Energy-Momentum in General Relativity

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Abstract

It is first shown that there is no exchange of energy-momentum during planetary motion. Then, starting with the field equations, it is shown that the gravitational field does not exchange energy-momentum with any form of matter. Conclusion: In general relativity, there is no gravitational energy, momentum, stress, force or power.

1. The energy-momentum vector

In the theory of general relativity, planetary motion is described by the geodesic equation [1]

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = 0 \tag{1}$$

where $u^\mu = dx^\mu/ds$ and

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}\, g^{\mu\rho}\Big(\frac{\partial g_{\nu\rho}}{\partial x^\lambda} + \frac{\partial g_{\rho\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\rho}\Big) \tag{2}$$

are the connection coefficients. The energy-momentum of the planet is given by $(c = 1)$ [2]

$$\mathbf{p} = m\mathbf{u} \tag{3}$$

and the question arises as to the rate of change of this four-vector. Expand $\mathbf{u} = \mathbf{e}_\mu u^\mu$ where the basis satisfies [3]

$$\nabla_\nu \mathbf{e}_\mu = \mathbf{e}_\lambda\, \Gamma^\lambda_{\mu\nu} \tag{4}$$

It follows that

$$\begin{aligned} \frac{d\mathbf{p}}{ds} &= m\frac{d}{ds}(\mathbf{e}_\mu u^\mu) = m\Big\{\mathbf{e}_\mu \frac{du^\mu}{ds} + \frac{d\mathbf{e}_\mu}{ds}\, u^\mu\Big\} \\ &= m\,\mathbf{e}_\mu\Big\{\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda\Big\} \end{aligned} \tag{5}$$

where $d\mathbf{e}_\mu = \mathbf{e}_\lambda\, \Gamma^\lambda_{\mu\nu} dx^\nu$. In light of the geodesic equation,

$$\frac{d\mathbf{p}}{ds} = 0 \tag{6}$$

The meaning of this equation may be stated in several ways:
(1) The energy-momentum of the planet does not change, as it moves along the geodesic path.
(2) There is no transfer of energy-momentum to or from the planet.
(3) There is no exchange of energy-momentum with the gravitational field.
In the following section, this result will be extended to include all forms of matter.

2. The stress-energy-momentum tensor

In the theory of special relativity, conservation of energy-momentum is expressed by the Lorentz covariant equation

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0 \tag{7}$$

This is a *differential* law, involving the flow of energy-momentum through an infinitesimal region. The equation is valid for any form of matter which has a well-defined stress-energy-momentum tensor $T^{\mu\nu}$. Finally, the coordinate system must be inertial and rectangular, $x^{\mu} = (x^0, x, y, z)$. However, suppose that the rectangular coordinates are replaced with ordinary spherical coordinates, $x^{\mu'} = (x^0, r, \theta, \phi)$. What will be the correct equation in the new coordinate system? To answer this question, begin with equation (7) and substitute the transformed quantities

$$T^{\mu\nu} = \frac{\partial x^{\mu}}{\partial x^{\alpha'}}\frac{\partial x^{\nu}}{\partial x^{\beta'}}\,T^{\alpha'\beta'} \qquad\qquad \frac{\partial}{\partial x^{\nu}} = \frac{\partial x^{\gamma'}}{\partial x^{\nu}}\frac{\partial}{\partial x^{\gamma'}} \tag{8}$$

This yields

$$\begin{aligned}
\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} &= \frac{\partial x^{\gamma'}}{\partial x^{\nu}}\frac{\partial}{\partial x^{\gamma'}}\left\{\frac{\partial x^{\mu}}{\partial x^{\alpha'}}\frac{\partial x^{\nu}}{\partial x^{\beta'}}\,T^{\alpha'\beta'}\right\}\\
&= \frac{\partial x^{\mu}}{\partial x^{\alpha'}}\left\{\frac{\partial T^{\alpha'\beta'}}{\partial x^{\beta'}} + \frac{\partial^2 x^{\nu}}{\partial x^{\beta'}\partial x^{\gamma'}}\frac{\partial x^{\gamma'}}{\partial x^{\nu}}\,T^{\alpha'\beta'} + \frac{\partial^2 x^{\nu}}{\partial x^{\beta'}\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^{\nu}}\,T^{\gamma'\beta'}\right\}\\
&= 0
\end{aligned} \tag{9}$$

The connection coefficients transform according to the formula [4]

$$\Gamma^{\alpha'}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial x^{\beta'}}\frac{\partial x^{\lambda}}{\partial x^{\gamma'}}\,\Gamma^{\mu}_{\nu\lambda} + \frac{\partial^2 x^{\nu}}{\partial x^{\beta'}\partial x^{\gamma'}}\frac{\partial x^{\alpha'}}{\partial x^{\nu}} \tag{10}$$

For rectangular coordinates, $\Gamma^{\mu}_{\nu\lambda} = 0$, and it follows that

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\alpha'}}\left\{\frac{\partial T^{\alpha'\beta'}}{\partial x^{\beta'}} + \Gamma^{\gamma'}_{\beta'\gamma'}T^{\alpha'\beta'} + \Gamma^{\alpha'}_{\beta'\gamma'}T^{\gamma'\beta'}\right\} = 0 \tag{11}$$

The expression in brackets is the covariant divergence $T^{\alpha'\beta'}_{\ \ ;\beta'}$. This proves that

$$T^{\alpha'\beta'}_{\ \ ;\beta'} = 0 \tag{12}$$

is the law of energy-momentum conservation in the spherical coordinate system. Q.E.D.

The generally covariant law of conservation holds true, not only for spherical coordinates, but for *all* systems of coordinates. It will now be applied to the field equations

$$R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu} R = -\frac{8\pi G}{c^4}\, T^{\mu\nu} \tag{13}$$

where $T^{\mu\nu}$ represents any form of matter. The covariant divergence of the left-hand side is identically zero [5], therefore the energy-momentum of matter is conserved

$$T^{\mu\nu}_{\ ;\nu} = 0 \tag{14}$$

There is no exchange of energy-momentum with the gravitational field.

3. Conclusion

The consequences of the planetary equation $d\mathbf{p}/ds = 0$ are unmistakable. They can be summarized by saying that the gravitational field does not interact dynamically with a planet. This is a particular case of the general result stated below.

The proof of the energy-momentum conservation law, $T^{\mu\nu}_{\ ;\nu} = 0$, is new. This law was explicitly rejected in the past [6-8], which gave rise to a host of ambiguous claims in the literature. Therefore, the proof given here is a crucial step forward. It makes the truth of this law abundantly clear. When combined with the field equations, it shows that the gravitational field is free of all dynamical interaction with matter.

Thus, there is no gravitational energy, momentum, stress, force or power. The field is purely geometric in character, comprising the metric tensor, the geodesic equation, the curvature tensor, etc. It is a four-dimensional extension of force-free Newtonian motion. [9,10]

It is fitting to close with a quotation from Einstein:

> "The general laws of nature are to be expressed by equations which hold good for all systems of coordinates, that is, are covariant with respect to any substitutions whatever (generally covariant)." [11]

References

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2. Ref. 1, page 53.

3. Ref. 1, page 259.

4. S. Weinberg, *Gravitation and Cosmology*, (Wiley, New York, 1972) page 100.

5. Ref. 4, page 147.

6. A. Einstein, "The Foundation of the General Theory of Relativity" in *The Principle of Relativity*, (Dover, New York, 1952) section 18.

7. L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Pergamon, Elmsford, 1975) 4th ed., section 96.

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11. Ref. 6, section 3.