((1+\gamma)^{\frac{1}{2}} + \gamma^{\frac{1}{2}}\right)\right], \quad (9b)$$

In the "periodic" initial-condition case that $0 > \gamma \geq -1$ we see from Eq. (8) that if $R(t_s) = 0$,

$$t_s = t_0 \mp (\gamma\omega)^{-1}\left[(1+\gamma)^{\frac{1}{2}} - (-\gamma)^{-\frac{1}{2}} \arcsin\left((-\gamma)^{\frac{1}{2}}\right)\right]. \quad (9c)$$

Having completed this comprehensive discussion of the character of the Friedmann equation and its solutions in the cases of "parabolic", "hyperbolic" and "periodic" types of dust-ball initial conditions, we now turn to the *extension* to *all* of these types of dust-ball initial conditions of the Oppenheimer-Snyder transformation—the work of Oppenheimer and Snyder themselves was *restricted* to the *single* "periodic" initial condition $\dot{R}(t_0) = 0$, which corresponds to the value -1 for the Friedmann-equation parameter γ.

The Oppenheimer-Snyder transformation for general initial conditions

The spherically-symmetric "comoving metric" for which the Einstein equation is solved in conjunction with uniform dust-ball energy density $\rho(t)$ is explicitly given by [20],

$$ds^2 = (cdt)^2 - U(r,t)dr^2 - V(r,t)((d\theta)^2 + (\sin\theta d\phi)^2), \quad (10a)$$

The *result* of solving the Einstein equation for this metric and the uniform dust-ball energy density $\rho(t)$ *within* the dust ball of radius a (namely for $r \leq a$) is [21],

$$V(r,t) = r^2(R(t))^2, \quad U(r,t) = (R(t))^2/[1+\gamma(\omega r/c)^2], \text{ and } \rho(t) = \rho(t_0)/(R(t))^3, \quad (10b)$$

where,

$$\omega^2 \stackrel{\text{def}}{=} (8\pi/3)G\rho(t_0)/c^2, \quad \gamma \stackrel{\text{def}}{=} (\dot{R}(t_0)/\omega)^2 - 1, \quad (10c)$$

and the dimensionless dynamical metric entity $R(t)$ *satisfies the Friedmann equation*,

$$(\dot{R}(t))^2 = \omega^2((1/R(t)) + \gamma), \quad (10d)$$

with the initial condition $R(t_0) = 1$.

To carry out the Oppenheimer-Snyder mapping of the spherically-symmetric "comoving" coordinates (r,t) to the spherically-symmetric "standard" coordinates $(\bar{r},\bar{t})$ we write the invariant differential line element ds^2 of Eq. (10a) in terms of the metric tensors of *both* coordinate systems [2, 11],

$$\begin{aligned} ds^2 &= B(\bar{r},\bar{t})(cd\bar{t})^2 - A(\bar{r},\bar{t})(d\bar{r})^2 - \bar{r}^2((d\theta)^2 + (\sin\theta d\phi)^2) \\ &= (cdt)^2 - U(r,t)(dr)^2 - V(r,t)((d\theta)^2 + (\sin\theta d\phi)^2). \end{aligned} \quad (11a)$$

Eq. (11a) *constrains* the mapping vector $(\bar{r}(r,t), \bar{t}(r,t))$; thus comparing the last terms on the left and right-hand sides respectively of Eq. (11a) immediately yields,

$$\bar{r}(r,t) = (V(r,t))^{\frac{1}{2}} = rR(t), \quad (11b)$$

where we have used the Eq. (10b) relation $V(r,t) = r^2(R(t))^2$. Next we would like to obtain $\bar{t}$ as a function of r and t, just as has been done in Eq. (11b) for $\bar{r}$. Inspection of Eq. (11a), however, reveals that that task is completely entwined with the determination of B and A as functions of r and t; moreover $\bar{t}$ *itself doesn't occur in relations that can be extracted from* Eq. (11a), *only its partial derivatives* $(\partial\bar{t}/\partial t)$ *and* $(c(\partial\bar{t}/\partial r))$ *do*. We are thus faced with solving *both* simultaneous algebraic *and* first-order partial differential equations merely to obtain $\bar{t}(r,t)$! These considerations give us our first small taste of the formidably long and arduous path, so masterly pioneered by Oppenheimer and Snyder, which lies ahead.

We now present in more explicit detail the part of Eq. (11a) which still must be solved to obtain B, A and $\bar{t}$,

$$B[(\partial\bar{t}/\partial t)(cdt) + c(\partial\bar{t}/\partial r)dr]^2 - A[(1/c)(\partial\bar{r}/\partial t)(cdt) + (\partial\bar{r}/\partial r)dr]^2 = (cdt)^2 - U(r,t)(dr)^2. \tag{11c}$$

Since the three bilinear differential forms $(cdt)^2$, $(2c\,dt\,dr)$ and $(dr)^2$ are linearly independent, Eq. (11c) produces *the three simultaneous equations*,

$$B(\partial\bar{t}/\partial t)^2 - A((1/c)(\partial\bar{r}/\partial t))^2 = 1, \tag{12a}$$

$$B(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)) - A((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r) = 0, \tag{12b}$$

$$B(c(\partial\bar{t}/\partial r))^2 - A(\partial\bar{r}/\partial r)^2 = -U. \tag{12c}$$

We begin by solving Eq. (12b) for B in terms of A, $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$,

$$B = \frac{A((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r)}{(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r))}. \tag{13a}$$

We substitute Eq. (13a) into Eq. (12a), and solve the result for A in terms of $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$,

$$A = \frac{1}{(\partial\bar{r}/\partial r)(1/c)(\partial\bar{r}/\partial t)(\partial\bar{t}/\partial t)/(c(\partial\bar{t}/\partial r)) - ((1/c)(\partial\bar{r}/\partial t))^2}. \tag{13b}$$

We now divide Eq. (12c) through by A, thus expressing it in terms of (B/A) and $(1/A)$. Into that we insert (B/A) obtained from Eq. (13a) and $(1/A)$ obtained from Eq. (13b). The resulting equation in terms of $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$ is,

$$\frac{(1/c)(\partial\bar{r}/\partial t)(\partial\bar{r}/\partial r)(c(\partial\bar{t}/\partial r))}{(\partial\bar{t}/\partial t)} - (\partial\bar{r}/\partial r)^2 + U\left[\frac{(\partial\bar{r}/\partial r)(1/c)(\partial\bar{r}/\partial t)(\partial\bar{t}/\partial t)}{(c(\partial\bar{t}/\partial r))} - ((1/c)(\partial\bar{r}/\partial t))^2\right] = 0. \tag{14a}$$

We now multiply Eq. (14a) through by the factor $(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r))$ to obtain the following homogeneous bilinear equation in the two variables $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$,

$$\begin{gathered}(1/c)(\partial\bar{r}/\partial t)(\partial\bar{r}/\partial r)(c(\partial\bar{t}/\partial r))^2 - (\partial\bar{r}/\partial r)^2(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)) + \\ U[(\partial\bar{r}/\partial r)(1/c)(\partial\bar{r}/\partial t)(\partial\bar{t}/\partial t)^2 - ((1/c)(\partial\bar{r}/\partial t))^2(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r))] = 0.\end{gathered} \tag{14b}$$

The structure of this homogeneous bilinear equation in the two variables $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$ becomes more transparent after *dividing it through by the factor* $U(\partial\bar{r}/\partial r)((1/c)(\partial\bar{r}/\partial t))$ and rearranging the order of the terms to obtain,

$$(\partial\bar{t}/\partial t)^2 - \left(\left[\frac{((1/c)(\partial\bar{r}/\partial t))}{(\partial\bar{r}/\partial r)}\right] + \left[\frac{(\partial\bar{r}/\partial r)}{U((1/c)(\partial\bar{r}/\partial t))}\right]\right)(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)) + \left[\frac{1}{U}\right](c(\partial\bar{t}/\partial r))^2 = 0. \tag{14c}$$

The homogeneous bilinear form in $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$ on the left-hand side of Eq. (14c) can now be seen to *factor* into the *product* of two homogeneous *linear* forms in $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$, namely,

$$\left((\partial\bar{t}/\partial t) - \left[\frac{((1/c)(\partial\bar{r}/\partial t))}{(\partial\bar{r}/\partial r)}\right](c(\partial\bar{t}/\partial r))\right)\left((\partial\bar{t}/\partial t) - \left[\frac{(\partial\bar{r}/\partial r)}{U((1/c)(\partial\bar{r}/\partial t))}\right](c(\partial\bar{t}/\partial r))\right) = 0. \tag{14d}$$

Now the *first* factor on the left-hand side of Eq. (14d) turns out to *also* be a factor of the *denominator* of the solution for A which is given by Eq. (13b). Indeed, it is readily verified from Eq. (13b) that $(1/A)$ can be written in the form,

$$(1/A) = \left((\partial\bar{t}/\partial t) - \left[\frac{((1/c)(\partial\bar{r}/\partial t))}{(\partial\bar{r}/\partial r)}\right](c(\partial\bar{t}/\partial r))\right)\left(\frac{((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r)}{(c(\partial\bar{t}/\partial r))}\right),$$

which implies that if the *first* factor on the left-hand side of Eq. (14d) vanished, then A would be *infinite*. Thus to obtain a finite value for the "standard" metric component A, it must be that the *second* factor on the left-hand side of Eq. (14d) vanishes, which implies the following homogeneous linear first-order partial differential equation for $\bar{t}(r,t)$,

$$((1/c)(\partial\bar{r}/\partial t))U(\partial\bar{t}/\partial t) = (\partial\bar{r}/\partial r)(c(\partial\bar{t}/\partial r)). \tag{15a}$$

Since $\bar{r}(r,t) = rR(t)$ from Eq. (11b) and $U(r,t) = (R(t))^2/(1+\gamma(\omega r/c)^2)$ from Eq. (10b), the Eq. (15a) partial differential equation for $\bar{t}(r,t)$ can be written,

$$[(r/c)/(1+\gamma(\omega r/c)^2)](\partial\bar{t}/\partial t) = [R(t)\dot{R}(t)]^{-1}(c(\partial\bar{t}/\partial r)). \tag{15b}$$

Now given a separable homogeneous linear first-order partial differential equation of the form,

$$\tau(r)(\partial\bar{t}/\partial t) = T(t)(c(\partial\bar{t}/\partial r)), \tag{16a}$$

there exists a large class of solutions for $\bar{t}(r,t)$. Indeed, for *any* differentiable dimensionless function $\chi(y)$ of dimensionless argument, Eq. (16a) is solved by,

$$\bar{t}(r,t) = t_1\chi\left((1/t_2)^2\int_{t_3}^{t}T(t')dt' + (1/c)(1/t_2)^2\int_{r_0}^{r}\tau(r')dr'\right), \tag{16b}$$

where r_0 is an arbitrary constant with the dimensions of length, and t_1, t_2 and t_3 are arbitrary constants with the dimensions of time. That the $\bar{t}(r,t)$ of Eq. (16b) actually solves the partial differential equation of Eq. (16a) can be straightforwardly verified.

In the *particular* case of Eq. (15b), which of course is the partial differential equation *of actual interest to us*, $\tau(r) = (r/c)/(1+\gamma(\omega r/c)^2)$ and $T(t) = 1/(R(t)\dot{R}(t))$. If we conveniently select the arbitrary constant r_0 of Eq. (16b) to be zero, then,

$$(1/c)\int_0^r[(r'/c)/(1+\gamma(\omega r'/c)^2)]dr' = \tfrac{1}{2}(\gamma\omega^2)^{-1}\ln(1+\gamma(\omega r/c)^2) = (\gamma\omega^2)^{-1}\ln[(1+\gamma(\omega r/c)^2)^{\frac{1}{2}}]. \tag{16c}$$

Similarly, if we conveniently select the arbitrary constant t_3 to be the time t_s of the "comoving" metric's GR-unphysical *singularity*, namely t_s is such that $R(t_s) = 0$ (the precise *value* of t_s is laid out in complete detail in Eqs. (9)), then,

$$\begin{gathered}\int_{t_s}^{t}[R(t')\dot{R}(t')]^{-1}dt' = \int_{t_s}^{t}[R(t')(\dot{R}(t'))^2]^{-1}\dot{R}(t')dt' = \\ (\omega^2)^{-1}\int_{t_s}^{t}[1+\gamma R(t')]^{-1}\dot{R}(t')dt' = (\omega^2)^{-1}\int_0^{R(t)}[1+\gamma R']^{-1}dR' = (\gamma\omega^2)^{-1}\ln(1+\gamma R(t)),\end{gathered} \tag{16d}$$

where we applied, in succession, the Eq. (10d) Friedmann equation of motion $(\dot{R}(t))^2 = \omega^2((1/R(t))+\gamma)$, the change of integration variable from t' to $R' = R(t')$ (for which $\dot{R}(t')dt' = dR'$), and the fact that $R(t_s) = 0$.

We now insert the integrations performed in Eqs. (16c) and (16d) into the Eq. (16b) solution prescription, and furthermore select the following convenient values for the two remaining arbitrary constants: $t_2 = (\gamma\omega^2)^{-\frac{1}{2}}$ and $t_1 = (1/\omega)$. This yields the solution $\bar{t}(r,t)$ of the Eq. (15b) partial differential equation as,

$$\bar{t}(r,t) = (1/\omega)\chi\left(\ln[(1+\gamma(\omega r/c)^2)^{\frac{1}{2}}(1+\gamma R(t))]\right), \tag{16e}$$

where $\chi(y)$ is an *arbitrary* differentiable dimensionless function of dimensionless argument. A tidier, more compact expression of this result is,

$$\bar{t}(r,t) = (1/\omega)\phi(u(r,t)), \tag{16f}$$

where $u(r,t)$ is defined as,

$$u(r,t) \stackrel{\text{def}}{=} (1+\gamma(\omega r/c)^2)^{\frac{1}{2}}(1+\gamma R(t)), \tag{16g}$$

and $\phi(u)$ is an *arbitrary* differentiable dimensionless function of positive dimensionless argument. It is well worth noting that the $\bar{t}(r,t)$ given by Eqs. (16f) and (16g) can straightforwardly be *verified* to satisfy Eq. (15b), provided that the Friedmann equation of motion $(\dot{R}(t))^2 = \omega^2((1/R(t))+\gamma)$ for $R(t)$ is taken into account.

In the region $0 \le r \le a$ we now have obtained the general form of $\bar{t}(r,t)$ given by Eqs. (16f) and (16g), albeit in terms of a function of one variable $\phi(u)$ *which hasn't yet been determined*, to which we can add the Eq. (11b) fact that $\bar{r}(r,t) = rR(t)$. With *those two pieces of information* we can *also* obtain the "standard" metric component functions A and B by applying Eqs. (13b) and (13a). The results for A and B aren't definitive *either* because they involve $\phi'(u)$ *which hasn't yet been determined*. Oppenheimer and Snyder insightfully *realized*, however, that A and B are in fact *uniquely determined at the* $r = a$ *surface of the dust ball* by the Birkhoff theorem. As a matter of fact, it turns out that A *conforms to the requirement of the Birkhoff theorem at* $r = a$ *regardless of what the function* $\phi(u)$ *is*. But that definitely *isn't* the case for B; the Birkhoff-theorem requirement for B at $r = a$ *determines* $|\phi'(u)|$, and thus *completes our knowledge* of the "comoving" to "standard" time transformation $\bar{t}(r,t)$. Not surprisingly, however, $\bar{t}(a,t)$ turns out to be *infinite* if $\bar{r}(a,t)$ isn't *larger* than the dust ball's Schwarzschild radius $a(\omega a/c)^2 = (8\pi/3)G\rho(t_0)a^3/c^4 = 2GM/c^2$ (since $M = (4\pi/3)\rho(t_0)a^3/c^2$), a *singular consequence* of the *restoration* in GR-physical "standard" coordinates of the *gravitational time dilation* that *simply doesn't exist* in the GR-unphysical "comoving coordinates". Gravitational time dilation *stymies* the dust ball's *access* in "standard" coordinates to a radius as small or smaller than its Schwarzschild radius, *even in the arbitrarily distant past*.

To calculate the "standard" metric components A and B we require the partial derivatives of $\bar{t}(r,t)$ and $\bar{r}(r,t)$ which enter into Eqs. (13b) and (13a) for A and B respectively. We use Eqs. (16f) and (16g) for $\bar{t}(r,t)$ to calculate its two partial derivatives,

$$(c(\partial\bar{t}/\partial r)) = \gamma(\omega r/c)(1 + \gamma(\omega r/c)^2)^{-\frac{1}{2}}(1 + \gamma R(t))\phi'(u(r,t)), \tag{17a}$$

and,

$$(\partial\bar{t}/\partial t) = (\gamma\dot{R}(t)/\omega)(1 + \gamma(\omega r/c)^2)^{\frac{1}{2}}\phi'(u(r,t)). \tag{17b}$$

We calculate the two partial derivatives of $\bar{r}(r,t)$ from its formula $\bar{r}(r,t) = rR(t)$,

$$(\partial\bar{r}/\partial r) = R(t), \tag{17c}$$

and,

$$((1/c)(\partial\bar{r}/\partial t)) = (r\dot{R}(t)/c), \tag{17d}$$

When Eqs. (17a) through (17d) are inserted into Eq. (13b) for A, *no* factors of $\phi'(u(r,t))$ survive. Indeed, with the help of the Friedmann equation, $(\dot{R}(t))^2 = \omega^2((1/R(t)) + \gamma)$, $A(r,t)$ is seen to have the relatively simple form,

$$A(r,t) = \frac{1}{1-[(\omega r/c)^2/R(t)]}. \tag{18a}$$

At the *surface* of the dust ball, namely at $r = a$, the $A(a,t)$ implied by Eq. (18a) *automatically* has the Birkhoff-theorem mandated *empty-space Schwarzschild-metric* A*-component form* as a function of $\bar{r}(a,t) = aR(t)$,

$$A(a,t) = \frac{1}{1-[(\omega a/c)^2/R(t)]} = \frac{1}{1-[(\omega^2a^3)/(c^2\bar{r}(a,t))]} = \frac{1}{1-[(2GM)/(c^2\bar{r})]}, \tag{18b}$$

where the last equality in Eq. (18b) is the consequence of $\omega^2 = (8\pi/3)G\rho(t_0)/c^2$, which is the definition of ω^2 given by Eq. (10c), together with the definition of the mass of the dust ball, namely $M = (4\pi/3)\rho(t_0)a^3/c^2$. The last expression on the right-hand side of Eq. (18b) is indeed the familiar classic A-component of *the empty-space Schwarzschild metric tensor*, which is exactly what is mandated for $A(r,t)$ at the $r = a$ surface of the dust ball by the Birkhoff theorem. Therefore the Eq. (18a) result for the A-component of the "standard" metric tensor *doesn't provide any information about the not-yet determined function* $\phi(u)$.

However when Eqs. (17a) through (17d) are inserted into Eq. (13a) for B, the result is,

$$B(r,t) = \frac{1}{((1/R(t))+\gamma)(1-[(\omega r/c)^2/R(t)])(\gamma\phi'(u(r,t)))^2}, \tag{18c}$$

which, in stark contrast with the $A(r,t)$ of Eq. (18a), *is explicitly dependent* on the not-yet determined function $\phi(u)$. The *presence* within the structure of the $B(r,t)$ of Eq. (18c) of a not-yet determined function is in fact *fortunate* because the Birkhoff theorem at the dust-ball's surface mandates that,

$$B(a,t) = (1 - [(2GM)/(c^2\bar{r}(a,t))]) = (1 - [(\omega a/c)^2/R(t)]), \tag{18d}$$

which is obliged to be be *consistent* with the result for $B(a,t)$ which is implied by $B(r,t)$ given by Eq. (18c). We therefore equate the second expression for $B(a,t)$ on the right-hand side of Eq. (18d) to the just discussed result for $B(a,t)$ which follows from the Eq. (18c) form for $B(r,t)$, and then solve the resulting equation for $\phi'(u(a,t))$. It turns out to be very convenient notationally to express that result for $\phi'(u(a,t))$ in terms of an intermediary definition, namely,

$$\phi'(u(a,t)) = F(R(t)), \tag{19a}$$

where,

$$F(s) \stackrel{\text{def}}{=} \pm \frac{s^{\frac{3}{2}}}{\gamma(1+\gamma s)^{\frac{1}{2}}(s-(\omega a/c)^2)}. \tag{19b}$$

We note from Eq. (16g) that $u(a,t)$ is a simple inhomogeneous linear function of $R(t)$ which is readily inverted, namely,

$$R(t) = \gamma^{-1}[(1+\gamma(\omega a/c)^2)^{-\frac{1}{2}}u(a,t) - 1]. \tag{19c}$$

Therefore, from Eq. (19a),

$$\phi'(u(a,t)) = F\left(\gamma^{-1}[(1+\gamma(\omega a/c)^2)^{-\frac{1}{2}}u(a,t) - 1]\right), \tag{19d}$$

which provides us *the functional form* of $\phi'(u)$ as,

$$\phi'(u) = F\left(\gamma^{-1}[(1+\gamma(\omega a/c)^2)^{-\frac{1}{2}}u - 1]\right), \tag{19e}$$

where *the functional form* of $F(s)$ is, of course, given by Eq. (19b).

In order to obtain $\bar{t}(r,t)$, however, we see from Eq. (16f) that we need *the functional form* of $\phi(u)$, which of course is,

$$\phi(u) = \phi(u_0) + \textstyle\int_{u_0}^{u} du'\, \phi'(u'). \tag{20a}$$

In Eq. (20a) we conveniently *fix* the arbitrary constant u_0 to have the value of $u(r,t)$ (which is defined by Eq. (16g)) at the initial time t_0 at surface of the dust ball (namely at $r = a$),

$$u_0 = u(a,t_0) = (1+\gamma(\omega a/c)^2)^{\frac{1}{2}}(1+\gamma R(t_0)) = (1+\gamma(\omega a/c)^2)^{\frac{1}{2}}(1+\gamma), \tag{20b}$$

with the second equality following from the fact that the initial time t_0 satisfies $R(t_0) = 1$ (see immediately below Eq. (10d)).

Of course the functional form of $\phi'(u')$ is obtained from Eq. (19e). We insert that functional form into Eq. (20a), which produces,

$$\phi(u) = \phi(u(a,t_0)) + \textstyle\int_{u(a,t_0)}^{u} du'\, F\left(\gamma^{-1}[(1+\gamma(\omega a/c)^2)^{-\frac{1}{2}}u' - 1]\right). \tag{20c}$$

We can formally greatly simplify the integrand in Eq. (20c) by making the simple inhomogenous linear change of the integration variable from u' to s, where,

$$s = \gamma^{-1}[(1+\gamma(\omega a/c)^2)^{-\frac{1}{2}}u' - 1], \tag{20d}$$

and therefore,

$$u' = [(1+\gamma(\omega a/c)^2)^{\frac{1}{2}}(1+\gamma s)] \text{ and } du' = (1+\gamma(\omega a/c)^2)^{\frac{1}{2}}\gamma ds. \tag{20e}$$

With this change of the integration variable, Eq. (20c) becomes,

$$\phi(u) = \phi(u(a,t_0)) + (1+\gamma(\omega a/c)^2)^{\frac{1}{2}}\gamma \textstyle\int_1^{\sigma(u)} ds\, F(s), \tag{20f}$$

where,

$$\sigma(u) \stackrel{\text{def}}{=} \gamma^{-1}[(1+\gamma(\omega a/c)^2)^{-\frac{1}{2}}u - 1],$$

and we have used the fact given in Eq. (20b) that $u(a,t_0) = (1+\gamma(\omega a/c)^2)^{\frac{1}{2}}(1+\gamma)$.

With the previously unknown function $\phi(u)$ now in hand via Eq. (20f), we recall from Eq. (16f) that the GR-unphysical Oppenheimer-Snyder transformation $\bar{t}(r,t)$ from GR-unphysical "comoving space-time" to GR-physical "standard" time is given by $(1/\omega)\phi(u(r,t))$, where $u(r,t) = (1+\gamma(\omega r/c)^2)^{\frac{1}{2}}(1+\gamma R(t))$ as per Eq. (16g). Furthermore, the explicit form of the integrand $F(s)$ in Eq. (20f) is, of course, given by Eq. (19b). Putting all this together, we obtain,

$$\bar{t}(r,t) = \bar{t}(a,t_0) \pm \left((1+\gamma(\omega a/c)^2)^{\frac{1}{2}}/\omega\right) \int_1^{S(r,t)} ds\, \frac{s^{\frac{3}{2}}}{(1+\gamma s)^{\frac{1}{2}}(s-(\omega a/c)^2)}, \tag{20g}$$

where,

$$S(r,t) \stackrel{\text{def}}{=} \sigma(u(r,t)) = \gamma^{-1}\left[(1+\gamma R(t))\left(\frac{1+\gamma(\omega r/c)^2}{1+\gamma(\omega a/c)^2}\right)^{\frac{1}{2}} - 1\right], \tag{20h}$$

and $R(t)$ satisfies the Friedmann equation $(\dot{R}(t))^2 = \omega^2((1/R(t)) + \gamma)$ with the initial condition $R(t_0) = 1$. Eq. (20g) is restricted to $0 \leq r \leq a$, where a is the dust ball's unchanging radius in "comoving coordinates". The parameter ω^2 equals by definition $(8\pi/3)G\rho(t_0)/c^2$, where $\rho(t_0)$ is the initial uniform energy density of the dust ball, while the parameter γ is defined as $[(\dot{R}(t_0)/\omega)^2 - 1]$; note that $\gamma \geq -1$. The $\pm$ sign in Eq. (20g) has the value of the *sign* of $\dot{R}(t)$ when $\dot{R}(t) \neq 0$, and has the value -1 when $\dot{R}(t) = 0$.

It is clear from Eq. (20g) that $\bar{t}(r,t)$ *diverges to infinity whenever* $S(r,t) \leq (\omega a/c)^2$ (it is readily checked that $(\omega a/c)^2$ is the Schwarzschild radius of the dust ball divided by its "comoving coordinate" radius a). Thus all "comoving" space-time points (r, t) for which $S(r,t) \leq (\omega a/c)^2$ are mapped to "standard" time *infinity*, which makes all such "comoving" space-time points are *inaccessible* in GR-physical "standard" coordinates.

We have pointed out that the "comoving time" t_s such that $R(t_s) = 0$ produces a singularity in the "comoving metric" described by Eqs. (10a) through (10d). It is readily checked from Eq. (20h) that $S(r,t_s) \leq 0 < (\omega a/c)^2$ for all r within the "comoving" dust ball, namely $0 \leq r \leq a$. Therefore the points of singularity of the "comoving" metric are *inaccessible* in GR-physical "standard" coordinates. That is how the GR-unphysical singular Oppenheimer-Snyder transformation *removes* the singularity present in the GR-unphysical "comoving" metric in the course of mapping it to the GR-physical nonsingular "standard" metric.

In the next section we shall use the extended Oppenheimer-Snyder transformation, namely $\bar{r}(r,t) = rR(t)$ along with Eq. (20g) for $\bar{t}(r,t)$, specialized to the *surface* of the dust ball, to work out in GR-physical "standard" coordinates the equation of motion of the radial coordinate of that surface, i.e., of the dust ball's radius.

Can a dust ball's radius undergo positive acceleration?

The dust ball's surface in "comoving coordinates" is, of course, $r_{\mathrm{surf}}(t) = a$; because dust three-velocity is always zero everywhere in GR-unphysical "comoving coordinates" the dust ball's surface is *unchanging* in GR-unphysical "comoving time". (However the *uniform energy density* of the dust *within* the $r \leq a$ dust ball *does* vary in GR-unphysical "comoving time" according to $\rho(r,t) = \rho(t_0)/(R(t))^3$, while $\rho(r,t) = 0$ for all $r > a$, namely outside the dust ball in "comoving coordinates".)

The dust ball's surface transformed to "standard" coordinates is,

$$\bar{r}_{\mathrm{surf}}(t) = \bar{r}(r_{\mathrm{surf}}(t),t) = \bar{r}(a,t) = aR(t), \tag{21a}$$

where we have used the (radial) space portion $\bar{r}(r,t) = rR(t)$ of the extended Oppenheimer-Snyder transformation given in Eq. (11b). We see from Eq. (21a) that in "standard" coordinates the dust ball's surface is *moving* radially instead of having the unchanging radius a that it has in "comoving coordinates". However that motion is presented in Eq. (21a) in terms of "comoving" time t instead of in terms of the "standard" time which obtains at that surface, namely,

$$\bar{t}_{\mathrm{surf}}(t) = \bar{t}(r_{\mathrm{surf}}(t),t) = \bar{t}(a,t). \tag{21b}$$

Of course $\bar{t}(a,t)$ is readily obtained as a special case of Eq. (20g). Since we see from Eq. (20h) that $S(a,t) = R(t)$ we obtain from Eqs. (21b) and (20g) that,

$$\bar{t}_{\mathrm{surf}}(t) = \bar{t}(a,t_0) \pm \left((1+\gamma(\omega a/c)^2)^{\frac{1}{2}}/\omega\right) \int_1^{R(t)} ds\, \frac{s^{\frac{3}{2}}}{(1+\gamma s)^{\frac{1}{2}}(s-(\omega a/c)^2)}. \tag{21c}$$

From Eq. (21c) we see that all "comoving times" t for which $R(t) \leq (\omega a/c)^2)$ are mapped at the dust ball's surface to "standard" time *infinity*, and therefore all such "comoving times" are *inaccessible* at the dust ball's surface in GR-physical "standard" coordinates.

To work out the radial velocity of the dust ball in "standard coordinates" it will be useful to have in hand the derivative of $\bar{t}_{\mathrm{surf}}(t)$ with respect to the "comoving time" t,

$$d\bar{t}_{\mathrm{surf}}(t)/dt = \pm((1+\gamma(\omega a/c)^2)^{\frac{1}{2}}/\omega) \left[\frac{R(t)^{\frac{3}{2}}}{(1+\gamma R(t))^{\frac{1}{2}}(R(t)-(\omega a/c)^2)}\right] \dot{R}(t). \tag{21d}$$

It will in fact be the *reciprocal* of $d\bar{t}_{\mathrm{surf}}(t)/dt$ which will be *directly* useful for calculating the radial velocity of the dust ball in "standard coordinates",

$$(d\bar{t}_{\mathrm{surf}}(t)/dt)^{-1} = \pm(1+\gamma(\omega a/c)^2)^{-\frac{1}{2}}\omega((1/(R(t))+\gamma)^{\frac{1}{2}}(1-[(\omega a/c)^2)/R(t)])/\dot{R}(t). \tag{21e}$$

From the paragraph below Eq. (20h) we can see that the $\pm$ sign in Eq. (21e) is such that,

$$\pm\dot{R}(t) = |\dot{R}(t)|,$$

and the Friedmann equation $(\dot{R}(t))^2 = \omega^2((1/(R(t)) + \gamma)$ of course implies that,

$$|\dot{R}(t)| = \omega((1/(R(t)) + \gamma)^{\frac{1}{2}}.$$

Therefore Eq. (21e) simplifies to,

$$(d\bar{t}_{\text{surf}}(t)/dt)^{-1} = (1 + \gamma(\omega a/c)^2)^{-\frac{1}{2}}(1 - [(\omega a/c)^2)/R(t)]), \tag{21f}$$

which is clearly the reciprocal of the relativistic-speed cum gravitational-field *time dilation factor* at the dust ball's surface—note that $(d\bar{t}_{\text{surf}}(t)/dt)^{-1}$ goes to *unity* in the nonrelativistic limit $c \to \infty$. Eq. (21f) makes it very clear that the GR-physical "standard" coordinates *supply extremely basic relativistic effects which are absent altogether from the GR-unphysical Newtonian-analogous "comoving coordinates"*.

When using Eq. (21f) it *absolutely must*, of course, be borne in mind that it is *only* applicable for *"comoving" t* such that $R(t) > (\omega a/c)^2$ because "comoving" t for which $R(t) \le (\omega a/c)^2$ are *inaccessible* in GR-physical "standard" coordinates at the dust ball's surface, as has been emphatically pointed out below Eq. (21c).

Eq. (21f) permits the radial velocity of the dust ball's surface to be calculated in GR-physical "standard" coordinates (albeit the result is, just as for the result of $\bar{r}_{\text{surf}}$ in Eq. (21a), presented in terms of "comoving time" t) by combining the result of Eq. (21a) with Eq. (21f),

$$\frac{d\bar{r}_{\text{surf}}}{d\bar{t}_{\text{surf}}}(t) = (d\bar{r}_{\text{surf}}(t)/dt)(d\bar{t}_{\text{surf}}(t)/dt)^{-1} = a\dot{R}(t)(1 + \gamma(\omega a/c)^2)^{-\frac{1}{2}}(1 - [(\omega a/c)^2)/R(t)]), \tag{21g}$$

which, by using the Friedmann equation in the form $|\dot{R}(t)| = \omega((1/R(t)) + \gamma)^{\frac{1}{2}}$, may also be presented as,

$$\frac{d\bar{r}_{\text{surf}}}{d\bar{t}_{\text{surf}}}(t) = \pm a\omega\left(\frac{(1/R(t)) + \gamma}{1 + \gamma(\omega a/c)^2}\right)^{\frac{1}{2}}(1 - [(\omega a/c)^2)/R(t)]), \tag{21h}$$

where $\pm$ is, as usual, the sign of $\dot{R}(t)$. A very interesting (and for GR physics *necessary*) feature of the Eq. (21h) expression for the "standard-coordinate" radius velocity of the expanding dust ball is that so long as we *stick* with those "comoving times" t which are *accessible* in "standard" coordinates at the dust ball's surface, namely those t such that $R(t) > (\omega a/c)^2$, then the *magnitude* of the Eq. (21h) velocity is less than c. In fact it is readily shown that,

$$a\omega\left(\frac{(1/R(t)) + \gamma}{1 + \gamma(\omega a/c)^2}\right)^{\frac{1}{2}} < c \quad \text{provided that} \quad R(t) > (\omega a/c)^2,$$

which implies from Eqs. (21h) and (21a) that,

$$|d\bar{r}_{\text{surf}}/d\bar{t}_{\text{surf}}| < c(1 - (r_S/\bar{r}_{\text{surf}})) \quad \text{provided that} \quad \bar{r}_{\text{surf}} > r_S, \tag{21i}$$

where $r_S = a(\omega a/c)^2 = 2GM/c^2$ is the Schwarzschild radius of the dust ball. Thus the *maximum* speed of the dust ball's radius in "standard" coordinates is actually reduced *below* c by the presence of the gravitational time dilation factor $(1 - (r_S/\bar{r}_{\text{surf}}))$.

We of course know that the condition $\bar{r}_{\text{surf}} > r_S$ *cannot be violated* on pain of forcing $\bar{t}_{\text{surf}}$ to *infinity*, as is seen from from Eqs. (21a) and (21c). Eq. (21i) points out *the exact mechanism* whereby $\bar{r}_{\text{surf}} > r_S$ *is enforced* in these "standard" coordinates: because of *gravitational time dilation* the *speed* in "standard" coordinates of the dust-ball radius $\bar{r}_{\text{surf}}$ *falls at least linearly toward zero* as $\bar{r}_{\text{surf}}$ approaches r_S; thus it takes an *infinite* "standard" time $\bar{t}_{\text{surf}}$ for $\bar{r}_{\text{surf}}$ to "reach" r_S. So it is *gravitational time dilation* which prevents the "standard coordinate" radius $\bar{r}_{\text{surf}}(\bar{t}_{\text{surf}})$ of the dust ball from ever reaching the dust ball's Schwarzschild radius r_S.

Slightly recasting Eq. (21h) gives us *the equation of motion* of the dust ball's radius in "standard" coordinates *instead* of a relation intermediated by "comoving time", and that equation of motion *automatically incorporates* the Eq. (21i) principle of *no access* to the Schwarzschild value r_S for the dust ball's radius

$\bar r_{\rm surf}(\bar t_{\rm surf})$ in "standard" coordinates. Upon squaring both sides of Eq. (21h) and using Eq. (21a) to eliminate $R(t)$ in favor of $\bar r_{\rm surf}$, we obtain the desired equation of motion for $\bar r_{\rm surf}(\bar t_{\rm surf})$,

$$(d\bar r_{\rm surf}/d\bar t_{\rm surf})^2 = 2GM\big((1/\bar r_{\rm surf}) + (\gamma/a)\big)\left[\tfrac{(1-(r_S/\bar r_{\rm surf}))^2}{1+\gamma(r_S/a)}\right], \tag{21j}$$

where a is the initial value of $\bar r_{\rm surf}$ at $\bar t_{\rm surf} = \bar t(a, t_0)$. Note that in the nonrelativistic limit $c \to \infty$, the Schwarzschild radius r_S goes to zero, so that Eq. (21j) reduces to Eq. (4b), which is the "test mass" version of the Friedmann equation. The difference between Eq. (21j) and the Eq. (4b) "test mass" Friedmann equation lies *solely* in the purely relativistic squared reciprocal *time dilation factor* in the square brackets on the right-hand side of Eq. (21j). This *pinpoints* the fact that the GR-unphysical "comoving coordinates" of Eq. (4b) *extinguish GR-physical relativistic time dilation*, which, of course, is *exactly* what the GR-unphysical "comoving coordinate" *fixing* of g_{00} to unity *patently* does in *the static gravitational field limit*.

At the *initial* $\bar t_{\rm surf}$, when $\bar r_{\rm surf} = a$, we know from Eq. (21i) that $\beta_0^2 < (1 - (r_S/a))^2$, where β_0 is *defined* to be the *initial* value of $(1/c)(d\bar r_{\rm surf}/d\bar t_{\rm surf})$. With that *restriction* on the *value* of β_0^2 in mind, we can use Eq. (21j) at the *initial* $\bar t_{\rm surf}$ to determine the value of γ. The result of doing so is,

$$\gamma = \tfrac{\beta_0^2 - (r_S/a)(1-(r_S/a))^2}{(r_S/a)((1-(r_S/a))^2 - \beta_0^2)}, \tag{21k}$$

where $0 \le \beta_0^2 < (1 - (r_S/a))^2$. Note that $\beta_0 = 0$ produces $\gamma = -1$ (the Oppenheimer-Snyder initial condition), while $\beta_0^2 = (r_S/a)(1 - (r_S/a))^2$ produces $\gamma = 0$ (the "parabolic" case). The $0 < \gamma < +\infty$ "hyperbolic" case corresponds to those values of β_0^2 which satisfy $(r_S/a)(1 - (r_S/a))^2 < \beta_0^2 < (1 - (r_S/a))^2$.

Finally, we can obtain *the second-order version* of the Eq. (21j) equation of the motion of the dust ball's surface radius in "standard" coordinates. The left-hand side of that version refers to the *acceleration* of the dust ball's surface radius *instead* of to it's *speed* as Eq. (21j) does. We obtain the acceleration version by differentiating both sides of Eq. (21j) with respect to $\bar t_{\rm surf}$, and then dividing both sides by $2(d\bar r_{\rm surf}/d\bar t_{\rm surf})$. The result comes out to be,

$$d^2\bar r_{\rm surf}/d\bar t^2_{\rm surf} = GM\left[\left((-1 + 2\gamma(r_S/a))/\bar r^2_{\rm surf}\right) + 3\left(r_S/\bar r^3_{\rm surf}\right)\right]\left[\tfrac{1-(r_S/\bar r_{\rm surf})}{1+\gamma(r_S/a)}\right]. \tag{21l}$$

In the nonrelativistic limit that $r_S \to 0$ we can see that Eq. (21l) *reduces* to the well-known *negative* Newtonian acceleration result for a "test mass",

$$d^2\bar r_{\rm surf}/d\bar t^2_{\rm surf} = -GM/\bar r^2_{\rm surf}. \tag{21m}$$

On the other hand, since $\gamma \ge -1$ and and $a > r_S$, we can see from Eq. (21l) that there will *always* exist a *range of values* of $\bar r_{\rm surf} > r_S$ such that the acceleration $d^2\bar r_{\rm surf}/d\bar t^2_{\rm surf}$ of the dust ball's surface is *positive*.

As one example, when $\gamma = 0$ (the "parabolic" case), the acceleration $d^2\bar r_{\rm surf}/d\bar t^2_{\rm surf}$ of the dust ball's radius is *positive* for $r_S < \bar r_{\rm surf} < 3r_S$. In that case the peak speed of the dust ball's radius occurs at the dust-ball radius value $\bar r_{\rm surf} = 3r_S$, and, as we can see from Eq. (21j), that peak speed is equal to $2c/3^{\frac{3}{2}} = 0.3849c$.

But the most fascinating result of Eq. (21l) is that in sufficiently highly "hyperbolic" cases where $\gamma \ge \frac{1}{2}(a/r_S)$, the acceleration $d^2\bar r_{\rm surf}/d\bar t^2_{\rm surf}$ of the dust ball's radius *always is positive regardless of the value* $\bar r_{\rm surf}(\bar t_{\rm surf})$ *of that radius*. Attempts to account for the observed acceleration of the expansion of the universe by means of an ad hoc nonzero cosmological constant thus seem to be quite unnecessary.

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