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Fermat's Last Theorem Proved on a Single Page

"5% of the people think; 10% of the people think that they think; and the other 85% would rather die than think."----Thomas Edison

"The simplest solution is usually the best solution"---Albert Einstein

Abstract

Honorable Pierre de Fermat was truthful. He could have squeezed the proof of his last theorem into a page margin. Fermat's last theorem has been proved on a single page. Two similar versions of the proof are presented, using a single page for each version. The proof is based on the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, or $\sin^2 x + \cos^2 x = 1$. It is shown by contradiction that the uniqueness of this identity excludes all other n-values, n > 2 from satisfying the equation $c^n = a^n + b^n$. One will first show that if $n = 2$, $c^n = a^n + b^n$ holds, followed by showing that if $n > 2$ (n an integer), $c^n = a^n + b^n$ does not hold. For the first version, let a, b, and c be three relatively prime positive integers which are the lengths of the sides of the right triangle, *ABC,* where c is the length of the hypotenuse, and a and b are the lengths of the other two sides. Also, let the acute angle at vertex A be denoted by θ. For the second version of the proof, ratio terms were used to begin the construction of the proof, without reference to a triangle. The second version confirmed the proof in the first version. It is also exemplified that if some of the lengths are not positive integers but positive radicals, the derived necessary condition for $c^n = a^n + b^n$ to hold is applicable. Each proof version is very simple, and even high school students can learn it. Perhaps, the proof in this paper is the proof that Fermat wished there were enough margin for it in his paper. With respect to prizes, if the prize for a 150-page proof were $715,000, then the prize for a single page proof (considering the advantages) using inverse proportion, would be $107,250,000.

Proof: Version 1

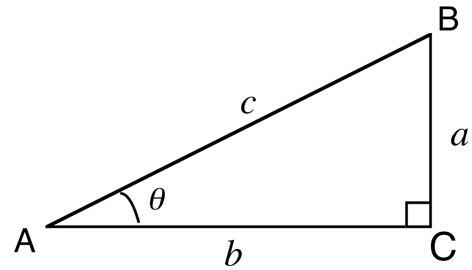
Pythagorean Identity Postulate: There exists only a single fundamental trigonometric identity such that $\sin^n\theta + \cos^n\theta = 1$ (n a positive integer).

Given: $c^n = a^n + b^n$ (n an integer; $a, b,$and c are relatively prime positive integers)

Required: To prove that $c^n = a^n + b^n$ does not hold if $n > 2$.

Plan: One will first show that if $n = 2$, $c^n = a^n + b^n$ holds,
followed by showing that if $n > 2$ (n an integer), $c^n = a^n + b^n$ does not hold.

Proof: Let $a, b,$and c be three relatively prime positive integers which are the lengths of the sides of the right triangle in the figure below, where c is the length of the hypotenuse, and a and b are the lengths of the other two sides. Also, let θ denote the acute angle at vertex A.



Then $a = c\sin\theta$ (1)

$b = c\cos\theta$ (2)

$c^n = a^n + b^n$ (3)

$c^n = (c\sin\theta)^n + (c\cos\theta)^n$

$c^n = c^n\sin^n\theta + c^n\cos^n\theta$ (4)

$c^n = c^n(\sin^n\theta + \cos^n\theta)$ (5).

Left-hand side (LHS) of equation (5) equals right-hand side (RHS) of (5) only if

$$\sin^n\theta + \cos^n\theta = 1$$

That is, a necessary condition for (5) to be true is that

$$\boxed{\sin^n\theta + \cos^n\theta = 1}$$

If $n = 2$, $c^2 = c^2(\sin^2\theta + \cos^2\theta)$ is true since $\sin^2\theta + \cos^2\theta = 1$.Therefore equations (5) and (3) are true.

Since there exists only a single Pythagorean identity (a postulate) such that $\sin^n\theta + \cos^n\theta = 1$, and $\sin^2\theta + \cos^2\theta = 1$, with $n = 2$, there are no other positive integers, n, such that $\cos^n\theta + \sin^n\theta = 1$.

Therefore, equations (5) and (3) will be true only if $n = 2$, and there are no other positive integers, $n > 2$ which will make equations (5) and (3) true.

Therefore, $c^n = a^n + b^n$ holds only if $n = 2$, and does not hold if $n > 2$. The proof is complete.

Conclusion

Fermat's last theorem has been proved in this paper. Note above that the main criterion is in equation (5) above, which requires that $\sin^n\theta + \cos^n\theta = 1$, if $c^n = c^n(\sin^n\theta + \cos^n\theta)$ and $c^n = a^n + b^n$ are to hold. Perhaps, the proof in this paper is the proof that Fermat wished there were enough margin for it in his paper.

About the Pythagorean Identity Postulate

Since $\sin^2\theta + \cos^2\theta = 1$ is true, for any other n–value such that $\sin^n\theta + \cos^n\theta = 1$, $\sin^n\theta + \cos^n\theta = \sin^2\theta + \cos^2\theta$, would imply that if $n \neq 2$, $\sin^n\theta + \cos^n\theta = \sin^2\theta + \cos^2\theta$ would be a false statement. For example, if $n = 3$, $\sin^3\theta + \cos^3\theta = \sin^2\theta + \cos^2\theta$ would imply that $3 = 2$ (equating the exponents), which is false; and by contradiction, $\sin^3\theta + \cos^3\theta \neq \sin^2\theta + \cos^2\theta$ and consequently, $\sin^3\theta + \cos^3\theta \neq 1$. Therefore, $\sin^n\theta + \cos^n\theta = 1$ only if $n = 2$. Thus, there exists only a single Pythagorean identity such that $\sin^n\theta + \cos^n\theta = 1$, and it is $\sin^2\theta + \cos^2\theta = 1$, with $n = 2$.

$a = c\sin\theta$

$b = c\cos\theta$

$c^n = a^n + b^n$

$c^n = (c\sin\theta)^n + (c\cos\theta)^n$

$c^n = c^n\sin^n\theta + c^n\cos^n\theta$

$c^n = .c^n(\sin^n\theta + \cos^n\theta)$. Equation (5) is true only if $\sin^n\theta + \cos^n\theta = 1$

For (5) to be true $\sin^n\theta + \cos^n\theta = 1$. If $n = 2$, $c^2 = c^2(\sin^2\theta + \cos^2\theta)$ is true since $\sin^2\theta + \cos^2\theta = 1$ and therefore, equations (5) and (3) hold. There exists a single identity such that $\sin^n\theta + \cos^n\theta = 1$, and $\sin^2\theta + \cos^2\theta = 1$ with $n = 2$, there are no other positive integers such that $\sin^n\theta + \cos^n\theta = 1$ Therefore, equations (5) and (3) will be true only if $n = 2$, and there are no other integers, $n > 2$ making eqns (5) and (3) true. $c^n = a^n + b^n$ holds only if $n = 2$, and does not hold if $n > 2$. QED

Proof: Version 1 in the Margin

Fermat was truthful. He could have squeezed the proof into the page margin.

If Fermat were reincarnated, he would be pleased.

Adonten

Proof: Version 2 (Using ratios)

Confirmation of Version 1 Proof

Pythagorean Identity Postulate: There exists only a single fundamental trigonometric identity such that $\sin^n x + \cos^n x = 1$ (n a positive integer).

Given: $c^n = a^n + b^n$ (n an integer; $a, b,$ and c are relatively prime positive integers)

Required: To prove that $c^n = a^n + b^n$ does not hold if $n > 2$

Plan: One will first show that if $n = 2$, $c^n = a^n + b^n$ holds, followed by showing that if $n > 2$ (n an integer), $c^n = a^n + b^n$ does not hold. One will begin by applying ratio terms.

$c^n = a^n + b^n$ (1) (Given)

$a^n + b^n = c^n$ (2) (rewriting)

$a^n = rc^n$ (3) (r is a ratio term)

$b^n = sc^n$ (4) (s is a ratio term) ($r + s = 1$)

$rc^n + sc^n = c^n$ (5) (substitute for a^n and b^n from (3) and (4))

$c^n(r + s) = c^n$ (6)

Now, by the substitution axiom, since $r + s = 1$, $r + s$ can be replaced by any quantity = 1. One can therefore replace $r + s$ by $\sin^2 x + \cos^2 x$, since $\sin^2 x + \cos^2 x = 1$. Then equation (6) becomes

$c^n(\sin^2 x + \cos^2 x) = c^n$ (7)

If $n = 2$, (7) becomes $c^2(\sin^2 x + \cos^2 x) = c^2$ (8)

$c^2 = c^2(\sin^2 x + \cos^2 x)$ (8) (rewriting)

Equation (8) is true since $\sin^2 x + \cos^2 x = 1$. Consequently, equations (8) and (1) hold. Therefore, if $n = 2$, $c^n = a^n + b^n$.

Generalizing equation (7), one obtains $c^n(\sin^n x + \cos^n x) = c^n$ (9)

in which the necessary condition for (9) to hold is $\sin^n x + \cos^n x = 1$. Since there exists only a single fundamental Pythagorean identity (a postulate) such that $\sin^n x + \cos^n x = 1$, and $\sin^2 x + \cos^2 x = 1$, with $n = 2$, there are no other positive integers, n, such that $\sin^n x + \cos^n x = 1$. Therefore, equations (9) and (1) will be true only if $n = 2$, and there are no other positive integers, $n > 2$ which will make equations (9) and (1) true. Therefore, $c^n = a^n + b^n$ holds only if $n = 2$. and does not hold if $n > 2$. The proof is complete.

Example on ratio terms

If $4 + 8 = 12$, and the ratio terms are $\frac{1}{3}$ and $\frac{2}{3}$, then

$4 = \frac{1}{3} \bullet 12$,

$8 = \frac{2}{3} \bullet 12$; and the sum of the ratio terms is

$\frac{1}{3} + \frac{2}{3} = 1$

Other equivalent identities
Note: "magic "number, 2.

$\sec^2 x - \tan^2 x = 1$

$\csc^2 x - \cot^2 x = 1$

$\cos 2x + 2\sin^2 x = 1$

$2\cos^2 x - \cos 2x = 1$

Elimination of the ratio terms r and s

The author was impressed and gratified by the substitution axiom which permitted the introduction of the much needed necessary condition $\sin^n x + \cos^n x = 1$ in Versions 1 of the proof.

Conclusion

Fermat's last theorem has been proved in this paper. Note above that the main criterion is that $\sin^n x + \cos^n x = 1$, if $c^n = c^n(\sin^n x + \cos^n x)$ and $c^n = a^n + b^n$ are to hold. Perhaps, the proof in this paper is the proof that Fermat wished there were enough margin for it in his paper.

About the Pythagorean Identity Postulate

Since $\sin^2 x + \cos^2 x = 1$, for any other n–value such that $\sin^n x + \cos^n x = 1$, $\sin^n x + \cos^n x = \sin^2 x + \cos^2 x$, would imply that if $n \neq 2$, $\sin^n x + \cos^n x = \sin^2 x + \cos^2 x$ would be a false statement. For example, if $n = 3$, $\sin^3 x + \cos^3 x = \sin^2 x + \cos^2 x$ would imply that $3 = 2$ (equating the exponents), which is false; and by contradiction, $\sin^3 x + \cos^3 x \neq \sin^2 x + \cos^2 x$ and $\sin^3 \theta + \cos^3 \theta \neq 1$. Thus, $\sin^n x + \cos^n x = 1$ only if $n = 2$.

Discussion

About Version 2 of the proof (Using ratios)
From equation (6) in Version 2 proof, one could replace $r + s$ by each of the equivalent identities as shown below. Note that $r + s = 1$; $c^n(r + s) = c^n$ (6)

Alternatives for equation (6) in Version 2 proof	**Other equivalent identities** Note the number, 2 in various positions
$c^n(\sec^2 x - \tan^2 x) = c^n$	$\sec^2 x - \tan^2 x = 1$
$c^n(\csc^2 x - \cot^2 x) = c^n$	$\csc^2 x - \cot^2 x = 1$
$c^n(\cos 2x + 2\sin^2 x) = c^n$	$\cos 2x + 2\sin^2 x = 1$
$c^n(2\cos^2 x - \cos 2x) = c^n$	$2\cos^2 x - \cos 2x = 1$

Justification for using a right triangle in Version 1 of proof

Version 1 proof began with reference to a right triangle and not an oblique triangle.
If it were an oblique triangle, the equation would be $c^2 = a^2 + b^2 - 2ab\cos C$ (A)
For Fermat's Last Theorem, three terms are involved: $c^n = a^n + b^n$ (B)
Comparing equations (A) and (B) , $2ab\cos C = 0$ in (B), which implies that $m\angle C = 90^\circ$. That is, $\angle C$ is a right angle ($\cos 90^\circ = 0$).

Uniqueness of $\sin^n x + \cos^n x = 1$ and $\sin^2 x + \cos^2 x = 1$

If $n = 2$ **,** $\sin^n x + \cos^n x = 1$ becomes $\sin^2 x + \cos^2 x = 1$, which is true. If $n = 3$, $\sin^n x + \cos^n x$ becomes $\sin^3 x + \cos^3 x$. Now, if it is assumed that $\sin^3 x + \cos^3 x = 1$. then, $\sin^3 x + \cos^3 x = \sin^2 x + \cos^2 x$ (C) (since $\sin^2 x + \cos^2 x = 1$), and equation (C) implies that $3 = 2$ (equating exponents), which is false; and by contradiction, $\sin^3 x + \cos^3 x \neq \sin^2 x + \cos^2 x$, and $\sin^3 x + \cos^3 x \neq 1$. Similarly if $n = 4$, one will obtain the false statement, $4 = 2$, and $\sin^4 x + \cos^4 x \neq 1$. Similarly, if $n = 5, 6, 7, \ldots.$ one would obtain respectively, the false statements $5 = 2,\ 6 = 2,\ 7 = 2,\ \ldots$, and by contradiction, each of these n-valies will not make $\sin^n x + \cos^n x$ equal to $\sin^2 x + \cos^2 x$ and consequently, will not make $\sin^n x + \cos^n x = 1$ true. Therefore, If $n \neq 2$, there are no positive integers which will satisfy $\sin^n x + \cos^n x = 1$ **Note:** $\sin^m x + \cos^m x = \sin^n x + \cos^n x$ only if $m = n$.	Algebraic example If $a^n + b^n = a^2 + b^2$, then $n = 2$ or If $n = 2$, then $a^n + b^n = a^2 + b^2$ Question: If $n = 3$, is $a^3 + b^3 = a^2 + b^2$? Answer: No. **Note**: If $A = B$ and $C \neq A$, then $C \neq B$.

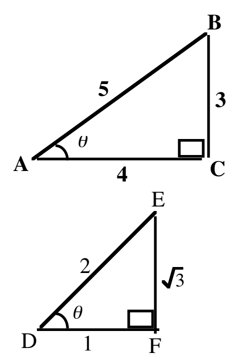
Overall Conclusion

Fermat's last theorem has been proved in this paper. In the first version of the proof, one began with reference to a right triangle; but in the second version of the proof, the proof construction began with ratio terms without reference to a triangle. The ratio terms were later on "miraculously" eliminated from the equations. The necessary condition for the relevant equations involved to be true is that $\sin^n x + \cos^n x = 1$ (or $\sin^n \theta + \cos^n \theta = 1$). Thus, if $c^n = c^n(\sin^n x + \cos^n x)$ and $c^n = a^n + b^n$ are to hold, $\sin^n x + \cos^n x = 1$ or $\sin^n \theta + \cos^n \theta = 1$ must be satisfied. First, the author determined, why the equation, $c^n = a^n + b^n$ is true if $n = 2$. It was determined that the necessary condition is $\sin^n x + \cos^n x = 1$ or $\sin^n \theta + \cos^n \theta = 1$, and this condition is satisfied only if $n = 2$, to produce $\sin^2 x + \cos^2 x = 1$. If $n = 3,4,5,...$, this necessary $\sin^n x + \cos^n x = 1$ or $\sin^n \theta + \cos^n \theta = 1$ is never satisfied. From the proof, the only condition for $c^n = a^n + b^n$ to hold is the necessary condition derived in this paper.

Therefore, $c^n = a^n + b^n$ holds only if $n = 2$, and does not hold if $n > 2$.
One should note above that version 2 proof confirmed the proof in version 1 of the proof.
Perhaps, the proof in this paper is the proof that Fermat wished there were enough margin for it in his paper.

About the numbers, *a, b,* c, being positive integers

The equation $c^n = a^n + b^n$ will be true if $\sin^n \theta + \cos^n \theta = 1$
If $n = 2$, $\sin^n \theta + \cos^n \theta = 1$ becomes $\sin^2 \theta + \cos^2 \theta = 1$.
Let one apply the necessary condition to the dimensions of the triangles in the figures to the right.



The lengths of the sides of triangle ABC are $3,\ 4,\ 5$ (pos.integers)

$\sin\theta = \frac{3}{5}$; $\sin^2\theta = \frac{9}{25}$; $\cos\theta = \frac{4}{5}$; $\cos^2\theta = \frac{16}{25}$; $\boxed{\sin^2\theta + \cos^2\theta = \frac{9}{25} + \frac{16}{25} = 1}$

The lengths of the sides of triangle DEF are $1, \sqrt{3},\ 2$ (one radical)

$\sin\theta = \frac{\sqrt{3}}{2}$; $\sin^2\theta = \frac{3}{4}$; $\cos\theta = \frac{1}{2}$; $\cos^2\theta = \frac{1}{4}$; $\boxed{\sin^2\theta + \cos^2\theta = \frac{3}{4} + \frac{1}{4} = 1}$

If $n = 3$, for triangle ABC, $\boxed{\sin^3\theta + \cos^3\theta = \frac{27}{125} + \frac{64}{125} = \frac{91}{125} \neq 1}$

If $n = 3$, for triangle DEF, $\boxed{\sin^3\theta + \cos^3\theta = \frac{3\sqrt{3}}{8} + \frac{1}{8} = \frac{1+3\sqrt{3}}{8} \neq 1}$

If $n = 2$, each of the sets of the dimensions of the two triangles satisfies the necessary condition, $\sin^n \theta + \cos^n \theta = 1$; but if $n = 3$ ($n > 2$), the necessary condition is not satisfied, that is, $\sin^n \theta + \cos^n \theta \neq 1$

The lengths of the sides of triangle ABC are all positive integers, but not all the lengths of triangle DEF are positive integer (one length is a radical). Therefore, the necessary condition is applicable even if some of the numbers involved are positive radicals.

Question for a mathematics final exam for the 2016 Fall semester.
Bonus Question: Prove Fermat's Last Theorem.

Adonten