Euler's proof of Fermat's Last Theorem for n = 3 is incorrect

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Abstract

We have spotted an error of Euler's proof, so that the used infinite descent is impossible in his proof (case A).

1 Euler's proof for n = 3

First, we rewrite a proof for n = 3, which was proven by Euler in 1770 as follows:
As Fermat did for the case n = 4, Euler used the technique of infinite descent.The proof assumes a solution(x,y,z) to the equation $x^3 + y^3 + z^3 = 0$, where the three non-zero integers x, y, z are pairwise coprime and not all positive. One of three must be even, whereas the other two are odd. Without loss of generality, z may be assumed to be even.
Since x and y are both odd, they cannot be equal, if x = y , then $2x^3 = -z^3$, which implies that x is even, a contradiction.
Since x and y are both odd, their sum and difference are both even numbers.

$$2u = x + y$$

$$2v = x - y$$

Where the non-zero integers u and v are coprime and have different parity (one is even, the other odd). Since x = u + v and y = u - v, it follows that

$$-z^3 = (u + v)^3 + (u - v)^3 = 2u(u^2 + 3v^2)$$

Since u and v have opposite parity, $u^2 + 3v^2$ is always an odd number. Therefore, since z even, u is even and v is odd. Since u and v are coprime, the greatest common divisor of 2u and $u^2 + 3v^2$ is either 1 (case A) or 3 (case B).

Proof for Case A

In this case, the two factors of $-z^3$ are coprime. This implies that 3 does not divide u and the two factors are cubes of two smaller numbers, r and s.

$$2u = r^3$$

$$u^2 + 3v^2 = s^3$$

Since $u^2 + 3v^2$ is odd, so is s. Then Euler claimed that it is possible to write:

$$s = e^2 + 3f^2$$

which e and f integers, so that

$$u = e(e^2 - 9f^2)$$

$$v = 3f(e^2 - f^2)$$

Since u is even and v is odd, then e is even and f is odd. Since

$$r^3 = 2u = 2e(e - 3f)(e + 3f)$$

The factors 2e, (e - 3f), (e + 3f) are coprime, since 3 can not divide e: if e were divisible by 3, then 3 would divide u, violating the designation of u and v as coprime. Since the three factors on the right- hand side are coprime, they must individually equal cubes of smaller integers

$$-2e = k^3$$

$$e - 3f = l^3$$

$$e + 3f = m^3$$

Which yields a smaller solution $k^3+l^3+m^3 = 0$. Therefore, by the argument of infinite descent, the original solution (x, y, z) was impossible.

Proof for Case B

In this case, the greatest common divisor of 2u and $u^2 + 3v^2$ is 3. That implies that 3 divides u, and one may express u = 3w in terms of a smaller integer w. Since u is divisible by 4, so is w, hence, w is also even. since u and v are coprime, so are v and w. Therefore, neither 3 nor 4 divide v.
Substituting u by w in the equation for z^3 yields

$$-z^3 = 6w(9w^2 + 3v^2) = 18w(3w^2 + v^2)$$

Because v and w are coprime, and because 3 does not divide v, then 18w and $3w^2 + v^2$ are also coprime. Therefore, since their product is a cube, they are each the cube of smaller integers, r and s

$$18w = r^3$$

$$3w^2 + v^2 = s^3$$

By the step as in case A, it is possible to write :

$$s = e^2 + 3f^2$$

which e and f integer, so that

$$v = e(e^2 - 9f^2)$$

$$w = 3f(e^2 - f^2)$$

Thus, e is odd and f is even, because v is odd. The expression for 18w then becomes

$$r^3 = 18w = 54f(e^2 - f^2) = 54f(e + f)(e - f)$$

Since 3^3 divides r^3 we have that 3 divides r, so $(r/3)^3$ is an integer that equals 2f(e + f)(e - f). Since e and f are coprime, so are the three factors 2e,e + f, and e - f , therefore, they are each the cube of smaller integers k, l, and m.

$$-2f = k^3$$

$$e + f = l^3$$

$$e - f = m^3$$

which yields a smaller solution $k^3+l^3+m^3 = 0$. Therefore, by the argument of infinite descent, the original solution (x, y, z) was impossible.

2 Arguments

Lemma. *if the equation $x^3 + y^3 + z^3 = 0$ is satisfied in integers, then one of the numbers x, y, and z must be divisible by 3*

*proof.*From the equation $x^3 + y^3 + z^3 = 0$, we obtain:

$$(x + y + z)^3 = 3(z + x)(z + y)(x + y)$$

Then, x + y + z is divisible by 3, $(x + y + z)^3$ is divisible by 3^3
So (z + x)(z + y)(x + y) must be divisible by 3:
If z + x is divisible by 3, then y is divisible by 3
If z + y is divisible by 3, then x is divisible by 3
If x + y is divisible by 3, then z is divisible by 3
Hence, one of x, y, and z must be divisible by 3.

Mistake in Euler's proof

For the case A
Since step,

$$u = e(e^2 - 9f^2)$$
$$v = 3f(e^2 - f^2)$$

Euler already considered only u, and passed over v , and it was a gap of proof as follows :
Since $v = 3f(e^2 - f^2)$, then v is divisible by 3.
Since

$$2v = x - y$$

Then, x - y is divisible by 3, hence, both of them are divisible by 3, or both not divisible by 3. Since x and y are coprime, then x and y have not common divisor, so both x and y are not divisible by 3. By lemma above, z must be divisible by 3, which implies that 2u and $u^2 + 3v^2$ have common divisor 3, a contradiction. Case A is impossible!
Or by other argument as follows:
$2u = r^3$ then $u = 2^2r'^3$, since in the case A, u is not divisible by 3, then r' is not divisible by 3
It gives:

$$2^2r'^3 = e(e^2 - 9f^2)$$
$$9ef^2 = e^3 - 2^2r'^3$$
$$9ef^2 = e^3 - r'^3 - 3r'^3$$

The term: $e^3 - r'^3 = (e - r')((e - r')^2 + 3er')$ is not divisible by 3, or is divisible by 3^2
Hence, Left hand side of equation : $9ef^2 = e^3 - r'^3 - 3r'^3$ is divisible by 3^2, right hand side is not. Case A is impossible!
These above arguments is the correct proof for case A if

$$u = e(e^2 - 9f^2)$$
$$v = 3f(e^2 - f^2)$$

is the **only way** for $u^2 + 3v^2$ to be expressed as a cube. However, Euler only showed that is the **possible way**.

References

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