

# Merging Mathematical Technologies by Applying the Reverse bra-ket Method

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## Abstract

Quaternionic Hilbert spaces can store discrete quaternions and quaternionic continuums in the eigenspaces of operators that reside in these Hilbert spaces. The reverse bra-ket method is an extension of the bra-ket notation that was introduced by P.M. Dirac. The reverse bra-ket method can create natural parameter spaces from quaternionic number systems and can relate the combinations of functions and their parameter spaces with eigenspaces and eigenvectors of corresponding operators that reside in non-separable Hilbert spaces. This also works for separable Hilbert spaces. The defining functions relate the separable Hilbert space with its non-separable companion. In this way, the method links Hilbert space technology with function technology, differential technology and integral technology. Quaternionic number systems exist in several versions that differ in the way that they are ordered. This is applied by defining multiple types of parameter spaces in the same Hilbert space. The set of closed subspaces of a separable Hilbert space has the relational structure of an orthomodular lattice. This fact makes the Hilbert space suitable for modelling quantum physical systems. The reverse bra-ket method is a powerful tool for generating quaternionic models that help investigating quantum physical models.

## 1 Introduction: The Why and How

### 1.1 Why

Current quantum physical models treat Hilbert spaces, function theory and differential calculus and integral calculus as separate entities. In the past nothing existed that directly relates these ingredients, which together constitute the quantum physical model. Thus, a need exists for a methodology that intimately binds these ingredients into a consistent description of the structure and the phenomena that occur in the model. Hilbert spaces are no more and no less than structured storage media. Quaternions can store progression and spatial location in a single data element [1]. Function theory relates these data. Differential calculus describes the change of data and integral calculus collects common characteristics of data.

The applied quaternionic calculus is treated in papers that are free accessible available on the author's electronic e-print archive. [http://vixra.org/author/j\\_a\\_j\\_van\\_leunen](http://vixra.org/author/j_a_j_van_leunen) .

### 1.2 How

A need exists to be able to treat sets of discrete dynamic data and related fields independent of the equations that describe their behavior. This is possible by exploiting the fact that Hilbert spaces can store discrete quaternions and quaternionic continuums in the eigenspaces of operators that reside in Hilbert spaces. The reverse bra-ket method can create natural parameter spaces from quaternionic number systems and can relate the combination of a mostly continuous function and its parameter space to the eigenspace and the eigenvectors of a corresponding operator that resides in a non-separable Hilbert space. This also works for separable Hilbert spaces. In addition the defining

functions relate the separable Hilbert space with an unique non-separable companion. This enables the view that the separable Hilbert space is embedded inside its non-separable companion.

Quaternionic number systems exist in several versions that differ in the ordering that determines their symmetry flavor. Thus, in Hilbert spaces several different versions of parameter spaces can coexist. It is possible that the members of a category of parameter spaces float over another parameter space. This can be used by modelling elementary objects, whose platforms float over a background space.

### 1.3 Modelling Example

The reverse bra-ket method is extensively applied in a mathematical test model. This model is named "The Hilbert Book Test Model". That model is based on a foundation that does not yet contain numbers. The foundation is a relational structure, which is in mathematical terminology known as an orthomodular lattice. Since the set of closed subspaces of a separable Hilbert space has exactly this relational structure, can this Hilbert space be interpreted as the direct extension of this foundation [2] [3] [4] [5].

The Hilbert space introduces number systems, but restricts the choice to division rings. Only three suitable division rings exist. The real numbers, the complex numbers and the quaternions. The Hilbert Book Test Model selects the quaternionic version of the Hilbert space and applies the reverse bra-ket method in order to extend the model such that also the non-separable companion of the separable Hilbert space is included.

The Hilbert Book Model interprets the orthomodular lattice as part of a recipe for modular construction. Not all closed subspaces of a separable Hilbert space represent modules or modular systems. Thus being a module involves more than just the ability to be represented by a closed subspace of the separable Hilbert space. The orthomodular lattice is an atomic lattice. The atoms are represented by subspaces, which are spanned by a single Hilbert vector. That Hilbert vector is eigenvector of a special normal operator, which has a quaternion as the corresponding eigenvalue. The real part of this eigen value can be represented by progression and the corresponding imaginary part can be interpreted as a location in a quaternionic parameter space.

This means that the eigenvector represents the elementary module at a single progression instant. At the next progression instant the elementary module is represented by another Hilbert vector and a different spatial location. In this way the elementary module hops as a function of progression. Its landing locations form a swarm.

Both the hopping path and the swarm represent the elementary module. A coherent location swarm owns a continuous location density distribution. The reverse bra-ket method links this function to a continuum that resides in the non-separable companion of the separable Hilbert space.

This continuum can be interpreted in two different ways. Either the continuum describes the location swarm in a smooth way or the view is taken that the continuum is deformed by the embedded locations. The selected view does not influence the underlying model.

By exploiting a selected interpretation the model is further extended into a feature rich dynamic structure and in this extension the reverse bra-ket method plays an important role.

The Hilbert Book Model is not subject of this paper [9].

#### 1.4 Restriction

The reverse bra-ket method does not link functions to all categories of Hilbert space operators. This paper gives an overview of operators that are covered and it mentions some operators that are not linked to functions or derived functions. Especially stochastic operators are not linked to the generating stochastic processes by the reverse bra-ket method.

#### 1.5 Dirac's bra-ket notation

The reverse bra-ket method relies heavily on Paul Dirac's bra-ket notation [6]. The author is astonished that Dirac did not discover the reverse bra-ket method. If he or someone else would have discovered the reverse bra-ket method in the first decades of the twentieth century, then quite probably the evolution of quantum physics would have taken a different path.

## 2 Quaternionic Hilbert spaces

Separable Hilbert spaces are linear vector spaces in which an inner product is defined. This inner product relates each pair of Hilbert vectors. The value of that inner product must be a member of a division ring. Suitable division rings are real numbers, complex numbers and quaternions. This paper uses quaternionic Hilbert spaces.

Paul Dirac introduced the bra-ket notation that eases the formulation of Hilbert space habits<sup>2</sup>.

$$\langle x|y\rangle = \langle y|x\rangle^* \quad (1)$$

$$\langle x + y|z\rangle = \langle x|z\rangle + \langle y|z\rangle \quad (2)$$

$$\langle \alpha x|y\rangle = \alpha \langle x|y\rangle \quad (3)$$

$$\langle x|\alpha y\rangle = \langle x|y\rangle \alpha^* \quad (4)$$

$\langle x|$  is a bra vector.  $|y\rangle$  is a ket vector.  $\alpha$  is a quaternion.

This paper considers Hilbert spaces as no more and no less than structured storage media for dynamic geometrical data that have an Euclidean signature. Quaternions are ideally suited for the storage of such data. Of course quaternions can also represent other kind of data.

The operators of separable Hilbert spaces have countable eigenspaces. Each infinite dimensional separable Hilbert space owns a Gelfand triple. This will be shown by the fact that the same defining functions can be used for defining a corresponding category of operators. The Gelfand triple embeds this separable Hilbert space and offers as an extra service operators that feature continuums as eigenspaces. In the corresponding subspaces the definition of dimension loses its sense.

Operators map Hilbert vectors onto other Hilbert vectors. Via the inner product the operator  $T$  may be linked to an adjoint operator  $T^\dagger$ .

$$\langle Tx|y\rangle \stackrel{\text{def}}{=} \langle x|T^\dagger y\rangle \quad (5)$$

$$\langle Tx|y\rangle = \langle y|Tx\rangle^* = \langle T^\dagger y|x\rangle^* \quad (6)$$

A linear quaternionic operator  $T$ , which owns an adjoint operator  $T^\dagger$  is normal when

$$T^\dagger T = T T^\dagger \quad (7)$$

$T_0 = (T + T^\dagger)/2$  is a self adjoint operator and  $\mathbf{T} = (T - T^\dagger)/2$  is an imaginary normal operator. Self adjoint operators are also Hermitian operators. Imaginary normal operators are also anti-Hermitian operators.

## 3 Representing operators and their eigenspaces by mostly continuous functions

By using what we will call **reverse bra-ket notation**, operators that reside in the Hilbert space and correspond to continuous functions, can easily be defined by starting from an orthonormal base of vectors. In this base the vectors are normalized and are mutually orthogonal. The vectors span a subspace of the Hilbert space. We will attach eigenvalues to these base vectors via the **reverse bra-ket notation**. This works both in separable Hilbert spaces as well as in non-separable Hilbert spaces.

Let  $\{q_i\}$  be the set of **rational** quaternions in a selected quaternionic number system and let  $\{|q_i\rangle\}$  be the set of corresponding base vectors. They are eigenvectors of a normal operator  $\mathcal{R}$ . Here we enumerate the base vectors with index  $i$ .

$$\mathcal{R} \stackrel{\text{def}}{=} |q_i\rangle q_i \langle q_i| = |q_i\rangle \Re(q_i) \langle q_i| \quad (4)$$

$\mathcal{R}$  is the configuration parameter space operator.  $\Re(q)$  is a quaternionic function, whose target equals its parameter space.

This notation must not be interpreted as a simple outer product between a ket vector  $|q_i\rangle$ , a quaternion  $q_i$  and a bra vector  $\langle q_i|$ . It involves a complete set of eigenvalues  $\{q_i\}$  and a complete orthomodular set of Hilbert vectors  $\{|q_i\rangle\}$ . It implies a summation over these constituents, such that for all bra's  $\langle x|$  and all ket's  $|y\rangle$  hold:

$$\langle x|\mathcal{R}|y\rangle = \sum_i \langle x|q_i\rangle q_i \langle q_i|y\rangle \quad (5)$$

$\mathcal{R}_0 = (\mathcal{R} + \mathcal{R}^\dagger)/2$  is a self-adjoint operator. Its eigenvalues can be used to arrange the order of the eigenvectors by enumerating them with the eigenvalues. After finishing the ordering, the ordered eigenvalues can be interpreted as **progression values**.

$\mathcal{R} = (\mathcal{R} - \mathcal{R}^\dagger)/2$  is an imaginary operator. Its eigenvalues can also be used to order the eigenvectors. The eigenvalues can be interpreted as **spatial values** and can be ordered in several different ways. By applying a Cartesian coordinate system the ordering can be achieved in eight mutually independent ways.

Let  $f(q)$  be a mostly continuous quaternionic function. Now the reverse bra-ket notation defines operator  $f$  as:

$$f \stackrel{\text{def}}{=} |q_i\rangle f(q_i) \langle q_i| \quad (6)$$

This defines a new operator  $f$  that is based on function  $f(q)$ . Here we suppose that the target values of  $f(q)$  belong to the same version of the quaternionic number system as its parameter space does.

Operator  $f$  has a countable set of discrete quaternionic eigenvalues.

For this operator the reverse bra-ket notation is a shorthand for

$$\langle x|f|y\rangle = \sum_i \langle x|q_i\rangle f(q_i) \langle q_i|y\rangle \quad (7)$$

Alternative formulations for the reverse bra-ket definition are:

$$\begin{aligned} f &\stackrel{\text{def}}{=} |q_i\rangle f(q_i) \langle q_i| \\ &= |q_i\rangle \langle f(q_i) q_i| = |q_i\rangle \langle f(q_i) q_i| = |f^*(q_i) q_i\rangle \langle q_i| = |f^\dagger q_i\rangle \langle q_i| \end{aligned} \quad (8)$$

Here we used the same symbol for the operator  $f$  and the function  $f(q_i)$ . The eigenspace is formed by the target values  $\{f(q_i)\}$ .

The left side of (7) only equals the right side when domain over which the summation is taken is restricted to the region of the parameter space  $\mathcal{R}$  where  $f(q)$  is sufficiently continuous. This includes point-like discontinuities.

### 3.1 Symmetry centers

We can define a category of anti-Hermitian operators  $\{\mathfrak{S}_n^x\}$  that have no Hermitian part and that are distinguished by the way that their eigenspace is ordered by applying a polar coordinate system. We call them symmetry centers  $\mathfrak{S}_n^x$ . A polar ordering always starts with a selected Cartesian ordering. The geometric center of the eigenspace of the symmetry center floats on a background parameter space of the normal reference operator  $\mathfrak{R}$ , whose eigenspace defines a full quaternionic parameter space. The eigenspace of the symmetry center  $\mathfrak{S}_n^x$  acts as a three dimensional spatial parameter space. The super script  $x$  refers to the symmetry flavor of  $\mathfrak{S}_n^x$ . The subscript  $n$  enumerates the symmetry centers. Sometimes we omit the subscript.

$$\mathfrak{S}^x \stackrel{\text{def}}{=} |\mathfrak{s}_i^x\rangle \mathfrak{s}_i^x \langle \mathfrak{s}_i^x| \quad (1)$$

$$\mathfrak{S}^{x\dagger} = -\mathfrak{S}^x \quad (2)$$

The concept of the symmetry center plays an important role in the representation of elementary modules. In the test model, symmetry centers are supposed to float over a background parameter space.

### 3.2 Continuum eigenspaces

In a non-separable Hilbert space, such as the Gelfand triple, the continuous function  $\mathcal{F}(q)$  can be used to define an operator, which features a continuum eigenspace. We start with defining a continuum parameter space. We use function  $\mathfrak{R}(q)$  in order to define the parameter space of  $\mathcal{F}(q)$ .

$$\mathfrak{R} \stackrel{\text{def}}{=} |q\rangle q \langle q| = |q\rangle \mathfrak{R}(q) \langle q| \quad (1)$$

This definition relates the separable Hilbert space and its companion Gelfand triple. The function  $\mathcal{F}(q)$  defines a continuum eigenspace.

$$\mathcal{F} \stackrel{\text{def}}{=} |q\rangle \mathcal{F}(q) \langle q| \quad (2)$$

Via the continuous quaternionic function  $\mathcal{F}(q)$ , the operator  $\mathcal{F}$  defines a curved continuum  $\mathcal{F}$ . This operator and the continuum reside in the Gelfand triple, which is a non-separable Hilbert space.

The function  $\mathcal{F}(q)$  uses the eigenspace of the reference operator  $\mathfrak{R}$  as a flat parameter space that is spanned by a quaternionic number system  $\{q\}$ . The continuum  $\mathcal{F}$  represents the target space of function  $\mathcal{F}(q)$ .

Here we no longer enumerate the base vectors with index  $i$ . We just use the name of the parameter. If no conflict arises, then we will use the same symbol for the defining function, the defined operator and the continuum that is represented by the eigenspace.

For the shorthand of the reverse bra-ket notation of operator  $\mathcal{F}$  the integral over  $q$  replaces the summation over  $q_i$ .

$$\langle x|\mathcal{F}y\rangle = \sum_i \langle x|q_i\rangle\mathcal{F}(q_i)\langle q_i|y\rangle \approx \int_q \langle x|q\rangle\mathcal{F}(q)\langle q|y\rangle dq \quad (3)$$

The integral only equals the sum if the function  $\mathcal{F}(q)$  is sufficiently continuous in the domain over which the integration takes place. Otherwise the left side only equals the right side when domain is restricted to the region of the parameter space  $\mathfrak{R}$  where  $\mathcal{F}(q)$  is sufficiently continuous. If  $\mathcal{F}(q)$  contains a series of point-like discontinuities, then these locations must be encapsulated and the encapsulated regions must be treated separately. Here the symmetry centers will play a significant role. ***Their symmetry flavor will influence the result of the overall integration.*** The overall integration covers all involved parameter spaces. This effect will attribute symmetry related properties to the symmetry centers.

The encapsulating boundary must be located in a region where the defining function is continuous. The shape of the boundary can be selected in a free fashion. For example it is possible to apply a cube rather than a sphere as encapsulating boundary. If the cube is taken parallel to the axes of the Cartesian coordinate system, which is used for defining the symmetry flavor, then the influence of the symmetry on the integration result can be easily determined. It leads to a short list of possible effects in which the three dimensions play a significant role. These effects can be divided in isotropic effects and three mutually independent categories of anisotropic effects.

Remember that quaternionic number systems exist in several versions, thus also the operators  $f$  and  $\mathcal{F}$  exist in these versions. The same holds for the parameter space operators. When relevant, we will use superscripts in order to differentiate between these versions.

Thus, operator  $f^x = |q_i^x\rangle f^x(q_i^x)\langle q_i^x|$  is a specific version of operator  $f$ . Function  $f^x(q_i^x)$  uses parameter space  $\mathcal{R}^x$ .

Similarly,  $\mathcal{F}^x = |q^x\rangle\mathcal{F}^x(q^x)\langle q^x|$  is a specific version of operator  $\mathcal{F}$ . Function  $\mathcal{F}^x(q^x)$  and continuum  $\mathcal{F}^x$  use parameter space  $\mathfrak{R}^x$ . If the operator  $\mathcal{F}^x$  that resides in the Gelfand triple  $\mathcal{H}$  uses the same defining function as the operator  $\mathcal{F}^x$  that resides in the separable Hilbert space, then both operators belong to the same quaternionic ordering version.

In general the dimension of a subspace loses its significance in the non-separable Hilbert space.

The continuums that appear as eigenspaces in the non-separable Hilbert space  $\mathcal{H}$  can be considered as quaternionic functions that also have a representation in the corresponding infinite dimensional separable Hilbert space  $\mathfrak{H}$ . Both representations use a flat parameter space  $\mathfrak{R}^x$  or  $\mathcal{R}^x$  that is spanned by quaternions.  $\mathcal{R}^x$  is spanned by rational quaternions.

The parameter space operators will be treated as reference operators. The rational quaternionic eigenvalues  $\{q_i^x\}$  that occur as eigenvalues of the reference operator  $\mathcal{R}^x$  in the separable Hilbert space map onto the rational quaternionic eigenvalues  $\{q_i^x\}$  that occur as subset of the quaternionic

eigenvalues  $\{q^x\}$  of the reference operator  $\mathfrak{R}^x$  in the Gelfand triple. In this way the reference operator  $\mathcal{R}^x$  in the infinite dimensional separable Hilbert space  $\mathfrak{H}$  relates directly to the reference operator  $\mathfrak{R}^x$ , which resides in the Gelfand triple  $\mathcal{H}$ .

All operators that reside in the Gelfand triple and are defined via a mostly continuous quaternionic function have a representation in the separable Hilbert space.

## 4 Types of operators

Only a special type of operators can directly be handled by the reverse bra-ket method. In that case the defining function must be available within the realm of the Hilbert space. All operators that are defined in the separable Hilbert space and that can be represented by a sufficiently continuous function, possess a smoothing companion in the non-separable Hilbert space. The integration process that is used by the reverse bra-ket method can handle point-like discontinuities and closed cavities in the parameter space of the defining function, where the defining function does not exist. These artifacts are handled by separating them from the **validity domain** [8].

Other types of operators are:

- Stochastic operators
  - These operators get their eigenvalues via mechanisms that reside outside of the realm of the Hilbert space and use stochastic processes in order to generate the eigenvalues.
- Density operators
  - If a stochastic operator generates a **coherent swarm** of eigenvalues that can be characterized by a continuous location density distribution, then the reverse bra-ket method can be used to define the corresponding density operator.
- Function operators
  - Function operators act on functions and in that way they produce new functions that can be used as defining functions of the corresponding operator.
- Smoothing operators
  - These operators have a defining function which is the convolution of an original defining function and a smoothing function that blurs the original defining function.
- Partial differential operators
  - These are special kinds of function operators.
  - The existence of partial differentials of quaternionic functions create the existence of partial differential operators that work in combination with the operators that define the function related operator.

In quaternionic differential calculus the partial differential operators work as multipliers.

$$g = g_0 + \mathbf{g} = \nabla f = (\nabla_0 + \nabla)(f_0 + \mathbf{f}) = \nabla_0 f_0 - \langle \nabla, \mathbf{f} \rangle + \nabla_0 \mathbf{f} + \nabla f_0 + \nabla \times \mathbf{f} \quad (1)$$

The terms of equation (1) are often given special names and symbols:

$$g_0 = \nabla_0 f_0 - \langle \nabla, \mathbf{f} \rangle \quad (2)$$

$$\mathbf{g} = \nabla_0 \mathbf{f} + \nabla f_0 + \nabla \times \mathbf{f} \quad (3)$$

$$\mathfrak{E} = -\nabla_0 f - \nabla f_0 \quad (4)$$

$$\mathfrak{B} = \nabla \times f \quad (5)$$

If  $\mathfrak{D}$  is a partial differential operator and  $\mathcal{G} = \mathfrak{D}\mathcal{F}$  for a category of functions  $\{\mathcal{F}\}$ , where  $\mathcal{G}$  is sufficiently continuous, then for all bra's  $\langle x|$  and all ket's  $|y\rangle$  hold:

$$\langle x|\mathcal{G}|y\rangle = \langle x|\mathfrak{D}\mathcal{F}|y\rangle \approx \int_q \langle x|q\rangle \mathfrak{D}\mathcal{F}(q) \langle q|y\rangle dq = \int_q \langle x|q\rangle \mathcal{G}(q) \langle q|y\rangle dq \quad (6)$$

Differential operators work on the category of operators that can be represented by defining functions, which can be differentiated. Especially the Hermitian kind of these operators appear to be of interest for application in physical theories.

Some Hermitian partial differential operators do not mix scalar and vector parts of functions. These are:

$$\begin{aligned} &\nabla_0 \\ &\nabla_0 \nabla_0 \\ &\langle \nabla, \nabla \rangle \end{aligned}$$

These operators can be combined in additions as well as in products. Two particular operators are:

$$\begin{aligned} \nabla \nabla^* &= \nabla^* \nabla = \nabla_0 \nabla_0 + \langle \nabla, \nabla \rangle \\ \mathfrak{D} &= -\nabla_0 \nabla_0 + \langle \nabla, \nabla \rangle \end{aligned}$$

The last one is the quaternionic version of d'Alembert's operator. The first one can be split into  $\nabla$  and  $\nabla^*$ . The second one cannot be split into quaternionic first order partial differential operators.

The field  $\mathfrak{F}$  is considered to be regular in spatial regions where the defining function  $\mathfrak{F}(q)$  obeys

$$\langle \nabla, \nabla \rangle \mathfrak{F} = 0 \quad (7)$$

Similar considerations hold for regions where:

$$\nabla \nabla^* \mathfrak{F} = (\nabla_0 \nabla_0 + \langle \nabla, \nabla \rangle) \mathfrak{F} = 0 \quad (8)$$

$$\mathfrak{D} \mathfrak{F} = (-\nabla_0 \nabla_0 + \langle \nabla, \nabla \rangle) \mathfrak{F} = 0 \quad (9)$$

## 5 Conclusion

The reverse bra-ket method is an extension of Dirac's bra-ket notation. The method involves summation and/or integration over a complete set of orthonormal base vectors, which are eigenvectors of a reference operator, whose eigenspace also acts as a parameter space. Hilbert spaces can inhabit multiple reference operators and in this way they inhabit multiple parameter spaces. This fact makes the reverse bra-ket method a very powerful tool that helps generating flexible mathematical test models. In these models the reverse bra-ket method links Hilbert space technology with function theory and with differentiation and integration technology.

### References

[1] In 1843 quaternions were discovered by Rowan Hamilton.

[http://en.wikipedia.org/wiki/History\\_of\\_quaternions](http://en.wikipedia.org/wiki/History_of_quaternions)

[2] Quantum logic was introduced by Garret Birkhoff and John von Neumann in their paper: G. Birkhoff and J. von Neumann, (1936) The Logic of Quantum Mechanics, *Annals of Mathematics*, **37**, 823–843

This paper also indicates the relation between this orthomodular lattice and separable Hilbert spaces.

[3] "Division algebras and quantum theory" by John Baez. <http://arxiv.org/abs/1101.5690> lists the restrictions that a selected foundation poses onto its extension.

[4] The Hilbert space was discovered in the first decades of the 20-th century by David Hilbert and others. See [http://en.wikipedia.org/wiki/Hilbert\\_space](http://en.wikipedia.org/wiki/Hilbert_space).

[5] In the sixties Israel Gelfand and Georgyi Shilov introduced a way to model continuums via an extension of the separable Hilbert space into a so called Gelfand triple. The Gelfand triple often gets the name rigged Hilbert space. It is a non-separable Hilbert space.

[http://www.encyclopediaofmath.org/index.php?title=Rigged\\_Hilbert\\_space](http://www.encyclopediaofmath.org/index.php?title=Rigged_Hilbert_space) .

[6] Paul Dirac introduced the bra-ket notation, which popularized the usage of Hilbert spaces. Dirac also introduced its delta function, which is a generalized function. Spaces of generalized functions offered continuums before the Gelfand triple arrived.

See: P.A.M. Dirac.(1958), The Principles of Quantum Mechanics, Fourth edition, Oxford University Press, ISBN 978 0 19 852011 5.

[7] Quaternionic function theory and quaternionic Hilbert spaces are treated in: J.A.J. van Leunen. (2015) Quaternions and Hilbert spaces. <http://vixra.org/abs/1411.0178> .

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[9] The Hilbert Book Test Model applies all capabilities of the reverse bra-ket method. And shows how this method merges several mathematical technologies. <http://vixra.org/abs/1603.0021> .